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A New Common Fixed Point Theorem for Suzuki Generalized (ψ, φ)-Weak Contractions in Ordered Metric Spaces

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Abstract In this paper, a new common fixed point theorem for two mappings which are satisfied the Suzuki's generalized weak contractive condition in the setting of partially ordered metric spaces is established. Some suitable examples are furnished to demonstrate the validity of the hypotheses of our results and reality of our generalizations. The results of this paper can be viewed as a generalization and improvement of some well-known results in this area.

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1. Introduction

It is well-known that the fixed point theory is an important and powerful tool to study nonlinear analysis and the Banach contraction principle which is a fundamental result in fixed point theory has been extended by some authors, see, for instance, [1–8] and the references therein.
The concept of weak contraction was introduced by Alber and Guerre Dlabriere [9] for single valued maps on Hilbert spaces in 1997. The study of fixed point in the setting of a partially ordered metric space was first started in 2004 by Ran and Reurings [10] and then by Nieto and López [11]. Subsequently, many authors obtained several interesting results in ordered metric spaces, for example, see [12–16].
During the last few decades many mathematical researchers have obtained a lot of results in common fixed point theory in ordered metric spaces, for example, see [11, 17–20].

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In 2010, Radenovic and Kadelberg [20] studied generalized weak contractions in partially ordered metric spaces and extended results of Dorić [3], Harjani and Sadarangani [21] as well as Zhang and Song [8].
Suzuki [22] obtained a powerful generalization of Banach contraction theorem in 2008. Using the idea of the Suzuki contraction, various fixed point results have been extended in many directions; see for instance [21–27]. Particularly, Singh et al [26] gave a weakly contractive version of Suzuki type in 2015 and generalized some results of Dorić [3], Zhang et al [8]. In this paper, a new version of Suzuki type contraction is introduced which class is larger than the class of weakly increasing maps in ordered metric spaces. The results of this paper extend and improve some famous results in this area, specially the results given in [10].

2. Preliminaries

In sequel, the following definitions and notations will be used in this paper.

Definition 2.1. [28] Let $(X, \preceq)$ be a partially ordered set and let T and S be two self-maps on X. then

1) the elements $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ holds and we denote it by $x \preceq\succeq y$.
2) a subset A of X is said to be well ordered if any two elements of A are comparable .
3) X is called regular if a nondecreasing sequence $\{x_n\}$ in $(X, \preceq)$ converges to $x \in X$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.
4) T is called nondecreasing w.r.t. $\preceq$ if $x \preceq y$ implies $Tx \preceq Ty$.
5) the mappings T and S are called weakly increasing if $Tx \preceq STx$ and $Sx \preceq TSx$ for all $x \in X$. In particular, if i_x is the identity function, T and i_x are weakly increasing maps if and only if $x \preceq Tx$ for each $x \in X$. In this case, T is called dominating.
If T and T are weakly increasing maps, then T is called weakly increasing map. It is clear that, T is a weakly increasing map if and only if $Tx \preceq T^2x$ for each $x \in X$.
There are some examples of weakly increasing maps (see [12]) when neither of such mappings T and S is nondecreasing w.r.t. $\preceq$.
6) T is said to be S-weakly isotone increasing if $Tx \preceq STx \preceq TSTx$ for all $x \in X$. Some examples of S-weakly isotone increasing maps can be found in [12].

The control functions were introduced by Doric [3] as follows:

Definition 2.2. A pair (ψ, φ) of self-maps on $[0, \infty)$ is called control functions if the following items are satisfied:

(1) ψ is a continuous nondecreasing function and $\psi(t) = 0$ if and only if $t = 0$.
(2) φ is lower semi-continuous with $\varphi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems which are based on control functions (see, e.g. [13, 29]).
For the sake of reader, we follow the following notations:

- $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\}$
- $n(x, y) = \max\{d(x, y), \frac{d(x,Tx)+d(y,Sy)}{2}, \frac{d(x,Sy)+d(y,Tx)}{2}\}$

- $m_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$
- $n_T(x,y) = \max\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$

for all $x, y \in X$, where T and S are two self-maps on the metric space (X,d).
We state the following lemma which is useful in proving our first main result.

Lemma 2.3 ([30]). *Let (X,d) be a metric space, and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence then there exists $\epsilon > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) > \epsilon$, $d(x_{m_k}, x_{n_k-1}) < \epsilon$ and*

(1) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$.
(2) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$.
(3) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon$.
(4) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$.

3. Main Results

In this section, a new concept of weakly increasing maps in ordered spaces is introduced. A common fixed point theorem and two uniqueness theorems are presented. These results can be viewed as a generalization and improvement of some results which have been appeared in this area, for instance, Radenovic and Kadelberg's results in ordered metric spaces in [20].

The following definition is a new version of the definition of weakly increasing maps.

Definition 3.1. Let $(X, \preceq)$ be an ordered set and (T,S) be a pair of self-maps on X. For any $x \in X$ and $n \geq 0$ put:

$$f_0(T,S;x) = Tx \text{ and } f_{n+1}(T,S;x) = T^{\delta_n} S^{1-\delta_n} f_n(T,S;x)$$

where $\delta_{2m} = 1$ and $\delta_{2m+1} = 0$, $\forall m \geq 0$.
The maps T and S are called partially-weakly isotone if there exist $x_0 \in X$ such that, at least one of the sequences $\{f_n(T,S;x_0)\}_{n\geq 0}$ and $\{f_n(S,T;x_0)\}_{n\geq 0}$ is nondecreasing w.r.t. $\preceq$.

In particular, if T and T are partially-weakly isotone maps, then T is called a partially-weakly isotone map. It is clear that T is a partially-weakly isotone map if and only if $T^n(x_0) \preceq T^{n+1}(x_0)$ for some $x_0 \in X$ and for any $n \in \mathbb{N}$.

Example 3.2. Let $(X, \preceq)$ be an ordered set.

(1) every two weakly increasing maps on X are partially-weakly isotone maps. Indeed, if (T,S) is a weakly increasing pair of self-maps on X, then both of the sequences $\{f_n(T,S;x)\}_{n\geq 0}$ and $\{f_n(S,T;x)\}_{n\geq 0}$ are nondecreasing w.r.t. $\preceq$, for any $x \in X$.
It is clear that, the converse is false.
(2) if T is S-weakly isotone increasing, then T and S are partially-weakly isotone maps. Indeed, For any $x \in X$, the sequence $\{f_n(T,S;x_0)\}_{n\geq 0}$ is nondecreasing w.r.t. $\preceq$.
(3) let $T : X \to X$ be a nondecreasing map such that $x_0 \preceq Tx_0$ for some $x_0 \in X$, then T is a partially-weakly isotone map. clearly,the converse is false.

Now, we present our first result.

Theorem 3.3. *Let $(X, \preceq, d)$ be an ordered complete metric space and (T, S) be a pair of partially-weakly isotone maps on X such that, for any $x, y \in X$, $\left(x \preceq\succeq y \text{ and } \frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \leq d(x,y)\right)$ implies*

$$\psi(d(Tx, Sy)) \leq \psi(m(x,y)) - \phi(m(x,y)) \tag{3.1}$$

where (ψ, φ) is a pair of control functions. Then T and S have a common fixed point provided that at least one of the following cases holds:

(i): *T or S is continuous.*
(ii): *X is regular.*

Proof. Since (T, S) is partially-weakly isotone, there exists $x_0 \in X$ such that, at least one of the sequences $\{f_n(T, S; x_0)\}_{n\geq 0}$ or $\{f_n(S, T; x_0)\}_{n\geq 0}$ is nondecreasing w.r.t. $\preceq$, for instance, the first one. We construct a recursive sequence $\{x_n\}_{n\geq 1}$ as follows:
For any $n \geq 0$, define:

$$x_{n+1} = f_n(T, S; x_0).$$

So $\{x_n\}$ is nondecreasing w.r.t. $\preceq$ and for all $n \geq 0$, we have:

$$\begin{aligned} x_{2n+1} &= f_{2n}(T, S; x_0) \\ &= T^{\delta_{2n}} S^{1-\delta_{2n}} f_{2n-1}(T, S; x_0) \\ &= T^1 S^0 f_{2n-1}(T, S; x_0) \\ &= Tx_{2n}. \end{aligned}$$

Also

$$\begin{aligned} x_{2n+2} &= f_{2n+1}(T, S; x_0) \\ &= T^{\delta_{2n+1}} S^{1-\delta_{2n+1}} f_{2n+1}(T, S; x_0) \\ &= T^0 S^1 f_{2n+1}(T, S; x_0) \\ &= Sx_{2n+1}. \end{aligned}$$

Now, We clam that if there exists $k_0 \in \mathbb{N}$ such that $x_{k_0} = x_{k_0+1}$, then $x_k = x_{k_0}$ for all $k \geq k_0$.
To see this, at first suppose that $k_0 = 2n$ for some $n \in \mathbb{N}$. In this case we have $x_{2n} = x_{2n+1}$ so
$x_{2n} \preceq x_{2n+1}$ and

$$\begin{aligned} \tfrac{1}{2}\min\{d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1})\} &= \tfrac{1}{2}\min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &\leq d(x_{2n}, x_{2n+1}). \end{aligned}$$

hence, by (3.1) we have:

$$\psi(d(x_{2n+1}, x_{2n+2})) = \psi(d(Tx_{2n}, Sx_{2n+1})) \leq \psi(m(x_{2n}, x_{2n+1})) - \phi(m(x_{2n}, x_{2n+1})).$$

Since $x_{2n} = x_{2n+1}$, we have:

$$\begin{aligned} m(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \\ &\qquad \tfrac{d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})}{2}\} \\ &= \max\{d(x_{2n+1}, x_{2n+2}), \tfrac{d(x_{2n}, x_{2n+2})}{2}\} \\ &= d(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Thus

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n+1}, x_{2n+2})) - \phi(d(x_{2n+1}, x_{2n+2}))$$

which is a contradiction unless $d(x_{2n+1}, x_{2n+2}) = 0$ i.e. $x_{2n+1} = x_{2n+2}$.
Hence $x_{k_0} = x_{k_0+1} = x_{k_0+2}$.
Similarly, if $k_0 = 2n + 1$ for some $n \geq 0$, we can prove that $x_{k_0+1} = x_{k_0+2}$.
Therefore, x_{k_0} is a common fixed point of T and S. we can then suppose that $d(x_n, x_{n+1}) > 0$ for all $n \geq 0$.
For convenience, we divide the rest of the proof into three steps.

Step (1): We prove that $\{x_n\}$ is asymptotically regular, i.e. $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$.

To prove it, at first we claim that

$$d(x_{k+1}, x_{k+2}) \leq m(x_k, x_{k+1}) = d(x_k, x_{k+1}), \ \forall k \in \mathbb{N}. \tag{3.2}$$

To see this, suppose that $k = 2n$ for some $n \in \mathbb{N}$. Since $x_{2n+1} \preceq\succeq x_{2n+2}$ and

$$\begin{aligned} \tfrac{1}{2}\min\{d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1})\} &= \tfrac{1}{2}\min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &\leq \ d(x_{2n}, x_{2n+1}), \end{aligned}$$

from (3.1) we have:

$$\begin{aligned} \psi(d(x_{2n+2}, x_{2n+1})) &= \psi(d(Sx_{2n+1}, Tx_{2n})) \\ &\leq \psi(m(x_{2n}, x_{2n+1})) - \phi(m(x_{2n}, x_{2n+1})) \\ &\leq \psi(m(x_{2n}, x_{2n+1})). \end{aligned} \tag{3.3}$$

So that

$$d(x_{2n+2}, x_{2n+1}) \leq m(x_{2n}, x_{2n+1}). \tag{3.4}$$

On the other hand,

$$\begin{aligned} m(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}) \\ &\quad , \tfrac{d(x_{2n}, Sx_{2n+1} + d(x_{2n+1}, Tx_{2n}))}{2}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \tfrac{d(x_{2n}, x_{2n+2})}{2}\} \\ &\leq \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \tfrac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}\} \\ &\leq \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

So if $d(x_{2n_0+1}, x_{2n_0+2}) \geq d(x_{2n_0}, x_{2n_0+1})$ for some $n_0 \in \mathbb{N}$, then

$$m(x_{2n_0}, x_{2n+1}) \leq d(x_{2n_0+1}, x_{2n_0+2}).$$

So, by (3.4)

$$m(x_{2n_0}, x_{2n_0+1}) = d(x_{2n_0+1}, x_{2n_0+2}).$$

But in this case (3.3) yields

$$\psi(d(x_{2n_0+2}, x_{2n_0+1})) \leq \psi(d(x_{2n_0+1}, x_{2n_0+2})) - \phi(d(x_{2n_0+1}, x_{2n_0+2})).$$

Which is a contradiction (because $d(x_{2n_0+1}, x_{2n_0+2}) > 0$).

Hence $d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1})$ and so $m(x_{2n}, x_{2n+1}) \leq d(x_{2n}, x_{2n+1})$. Also we have

$$m(x_{2n}, x_{2n+1}) \geq d(x_{2n}, x_{2n+1}).$$

Consequently, (3.2) is proved when $k > 0$ is an even number. By the same argument, one can verify that (3.2) holds when k is an odd number. Thus, the sequence $\{d(x_n, x_{n+1})\}_{n\geq 1}$ is nondecreasing and bounded below, so it converges

to a real number $r \geq 0$.
We have:

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} m(x_n, x_{n+1}) = r.$$

Taking limit(upper limit) on both side of (3.3), one can conclude that

$$\psi(r) \leq \psi(r) - \phi(r).$$

Which is a contradiction unless $r = 0$.
Consequently, we showed that:

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} m(x_n, x_{n+1}) = 0. \tag{3.5}$$

Step (2): $\{x_n\}$ is a Cauchy sequence.
At first, note that $(X, \preceq)$ is partially ordered and $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$. Thus $x_n \preceq x_m$ for all $m \geq n$ and so $x_m \preceq\succeq x_n$ for any $m, n \in \mathbb{N}$.
Now, to show that $\{x_n\}$ is a Cauchy sequence, because of (3.5), it is enough to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence.
On contrary, Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then by Lemma 2.3 there exists $\epsilon_0 > 0$ and subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and $d(x_{2m_k}, x_{2n_k})) \geq \epsilon_0$ and

(l_1) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \epsilon_0$.
(l_2) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k}) = \epsilon_0$.
(l_3) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \epsilon_0$.
(l_4) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k+1}) = \epsilon_0$.

Therefore, from the definition of $m(x, y)$ we have:

$$\begin{aligned}\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k-1}) &= \lim_{k\to\infty} \max\{d(x_{2n_k}, x_{2m_k-1}), d(x_{2n_k}, x_{2n_k+1}), d(x_{2m_k-1}, x_{2m_k}) \\ &\quad , \tfrac{d(x_{2n_k}, x_{2m_k} + d(x_{2m_k-1}, x_{2n_k+1}))}{2}\} \\ &= \max\{\epsilon_0, 0, 0, \tfrac{\epsilon_0+\epsilon_0}{2}\} \\ &= \epsilon_0.\end{aligned}$$

So

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \lim_{k\to\infty} m(x_{2n_k}, x_{2m_k-1}) = \epsilon_0. \tag{3.6}$$

Now we claim that for sufficiently large $k \in \mathbb{N}$, if $n_k > m_k > k$ then

$$\frac{1}{2}\min\{d(x_{2n_k}, Tx_{2n_k}), d(x_{2m_k-1}, Sx_{2m_k-1})\} \leq d(x_{2n_k}, x_{2m_k-1}). \tag{3.7}$$

Indeed, since $n_k > m_k$ and $\{d(x_n, x_{n+1})\}$ is non-increasing we have

$$\begin{aligned}d(x_{2n_k}, Tx_{2n_k}) &= d(x_{2n_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k+1}, x_{2m_k}) \leq d(x_{2m_k}, x_{2m_k-1}) = d(x_{2m_k-1}, Sx_{2m_k-1}).\end{aligned}$$

And so, the left hand side of inequality (3.7) is equal to $\frac{1}{2}d(x_{2n_k}, Tx_{2n_k}) = \frac{1}{2}d(x_{2n_k}, x_{2n_k+1})$.
Therefore we must show that, for sufficiently large $k \in \mathbb{N}$, if $n_k > m_k > k$ then

$$d(x_{2n_k}, x_{2n_k+1}) \leq d(x_{2n_k}, x_{2m_k-1})$$

According to (3.5), there exist $k_1 \in \mathbb{N}$ such that for any $k > k_1$,

$$d(x_{2n_k}, x_{2n_k+1}) < \frac{1}{2}\epsilon_0.$$

Also, there exist $k_2 \in \mathbb{N}$ such that for any $k > k_2$,

$$d(x_{2m_k-1}, x_{2m_k}) < \frac{1}{2}\epsilon_0.$$

Hence for any $k > \max\{k_1, k_2\}$ and $n_k > m_k > k$,

$$\begin{array}{rcl} \epsilon_0 & \leq & d(x_{2n_k}, x_{2m_k}) \\ & \leq & d(x_{2n_k}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2m_k}) \\ & \leq & d(x_{2n_k}, x_{2m_k-1}) + \frac{\epsilon_0}{2}. \end{array}$$

So one conclude that

$$\frac{\epsilon_0}{2} \leq d(x_{2n_k}, x_{2m_k-1})$$

Thus we obtain that for any $k > \max\{k_1, k_2\}$ and $n_k > m_k > k$

$$d(x_{2n_k}, x_{2n_k+1}) \leq \frac{\epsilon_0}{2} \leq d(x_{2n_k}, x_{2m_k-1}).$$

And (3.7) is prove. beside, we know that $x_{2n_k} \preceq\succeq x_{2m_k-1}$, so, (3.1) implies that

$$\begin{array}{rcl} \psi(d(x_{2n_k+1}, x_{2m_k})) & = & \psi(d(Tx_{2n_k}, Sx_{2m_k-1})) \\ & \leq & \psi(m(x_{2n_k}, x_{2m_k-1})) - \phi(m(x_{2n_k}, x_{2m_k-1})). \end{array} \tag{3.8}$$

Taking upper limit on both side of (3.8) and applying (3.6), one can conclude that

$$\psi(\epsilon_0) \leq \psi(\epsilon_0) - \phi(\epsilon_0).$$

Which contradicts $\epsilon_0 > 0$. So $\{x_n\}$ is a Cauchy sequence and since X is complete, there exists $u \in X$ such that $x_n \to u$, as $n \to \infty$.

Step (3): u is a common fixed point of T and S.

We shall distinguish the cases (i) and (ii) of the theorem.

(i)**:** Suppose that S is continuous. Since $x_n \to u$, we have $Sx_{2n+1} \to Su$, i.e. $x_{2n+2} \to Su$.

On the other hand, since $x_n \to u$, so $x_{2n+2} \to u$. Hence $Su = u$. Now, we have $u \preceq\succeq u$, also

$$\begin{array}{rcl} \frac{1}{2}\min\{d(u, Tu), d(u, Su)\} & = & \frac{1}{2}\min\{d(u, Tu), 0\} \\ & = & 0 \\ & \leq & d(u, u). \end{array} \tag{3.9}$$

Thus, (3.1) implies that

$$\psi(d(Tu, Su)) \leq \psi(m(u, u)) - \phi(m(u, u)).$$

Where

$$\begin{array}{rcl} m(u, u) & = & \max\{d(u, u),\ d(u, Tu),\ d(u, Su),\ \frac{d(u,Su)+d(u,Tu)}{2}\} \\ & = & \max\{0,\ d(u, Tu), 0,\ \frac{d(u,Tu)}{2}\} \\ & = & d(u, Tu). \end{array}$$

Consequently

$$\psi(d(Tu, u)) = \psi(d(Tu, Su)) \leq \psi(d(Tu, u)) - \phi(d(Tu, u)).$$

This is a contradiction unless $d(Tu, u) = 0$ i.e. $Tu = u$. So we obtain that $Su = Tu = u$.

Similarly, If T is continuous, one can prove that $Su = Tu = u$.

(ii)**:** Assume that X is regular. Then, since the sequence $\{x_n\}$ is nondecreasing with respect to $\preceq$, and $x_n \to u$ as $n \to \infty$, it follows that $x_n \preceq u$, for all $n \geq 0$ Now, at first we want to prove that

$$\lim_{n\to\infty} m(u, x_{2n}) = d(Su, u). \tag{3.10}$$

For this purpose, notice that

$$m(x_{2n}, u) = \max\{d(x_{2n}, u), d(x_{2n}, Tx_{2n}), d(u, Su), \frac{d(x_{2n}, Su) + d(u, Tx_{2n})}{2}\}.$$

Hence

$$\begin{aligned} d(u, Su) &\leq m(x_{2n}, u) \\ &= \max\{d(x_{2n}, u), d(x_{2n}, x_{2n+1}), d(u, Su), \frac{d(x_{2n}, Su) + d(u, x_{2n+1})}{2}\}. \end{aligned}$$

Letting $n \to \infty$ we obtain that

$$\begin{array}{rcl} d(u, Su) & \leq & \lim_{n\to\infty} m(u, x_{2n}) \\ & \leq & \max\{0, 0, d(u, Su), \frac{d(u,Su)+0}{2}\} \\ & = & d(u, Su). \end{array}$$

Thus, (3.10) is proved. In the same manner, one can conclude that

$$\lim_{n\to\infty} m(u, x_{2n+1}) = d(Tu, u) \tag{3.11}$$

Now, we claim that for all $n \geq 0$

$$\frac{1}{2}d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, u) \ or \ \frac{1}{2}d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, u).$$

If, for some $n_0 \geq 0$, both of them are false we will have

$$\begin{array}{rcl} d(x_{2n_0}, x_{2n_0+1}) & \leq & d(x_{2n_0}, u) + d(u, x_{2n_0+1}) \\ & < & \frac{1}{2}d(x_{2n_0}, x_{2n_0+1}) + \frac{1}{2}d(x_{2n_0+1}, x_{2n_0+2}) \\ & \leq & \frac{1}{2}d(x_{2n_0}, x_{2n_0+1}) + \frac{1}{2}d(x_{2n_0}, x_{2n_0+1}) \\ & = & d(x_{2n_0}, x_{2n_0+1}). \end{array}$$

Which is a contradiction and the claim is proved.

Now suppose that

$$\frac{1}{2}d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, u).$$

Therefore

$$\begin{array}{rcl} \frac{1}{2}\min\{d(x_{2n}, Tx_{2n}), d(u, Su)\} & = & \frac{1}{2}\min\{d(x_{2n}, x_{2n+1}), d(u, Su)\} \\ & \leq & \frac{1}{2}d(x_{2n}, x_{2n+1}) \\ & \leq & d(x_{2n}, u). \end{array}$$

Furthermore $x_n \preceq u, \forall n \geq 0$. So (3.1) implies that

$$\begin{array}{rcl} \psi(d(x_{2n+1}Su)) & = & \psi(d(Tx_{2n}, Su)) \\ & \leq & \psi(m(x_{2n}, u)) - \phi(m(x_{2n}, u)). \end{array}$$

Letting $n \to \infty$, taking into account (3.10), one can conclude that

$$\psi(d(u, Su)) \leq \psi(d(u, Su)) - \phi(d(u, Su)).$$

Which is a contradiction unless $d(u, Su) = 0$. i.e. $Su = u$
Also, we have $u \preceq u$ and

$$\begin{aligned} \tfrac{1}{2}\min\{d(u,Tu), d(u,Su)\} &= \tfrac{1}{2}\min\{d(u,Tu), 0\} \\ &= 0 \\ &\leq d(u,u). \end{aligned}$$

Thus, from (3.1) we obtain that

$$\begin{aligned} \psi(d(Tu,u)) &= \psi(d(Tu,Su)) \\ &\leq \psi(m(u,u)) - \phi(m(u,u)). \end{aligned}$$

Where

$$\begin{aligned} m(u,u) &= \max\{d(u,u),\ d(u,Tu),\ d(u,Su),\ \tfrac{d(u,Su)+d(u,Tu)}{2}\} \\ &= \max\{0,\ d(u,Tu), 0,\ \tfrac{d(u,Tu)}{2}\} \\ &= d(u,Tu). \end{aligned}$$

Therefor, we observe that

$$\psi(d(Tu,u)) \leq \psi(d(u,Tu)) - \phi(d(u,Tu)).$$

Which is a contradiction unless $Tu = u$. Hence we obtain that $Tu = Su = u$. Similarly, if we consider (3.11), we can proved that the second part of our clam leads to contradiction, unless $Su = Tu = u$.
So in any case, u is a common fixed point of T and S. And proof is completed.

■

Corollary 3.4. *Let all the conditions of Theorem 3.4 be satisfied, except (3.1) which is replaced by the following condition :*
There exists a positive Lebesque integrable function f on $\mathbb{R}^+$ such that $\int_0^\epsilon f(t)dt > 0$ for each $\epsilon > 0$, and for every $x, y \in X$, $\Big(x \preceq\succeq y$ and $\frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \leq d(x,y)\Big)$ implies

$$\int_0^{\psi(d(Tx,Sy))} f(t)dt \leq \int_0^{\psi(m(x,y))} f(t)dt - \int_0^{\varphi(m(x,y))} f(t)dt.$$

Then T and S have at least one common fixed point.

Proof. Let $G(u) := \int_0^u f(t)dt\ \forall u > 0$. Then
$\Big(x \preceq\succeq y$ *and* $\frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \leq d(x,y)\Big)$ implies

$$Go\psi(d(Tx,Sy)) \leq Go\psi(m(x,y)) - Go\varphi(m(x,y)).$$

It is easy verify that $\psi_1 := Go\psi$ and $\varphi_1 := Go\varphi$ are control functions and all conditions of Theorem 3.4 are satisfied (for ψ_1 and φ_1). Therefore T and S have at least one common fixed point. ■

If we take $S = T$ in Theorem 3.4 then we can obtain a fixed point theorem for T.

Corollary 3.5. *Let $(X, \preceq, d)$ be an ordered complete metric space and $T : X \to X$ be a partially-weakly isotone map such that,*

for any $x, y \in X$, $\left(x \preceq\succeq y \text{ and } \frac{1}{2}d(x,Tx) \leq d(x,y)\right)$ implies

$$\psi(d(Tx,Ty)) \leq \psi(m_T(x,y)) - \phi(m_T(x,y)). \tag{3.12}$$

Where (ψ, φ) is a pair of control functions .
Then, in each of the following two cases, T has a fixed point.

- **(i):** *T is continuous.*
- **(ii):** *X is regular.*

Taking into account part (3) of Example 3.1, one can obtain the following corollary:

Corollary 3.6. *Let $T : X \to X$ be a nondecreasing map such that $x_0 \preceq Tx_0$ for some $x_0 \in X$ and other conditions of Corollary 3.5 be fulfilled. Then T has a fixed point.*

The following two results are immediately derived from Theorem 3.4.

Corollary 3.7 ([10] Theorem 3.1)**.** *Let $(X, \preceq, d)$ be an ordered complete metric space and (T, S) be a pair of weakly increasing maps on X such that for any two comparable elements $x, y \in X$*

$$\psi(d(Tx,Sy)) \leq \psi(m(x,y)) - \phi(m(x,y)).$$

where (ψ, φ) is a pair of control functions. Then T and S have a common fixed point provided by at least one of the following cases hold:

- **(i):** *T or S is continuous.*
- **(ii):** *X is regular.*

Corollary 3.8 ([10] Corollary 3.3)**.** *Let $(X, \preceq, d)$ be an ordered complete metric space and $T : X \to X$ be a nondecreasing map such that $x_0 \preceq Tx_0$, for some $x_0 \in X$. If for every comparable elements $x, y \in X$ the following inequality holds*

$$\psi(d(Tx,Ty)) \leq \psi(m_T(x,y)) - \phi(m_T(x,y)). \tag{3.13}$$

Where (ψ, φ) is a pair of control functions .
Then, in each of the following two cases, T has a fixed point.

- **(i):** *T is continuous.*
- **(ii):** *X is regular.*

Theorem 3.9. *Assume that all the conditions of Corollary 3.5 are satisfied. Then T has a unique fixed point if and only if the set of all fixed points of T is well ordered.*

Proof. By Corollary 3.5, T has at least a fixed point. Now, if the fixed point of T is unique then the set of all fixed points of T is a singleton and so is well ordered.

Conversely, suppose that the set of all fixed points of T is well ordered, and u and v are two distinct fixed point of T. Then $u \preceq\succeq v$ and

$$\frac{1}{2}d(u,Tu) = 0 \leq d(u,v).$$

Hence

$$\begin{aligned}\psi(d(u,v)) &= \psi(d(Tu,Tv) \\ &\leq \psi(m_T(u,v)) - \phi(m_T(u,v)).\end{aligned}$$

Where

$$\begin{aligned} m_T(u,v) &= \max\{d(u,v),\ d(u,Tu),\ d(v,Tv),\ \tfrac{d(u,Tv)+d(v,Tu)}{2}\} \\ &= \max\{d(u,v),0,0,\ \tfrac{d(u,v)+d(v,u)}{2}\} \\ &= d(u,v). \end{aligned}$$

Thus we obtain that

$$\psi(d(u,v)) \leq \psi(d(u,v)) - \phi(d(u,v))$$

Which is a contradiction unless $d(u,v) = 0$ .i.e. $u = v$. ■

The next result establishes a sufficient condition for uniqueness of fixed point.

Theorem 3.10. *If the nondecreasing map $T : X \to X$ satisfies the conditions of Corollary 3.5 and the following assumption:*
(a) for arbitrary non-comparable two points $x, y \in X$ there exists $z \in X$ which is comparable with x and y, and also $z \preceq Tz$.
Then T has a unique fixed point.

Proof. At first, we claim that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.
Actually, if X is a singleton, say $X = \{x_0\}$, then $x_0 = T(x_0)$ and x_0 is the unique fixed point of T. Also if any two elements of X are comparable then $x \preceq Tx$ for any $x \in X$. In other wise, our claim is proved by the condition (a).
So, By Corollary 3.5, T has a fixed point.
Now let u and v be two fixed points of T. One of the following two cases can occur:

(1) $u \preceq\succeq v$.
Similarly as in the proof of Theorem 3.11, it can be shown that $u = v$.

(2) u and v are not comparable .
In this case by the hypothesis, there exists $z \in X$ such that $z \preceq\succeq u$, $z \preceq\succeq v$ and $z \preceq Tz$.
Put $y_n := T^n y$, for any $y \in X$ and $n \geq 0$.
Since T is nondecreasing, we obtain that $u = u_n \preceq\succeq z_n$, $v = v_n \preceq\succeq z_n$ for each $n \geq 0$.
If there exists $n_0 \geq 0$ such that, $z_{n_0} = u$ then $v \preceq\succeq z_{n_0} = u$ and so from item (1), $u = v$.
Thus, we can assume $z_n \neq u$, $\forall n \geq 0$.

Since $z \preceq Tz$, similarly as in the proof of Theorem 3.4, it can be shown that

$$\lim_{n\to\infty} d(z_{n-1}, z_n) = 0, \tag{3.14}$$

and $\{z_n\}$ is a convergent sequence.
Now, we claim that

$$\lim_{n\to\infty} d(u, z_n) = 0.$$

Indeed, for any $n \geq 1$, we have $u = u_n \preceq\succeq z_n$ and $\frac{1}{2}d(u,Tu) = 0 \leq d(u, z_{n-1})$, hence,

$$\begin{aligned} \psi(d(u,z_n)) &= \psi(d(Tu_{n-1}, Tz_{n-1}) \\ &\leq \psi(m_T(u_{n-1}, z_{n-1})) - \phi(m_T(u_{n-1}, z_{n-1})) \\ &\leq \psi(m_T(u, z_{n-1})) - \phi(m_T(u, z_{n-1})). \end{aligned}$$

Where

$$m_T(u, z_{n-1}) = \max\{d(u, z_{n-1}),\ d(u, Tu),\ d(z_{n-1}, Tz_{n-1}), \frac{d(u,Tz_{n-1})+d(z_{n-1},Tu)}{2}\} = \max\{d(u, z_{n-1}),\ d(z_{n-1}, z_n),\ \frac{d(u,z_n)+d(z_{n-1},u)}{2}\}.$$

And since ψ is nondecreasing we obtain that

$$d(u, z_n) \leq \max\{d(u, z_{n-1}),\ d(z_{n-1}, z_n),\ \frac{d(u, z_n) + d(z_{n-1}, u)}{2}\}.$$

One can consider the following two cases:

(a) there exists a sequence$\{n_k\}_{k\geq 0}$ of distinct positive integers that

$d(u, z_{n_k-1}) \leq d(z_{n_k-1}, z_{n_k})$.

In this case, (3.14) implies that

$$\lim_{k\to\infty} d(u, z_{n_k-1}) = 0$$

and since $\{z_n\}$ is a convergent sequence, one can conclude that

$$\lim_{n\to\infty} d(u, z_n) = 0.$$

(b) there exists $n_0 \geq 1$ such that, for any $n \geq n_0$

$d(u, z_n) > d(z_{n-1}, z_n)$.

In this case, for any $n \geq n_0$, we have

$$d(u, z_n) \leq \max\{d(u, z_{n-1}),\ \frac{d(u,z_n)+d(z_{n-1},u)}{2}\} \leq \max\{d(u, z_{n-1}), d(u, z_n)\}.$$

It is easily seen that,

$d(u, z_n) \leq m_T(u, z_{n-1}) = d(u, z_{n-1})$

Thus, for any $n \geq n_0$, the sequence $\{d(u, z_n)\}$ is non-increasing and so, it has a limit $l \geq 0$. In addition, we have:

$$\lim_{n\to\infty} m_T(u, z_{n-1}) = l.$$

Passing to (upper)limit in the relation

$\psi(d(u, z_n)) \leq \psi(m_T(u, z_{n-1})) - \phi(m_T(u, z_{n-1}))$

we obtain that

$\psi(l)) \leq \psi(l) - \phi(l)$.

Which is a contradiction unless $l = 0$.

So, in any case, we proved that

$$\lim_{n\to\infty} d(u, z_n) = 0.$$

In the same way, one can show that

$$\lim_{n\to\infty} d(v, z_n) = 0.$$

Finally, for any $n \geq 0$, we have

$0 \leq d(u, v) \leq d(u, z_n) + d(v, z_n)$.

Letting $n \to \infty$, we obtain that $d(u,v) = 0$ i.e. $u = v$.

Hence, in any case, the fixed point of T is unique.

■

It is clear that the Theorem 3.4 is a real generalization of Corollary 3.7. The following example shows that Corollary 3.6 is a generalization of the Corollary 3.8.

Example 3.11. Let $X = \{(0,0),(0,4),(5,0),(4,5),(5,4)\}$ be endowed with the metric d defined by

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2| \ .$$

Suppose that relation $\preceq$ is defined on X as follows:

$$(5,0) \preceq (0,4) \ , \ (4,5) \preceq (5,4),$$

$$(0,0) \preceq (x,y) \preceq (x,y) \ \forall (x,y) \in X.$$

It is easy to see that $(X, \preceq, d)$ is an regular ordered complete metric space.
Furthermore, suppose that $T : X \to X$ is defined as follows :

$$T(0,0) = T(5,0) = T(0,4) = (0,0),$$

$$T(4,5) = (5,0) \ , \ T(5,4) = (0,4).$$

It is obvious that T is nondecreasing with respect to $\preceq$ and $(0,0) \preceq (0,0) = T(0,0)$.
Now, we can verify that for any pair of control functions, T does not satisfy the condition (3.13) of Corollary 3.8, at $u = (4,5)$ and $v = (5,4)$.
Indeed, we have $u \preceq\succeq v$ and

$$d(Tu,Tv) = d((5,0),(0,4)) = 9.$$

Also

$$\begin{aligned} m_T(u,v) &= \max\{d(u,v), d(u,Tu), d(v,Tv), \tfrac{d(u,Tv)+d(v,Tu)}{2}\} \\ &= \max\{d((4,5),(5,4)), \ d((4,5),(5,0)), d((5,4),(0,4)) \\ &\quad , \tfrac{d((4,5),(0,4))+d((5,4),(5,0))}{2}\} \\ &= \max\{2, 6, 5, \tfrac{5+4}{2}\} \\ &= 6. \end{aligned}$$

And it is obvious that

$$\psi(9) \nleq \psi(6) - \varphi(6),$$

because ψ is nondecreasing and $\varphi(t) > 0 \ \forall t > 0$. Thus T does not satisfy the condition (3.13) of Corollary 3.8, at $u = (4,5)$ and $v = (5,4)$.
However, all the hypotheses of Corollary 3.6 are satisfied for T, with $\psi(t) = t$ and $\varphi(t) = \frac{1}{10}t$. In fact, for $u = (4,5)$ and $v = (5,4)$ we have

$$\frac{1}{2}d(u,Tu) = \frac{1}{2}d((4,5),(5,0)) = 3.$$

But $d(u,v) = d((4,5),(5,4)) = 2$. So, we obtain that

$$\frac{1}{2}d(u,Tu) \nleq d(u,v).$$

Also

$$\frac{1}{2}d(v,Tv) = \frac{1}{2}d((5,4),(0,4)) = \frac{5}{2}.$$

But $d(u,v) = 2$. So, we obtain that

$$\frac{1}{2}d(v,Tv) \nleq d(u,v).$$

It is easily seen that, for every two comparable elements $x, y \in X$

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \textit{ implies } \psi(d(Tx,Ty)) \leq \psi(m_T(x,y)) - \phi(m_T(x,y)).$$

For example, for $u = (4,5)$ and $z = (0,0)$, which are comparable, we have:

$$\psi(d(Tu,Tz)) = \psi(d((5,0),(0,0))) = 5.$$

On the other hand

$$\begin{aligned} m_T(u,z) &= \max\{d(u,z), d(u,Tu), d(z,Tz), \tfrac{d(u,Tz)+d(z,Tu)}{2}\} \\ &= \max\{d((4,5),(0,0)),\ d((4,5),(5,0)), d((0,0),(0,0)) \\ &\quad, \tfrac{d((4,5),(0,0))+d((0,0),(5,0))}{2}\} \\ &= \max\{9, 6, 0, \tfrac{9+5}{2}\} \\ &= 9. \end{aligned}$$

and we have:

$$\begin{aligned} \psi(m_T(u,z)) - \varphi(m_T(u,z)) &= \psi(9) - \varphi(9) \\ &= 9 - \tfrac{9}{10} \\ &\geq 5 \end{aligned}$$

Consequently we have:

$$\psi(d(Tu,Tz)) \leq \psi(m_T(u,z)) - \varphi(m_T(u,z))$$

Similarly, one can obtain the same inequalities for other comparable elements of X. Therefore all the hypotheses of Corollary 3.6 are satisfied.

Remark 3.12. In the Example 3.11, all the conditions of Theorem 3.11 and Theorem 3.10 are satisfied too, so by this theorems, the fixed point of T must be unique, and we see that $(0,0)$ is the unique fixed point of T.

The following example shows that the extra conditions in these theorems, are essential in order to guarantee the uniqueness of the fixed point. Furthermore, This example shows that, in Theorem 3.10 even if the inequality

$$\psi(d(Tx,Ty)) \leq \psi(m_T(x,y)) - \phi(m_T(x,y)),$$

is yield for all comparable elements $x, y \in X$, the condition (a) can not be replaced with the following:
(b) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, and for arbitrary non-comparable two points $x, y \in X$ there exists $z \in X$ which is comparable with x and y.

Example 3.13. Let $X = \{O(0,0), A(2,2), B(0,2), C(2,0)\} \subseteq \mathbb{R}^2$ be endowed with the metric d defined by

$$d((x_1,x_2),(y_1,y_2)) = \sqrt{|x_1-y_1|^2 + |x_2-y_2|^2}\ .$$

Suppose that relation $\preceq$ is defined on X as follows:

$$\preceq = \{(O,O),(A,A),(B,B),(C,C),(B,O),(B,A),(C,O),(C,A)\}.$$

Then $(X,\preceq,d)$ is a regular ordered complete metric space.
Suppose that $T : X \to X$ is defined as follow :

$$T(O) = O,\ T(A) = A,\ T(B) = C,\ T(C) = B.$$

Then, T is nondecreasing with respect to $\preceq$ and $O \preceq O = TO$.
Choosing $\psi(t) = t$ and $\varphi(t) = \frac{t}{a}$ for any $t \geq 0$, where $a \geq 2+\sqrt{2}$ is a real number, one can verify that, all conditions of Theorem 3.10 are satisfied, except condition (a) which is not established.
Indeed, for A and B, notice that $A \preceq\succeq B$ and

$$\psi(d(TA,TB)) = \psi(d(A,C)) = 2.$$

On the other hand

$$\begin{aligned} m_T(A,B) &= \max\{d(A,B), d(A,A), d(B,C), \tfrac{d(A,C)+d(B,A)}{2}\} \\ &= \max\{2,0,2\sqrt{2},2\} \\ &= 2\sqrt{2}. \end{aligned}$$

Thus

$$\begin{aligned} \psi(m_T(A,B)) - \varphi(m_T(A,B)) &= \psi(2\sqrt{2}) - \varphi(2\sqrt{2}) \\ &= 2\sqrt{2} - \tfrac{2\sqrt{2}}{a} \\ &\geq 2 \text{ (because } a \geq 2+\sqrt{2}) \end{aligned}$$

Consequently

$$\psi(d(TA,TB)) \leq \psi(m_T(A,B)) - \varphi(m_T(A,B)).$$

Similarly, the same inequalities for other comparable elements of X can be obtained .
Hence, all conditions of Theorem 3.10 are satisfied, except condition (a) which is not established. (because, there is no $Z \in X$
comparable with O and A and TZ).
However, it is clear that the condition (b) of Remark 3.12 is reliable, and O and A are distinct fixed points of T .
Also it is clear that the set of all fixed points of T is not well ordered, so conditions of Theorem 3.11 are not satisfied too.

Remark 3.14. In a similar way as in the proof of Theorem 3.4, one can prove that, Theorem 3.4 and its corollaries remain valid if $m(x,y)$ and $m_T(x,y)$ are replaced with $n(x,y)$ and $n_T(x,y)$, respectively.
Furthermore, it is interesting that, if $m_T(x,y)$ is replaced with $n_T(x,y)$ in the Theorem 3.10, then we can replace the condition (a) with the condition (b) of Remark 3.12. i.e. we have the following theorem:

Theorem 3.15. *Let $(X,\preceq,d)$ be an ordered complete metric space and $T : X \to X$ be a nondecreasing map such that $x_0 \preceq Tx_0$, for some $x_0 \in X$. And for any $x,y \in X$,* $\left(x \preceq\succeq y \text{ and } \frac{1}{2}d(x,Tx) \leq d(x,y)\right)$ *implies*

$$\psi(d(Tx,Ty)) \leq \psi(n_T(x,y)) - \phi(n_T(x,y)) \tag{3.15}$$

Where (ψ, φ) is a pair of control functions.

Furthermore, let the following condition hold:
(b) for arbitrary non-comparable $x, y \in X$ there exists $z \in X$ which is comparable with x and y.
If T is continuous or X is regular, then T has a unique fixed point.

Proof. Firstly, by Remark 3.14, T has a fixed point.
Now, let u and v be two fixed points of T. One can consider the following two cases:

(1) $u \preceq\succeq v$.
In this case, similarly as in the proof of Theorem 3.11, it can be shown that $u = v$.

(2) u and v are not comparable . In this case by the hypothesis, there exist $z \in X$ such that $z \preceq\succeq u$ and $z \preceq\succeq v$.
By using the notations which have been employed in the proof of Theorem 3.10, one can see that $u = u_n \preceq\succeq z_n$ and $v = v_n \preceq\succeq z_n$ for each $n \geq 0$, and $z_n \neq u, \ \forall n \geq 0$.
Now, for any $n \geq 1$, $u = u_n \preceq\succeq z_n$ and $\frac{1}{2}d(u, Tu) = 0 \leq d(u, z_{n-1})$, hence,

$$\begin{aligned}
\psi(d(u, z_n)) &= \psi(d(Tu_{n-1}, Tz_{n-1}) \\
&\leq \psi(n_T(u_{n-1}, z_{n-1})) - \phi(n_T(u_{n-1}, z_{n-1})) \\
&\leq \psi(n_T(u, z_{n-1})) - \phi(n_T(u, z_{n-1}))
\end{aligned}$$

Where

$$\begin{aligned}
n_T(u, z_{n-1}) &= \max\{d(u, z_{n-1}), \ \tfrac{d(u,Tu)+d(z_{n-1},Tz_{n-1})}{2}, \\
&\quad \tfrac{d(u,Tz_{n-1})+d(z_{n-1},Tu)}{2}\} \\
&= \max\{d(u, z_{n-1}), \ \tfrac{d(z_{n-1},z_n)}{2}, \ \tfrac{d(u,z_n)+d(z_{n-1},u)}{2}\} \\
&\leq \max\{d(u, z_{n-1}), \ \tfrac{d(z_{n-1},u)+d(u,z_n)}{2}, \ \tfrac{d(u,z_n)+d(z_{n-1},u)}{2}\} \\
&\leq \max\{d(u, z_{n-1}), \ \tfrac{d(z_{n-1},u)+d(u,z_n)}{2}\} \\
&\leq \max\{d(u, z_{n-1}), d(u, z_n)\}
\end{aligned}$$

And the proof is completed similar to the proof of Theorem 3.10. ∎

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