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Bounds on Multivalent Functions Associated with Quasi-Subordination

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Abstract In this paper, we introduce certain new subclasses of multivalent functions by using the concept of quasi subordination. We prove certain bounds and Fekete-Szego inequality for these classes of functions.

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. Clearly for $p = 1$, then $\mathcal{A}(p) = \mathcal{A}$, the class of normalized analytic functions. The subclass $\mathcal{R}_p$ of multivalent functions was defined and studied by Noor et-al. [1] as follows:

$$\mathcal{R}_p = \left\{ f \in A : \Re\left(\frac{f^{(p)}(z) + z f^{(p+1)}(z)}{p!} \right) > 0 \right\}, \tag{1.2}$$

where $f^{(p)}(z)$ denotes p^{th} derivative of $f(z)$. The class $\mathcal{R}_1 = \mathcal{R}$ was studied by Singh and Singh [2] in 1989.

Moreover, the subclass of $\mathcal{A}(p)$ consisting of all analytic functions and has positive real part in $\mathbb{E}$ is denoted by $\mathcal{P}$. An analytic description of $\mathcal{P}$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \quad z \in \mathbb{E}.$$

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If f and g are analytic functions in $\mathbb{E}$, we say that the function f is said to be *subordinate to the function* g and written as:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w in $\mathbb{E}$ with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{E}$, such that

$$f(z) = g(w(z)), \quad z \in \mathbb{E}.$$

Furthermore, if the function g is univalent in $\mathbb{E}$, then the subordination is equivalent to

$$f(z) \prec g(z) \Rightarrow f(0) = g(0) \text{ and } f(\mathbb{E}) \subset g(\mathbb{E}).$$

In 1970, Robertson [3] introduced the concept of quasi-subordination. For two analytic functions f and g, the function f is *quasi-subordinate to* g *in an open unit disc* $\mathbb{E}$, if there exist analytic functions $\varphi(z)$ with $|\varphi(z)| \leq 1$ such that $\frac{f(z)}{\varphi(z)}$ is analytic in $\mathbb{E}$ and

$$\frac{f(z)}{\varphi(z)} \prec g(z), \quad z \in \mathbb{E}. \tag{1.3}$$

The expression (1.3) can also be written as:

$$f(z) \prec_q g(z), \quad z \in \mathbb{E}. \tag{1.4}$$

Note that the quasi-subordination (1.4) is equivalent to $f(z) = \varphi(z)g(w(z))$. We also note that if $\varphi(z) = 1$, then the quasi-subordination $\prec_q$ becomes a usual subordination that is

$$f(z) \prec g(z), \quad \text{so that} \quad f(z) = g(w(z)).$$

Now we define the following definition:

Definition 1.1. A function $f(z)$ defined by (1.1) is said to be in the class $\mathcal{R}_p(q, \phi, \varphi)$, $p \in \mathbb{N} = \{1, 2, 3, ...\}$, if the following quasi subordination holds:

$$\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) - 1 \prec_q (\phi(z) - 1), \quad z \in \mathbb{E}. \tag{1.5}$$

where $\phi \in \mathcal{P}$ be univalent in E.
The above subordination condition (1.5) can also be written as:

$$\frac{\left(\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}\right) - 1}{\varphi(z)} \prec (\phi(z) - 1), \quad z \in \mathbb{E}. \tag{1.6}$$

In the subordination condition (1.6), if $\varphi(z) \equiv 1$, then the class $\mathcal{R}_p(q, \phi, \varphi)$ denoted by $\mathcal{R}_p(\phi)$ and satisfy the condition

$$\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) \prec \phi(z), \quad z \in \mathbb{E}. \tag{1.7}$$

Special Cases:
i) For $\varphi(z) \equiv 1$ and $\phi(z) = \frac{1+z}{1-z}$, the class $\mathcal{R}_p(q, \phi, \varphi) = \mathcal{R}_p$, defined and studied in [1].
ii) For $\varphi(z) \equiv 1$, $p = 1$ and $\phi(z) = \frac{1+z}{1-z}$, the class $\mathcal{R}_p(q, \phi, \varphi) = \mathcal{R}$, defined and studied in [2].
iii) For $\varphi(z) \equiv 1$, $p = 1$ and $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$, $(0 \leq \alpha < 1)$ the class $\mathcal{R}_p(q, \phi, \varphi) = \mathcal{N}_{\Sigma}^{\alpha,0}$, defined and studied in [4].

It is well known that the Fekete-Szegö functional is $\left|a_3 - a_2^2\right|$ was obtained by Fekete and Szegö [5]. Fekete and Szegö further generalized the estimate $\left|a_3 - \mu a_2^2\right|$ where μ is real and $f \in S$, the class of univalent functions. Since then, the problem of finding the sharp bounds for this functional of any compact family of function $f \in S$ with any complex μ is generally knows as the classical Fekete and Szegö problems inequality. Fekete and Szegö problem for several subclasses of $\mathcal{A}$ have been studied by many authors (see, e.g [6–11]) also recently by [12–14].

In this paper we mainly concentrate in determining the coefficient estimates including a Fekete and Szegö inequality of functions belonging to the classes $\mathcal{R}_p(q, \phi, \varphi)$, $\mathcal{R}_p(\phi)$ and $\mathcal{R}_1(q, \phi, \varphi)$. Some consequences of our main results are also given.
Throughout in this paper it is assumed that $\phi \in \mathcal{P}$ is analytic in E, and $\varphi(z)$ is also analytic in $\mathbb{E}$ and has the form given by:

$$\varphi(z) = d_0 + d_1 z + d_2 z^2 + ..., (|\varphi(z)| \leq 1; \quad z \in \mathbb{E}). \tag{1.8}$$

2. Preliminary Results

Lemma 2.1 ([15])**.** *Let the Schwarz function $w(z)$ be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + ... \qquad (z \in \mathbb{E}). \tag{2.1}$$

Then

$$|w_1| \leq 1, \quad \left|w_2 - t w_1^2\right| \leq 1 + (|t| - 1)\,|w_1|^2 \leq \max\{1, \ |t|\}.$$

Where $t \in C$, the result is sharp for the function $w(z) = z$ or $w(z) = z^2$.

Lemma 2.2 ([16])**.** *If w is analytic in $\mathbb{E}$, then*

$$\left|w_2 - \mu w_1^2\right| \leq \begin{cases} -\mu, & \text{if} \quad \mu \leq -1, \\ 1, & \text{if} \quad -1 \leq \mu \leq 1, \\ \mu, & \text{if} \quad \mu \geq 1. \end{cases} \tag{2.2}$$

When $\mu < -1$ or $\mu > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < \mu < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $\mu = -1$ if and only if

$$w(z) = \frac{z(t+z)}{1+tz} \quad (0 \leq t \leq 1),$$

or one of its rotations while for $\mu = 1$, equality holds if and only if

$$w(z) = \frac{-z(t+z)}{1+tz} \; (0 \leq t \leq 1),$$

or one of its rotations.

3. Main Results

Theorem 3.1. *Let the function $f(z) \in \mathcal{A}(p)$ defined by (1.1) be in the class $\mathcal{R}_p(q,\phi,\varphi)$, then*

$$|a_{p+1}| \leq \frac{p!B_1}{2(p+1)!}, \tag{3.1}$$

$$\left|a_{p+2} - ca_{p+1}^2\right| \leq \frac{2!p!B_1}{3(p+2)!}\begin{cases} \frac{B_2}{B_1} - c\frac{3p!(p+2)!B_1}{2!4((p+1)!)^2}, & \text{if} \quad c \leq \rho, \\ 1, & \text{if} \quad \rho \leq c \leq \sigma, \\ c\frac{3p!(p+2)!B_1}{2!4((p+1)!)^2} - \frac{B_2}{B_1}, & \text{if} \quad c \geq \sigma, \end{cases} \tag{3.2}$$

where

$$\rho = 2!\left(\frac{4(B_2 - B_1)\left((p+1)!\right)^2}{3p!(p+2)!B_1^2}\right) \text{ and } \sigma = 2!\left(\frac{4(B_2 + B_1)\left((p+1)!\right)^2}{3p!(p+2)!B_1^2}\right).$$

Proof. Let $f \in \mathcal{R}_p(q,\phi,\varphi)$, then for Schwarz function $w(z)$ given by (2.1) and for an analytic function $\varphi(z)$ given by (1.8), we have

$$\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) - 1 = \varphi(z)\left(\phi(w(z)) - 1\right), \tag{3.3}$$

where the series expansions of the right hand side and left hand side of (3.3) are given as:

$$\begin{aligned} \varphi(z)\left(\phi(w(z)) - 1\right) &= \left(d_0 + d_1 z + d_2 z^2 + ...\right)\left\{B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + ...\right\}, \\ &= d_0 B_1 w_1 z + \left\{d_0(B_1 w_2 + B_2 w_1^2) + d_1 B_1 w_1\right\} z^2 + ..., \end{aligned} \tag{3.4}$$

and

$$\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) - 1 = \frac{2(p+1)!}{p!1!}a_{p+1}z + \frac{3(p+2)!}{p!2!}a_{p+2}z^2 + ... \tag{3.5}$$

From the expansion (3.4) and (3.5), on equating the coefficients of z and z^2 in (3.3), we have

$$\frac{2(p+1)!}{p!1!}a_{p+1} = d_0 B_1 w_1, \tag{3.6}$$

$$\frac{3(p+2)!}{p!2!}a_{p+2} = \left\{d_0(B_1 w_2 + B_2 w_1^2) + d_1 B_1 w_1\right\}. \tag{3.7}$$

Now from (3.6) we have

$$a_{p+1} = \frac{p!d_0 B_1 w_1}{2(p+1)!}, \tag{3.8}$$

Taking modulus on (3.8), we have

$$|a_{p+1}| \leq \frac{p!B_1}{2(p+1)!}. \tag{3.9}$$

Which is required inequality (3.1).
Now in the view of (3.7),

$$a_{p+2} = \frac{2!p!B_1}{3(p+2)!}\left\{d_1 w_1 + d_0(w_2 + \frac{B_2}{B_1}w_1^2)\right\}. \tag{3.10}$$

For some $c \in \mathbb{C}$, we obtain from (3.8) and (3.10)

$$a_{p+2} - c a_{p+1}^2 = \frac{2!p!B_1}{3(p+2)!}\left[d_1 w_1 + d_0\left\{w_2 + \left(\frac{B_2}{B_1} - c\frac{3p!(p+2)!B_1}{2!4\left((p+1)!\right)^2}d_0\right)w_1^2\right\}\right]. \tag{3.11}$$

Since $\varphi(z)$ given by (1.8) is analytic and bounded in $\mathbb{E}$, therefore using the result given in [17, page 172], we have some y ($|y| \leq 1$).

$$|d_0| \leq 1 \text{ and } d_1 = (1 - d_0^2)y. \tag{3.12}$$

On putting the value of d_1 from (3.12) into (3.11), we have

$$a_{p+2} - c a_{p+1}^2 = \frac{2!p!B_1}{3(p+2)!}\left[(1-d_0^2)y w_1 + d_0\left\{w_2 + \left(\frac{B_2}{B_1} - c\frac{3p!(p+2)!B_1}{2!4\left((p+1)!\right)^2}d_0\right)w_1^2\right\}\right] \tag{3.13}$$

If $d_0 = 0$ in (3.13) and using the Lemma 2.1, we have

$$\left|a_{p+2} - c a_{p+1}^2\right| \leq \frac{2!p!B_1}{3(p+2)!}. \tag{3.14}$$

But if $d_0 \neq 0$, let us suppose that

$$F(d_0) = (1-d_0^2)y w_1 + d_0\left\{w_2 + \left(\frac{B_2}{B_1} - c\frac{3p!(p+2)!B_1}{2!4\left((p+1)!\right)^2}d_0\right)w_1^2\right\},$$

which is a polynomial in d_0 and hence analytic in $|d_0| \leq 1$ and maximum of $|F(d_0)|$ is attained at $d_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. We find that $\max_{(0\leq\theta<2\pi)}\left|F(e^{i\theta})\right| = |F(1)|$ and

$$\left|a_{p+2} - c a_{p+1}^2\right| \leq \frac{2!p!B_1}{3(p+2)!}\left|w_2 - \left(c\frac{3p!(p+2)!B_1}{2!4\left((p+1)!\right)^2} - \frac{B_2}{B_1}\right)w_1^2\right|. \tag{3.15}$$

By using the Lemma 2.2 on (3.15), we have the required inequality (3.2).
Sharpness of this result can be verified for the functions $f(z)$ given by

$$\frac{f^{(p)}(z) + z f^{(p+1)}(z)}{p!} = \phi(z^2), \tag{3.16}$$

or

$$\frac{f^{(p)}(z) + z f^{(p+1)}(z)}{p!} - 1 = z(\phi(z) - 1). \tag{3.17}$$

■

For $p = 1$, in Theorem 3.1, we have the following result.

Corollary 3.2. *Let the function* $f \in \mathcal{A}$ *be in the class* $\mathcal{R}_1(q, \phi, \varphi)$*, then*

$$|a_2| \leq \frac{B_1}{4},$$

$$\left|a_3 - ca_2^2\right| \leq \frac{B_1}{9}\begin{cases} \frac{B_2}{B_1} - c\frac{9B_1}{16}, & c \leq \rho_1, \\ 1, & \rho_1 \leq c \leq \sigma_1, \\ c\frac{9B_1}{16} - \frac{B_2}{B_1}, & c \geq \sigma_1, \end{cases}$$

where

$$\rho_1 = \frac{16(B_2 - B_1)}{9B_1^2} \text{ and } \sigma_1 = \frac{16(B_2 + B_1)}{9B_1^2}.$$

Theorem 3.3. *Let the function* $f \in \mathcal{A}(p)$ *be in the class* $\mathcal{R}_p(\phi)$ *for* $p \in \mathbb{N} = \{1, 2, 3, ...\}$*, then*

$$|a_{p+1}| \leq \frac{p!B_1}{2(p+1)!},$$

$$\left|a_{p+2} - ca_{p+1}^2\right| \leq \frac{2!p!B_1}{3(p+2)!}\begin{cases} \frac{B_2}{B_1} - c\frac{3p!(p+2)!B_1}{2!4((p+1)!)^2}, & \text{if} \quad c \leq \rho, \\ 1, & \text{if} \quad \rho \leq c \leq \sigma, \\ c\frac{3p!(p+2)!B_1}{2!4((p+1)!)^2} - \frac{B_2}{B_1}, & \text{if} \quad c \geq \sigma, \end{cases}$$

where

$$\rho = 2!\left(\frac{4(B_2 - B_1)\left((p+1)!\right)^2}{3p!(p+2)!B_1^2}\right) \text{ and } \sigma = 2!\left(\frac{4(B_2 + B_1)\left((p+1)!\right)^2}{3p!(p+2)!B_1^2}\right).$$

Proof. The proof of above theorem is similar to that of Theorem 3.1, when we take $\varphi(z) \equiv 1$, then (1.8) evidently implies that $d_0 = 1$ and $d_n = 0$, $n \in \mathbb{N}$, hence in view of (3.8), (3.11) and using Lemma 2.2, we obtain the required result. Sharpness can be verified for the function given by (3.16) or (3.17). ∎

For $p = 1$ and $\varphi(z) \equiv 1$ in Theorem 3.3, then we have the following result.

Corollary 3.4. *Let the function* $f \in \mathcal{A}$*, be in the class* $\mathcal{R}(\phi)$*, then*

$$|a_2| \leq \frac{B_1}{4},$$

$$\left|a_3 - ca_2^2\right| \leq \frac{B_1}{9}\begin{cases} \frac{B_2}{B_1} - c\frac{9B_1}{16}, & c \leq \rho_1, \\ 1, & \rho_1 \leq c \leq \sigma_1, \\ c\frac{9B_1}{16} - \frac{B_2}{B_1}, & c \geq \sigma_1, \end{cases}$$

where

$$\rho_1 = \frac{16(B_2 - B_1)}{9B_1^2} \text{ and } \sigma_1 = \frac{16(B_2 + B_1)}{9B_1^2}.$$

Theorem 3.5. *Let the function $f \in \mathcal{A}$, be in the class $\mathcal{R}_p(q,\phi,\varphi)$, then*

$$\left|a_{p+2} - ca_{p+1}^2\right| \leq \frac{2!p!B_1}{3(p+2)!} \max\left\{1, \left|c\frac{3p!(p+2)!B_1}{2!4\left((p+1)!\right)^2} - \frac{B_2}{B_1}\right|\right\}.$$

Proof. Let $f \in \mathcal{R}_p(q,\phi,\varphi)$, then we have from (3.15).

$$\left|a_{p+2} - ca_{p+1}^2\right| \leq \frac{2!p!B_1}{3(p+2)!} \left|w_2 - \left(c\frac{3p!(p+2)!B_1}{2!4\left((p+1)!\right)^2} - \frac{B_2}{B_1}\right) w_1^2\right|. \tag{3.18}$$

Applying the Lemma 2.1 on (3.18), we have

$$\left|a_{p+2} - ca_{p+1}^2\right| \leq \frac{2!p!B_1}{3(p+2)!} \max\left\{1, \left|c\frac{3p!(p+2)!B_1}{2!4\left((p+1)!\right)^2} - \frac{B_2}{B_1}\right|\right\}. \tag{3.19}$$

This complete the proof. Sharpness of this result can be verified in (3.16) and (3.17). ■

Theorem 3.6. *Let the function $f \in \mathcal{A}(p)$ be in the class $\mathcal{R}_p(\phi)$, then*

$$\left|a_{p+2} - ca_{p+1}^2\right| \leq \frac{2!p!B_1}{3(p+2)!} \max\left\{1, \left|c\frac{3p!(p+2)!B_1}{2!4\left((p+1)!\right)^2} - \frac{B_2}{B_1}\right|\right\}.$$

Proof. The proof of above theorem is similar to that of Theorem 3.5, when we take $\varphi(z) \equiv 1$, then (1.8) evidently implies that $d_0 = 1$ and $d_n = 0$, hence in view of (3.18) and using Lemma 2.1, we obtain the desired result. Sharpness can be verified for the function f given by (3.16). ■

For $p = 1$ in the Theorem 3.5, then we have the following result:

Corollary 3.7. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{R}_1(q,\phi,\varphi)$, then*

$$\left|a_{p+2} - ca_{p+1}^2\right| \leq \frac{B_1}{9} \max\left\{1, \left|\frac{9cB_1}{16} - \frac{B_2}{B_1}\right|\right\}.$$

For $p = 1$ and $\varphi(z) \equiv 1$ in the Theorem 3.6, then we have the following result:

Corollary 3.8. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{R}(\phi)$, then*

$$\left|a_{p+2} - ca_{p+1}^2\right| \leq \frac{B_1}{9} \max\left\{1, \left|\frac{9cB_1}{16} - \frac{B_2}{B_1}\right|\right\}.$$

Theorem 3.9. *Let the function $f \in \mathcal{A}(p)$ be in the class $\mathcal{R}_p(q,\phi,\varphi)$, then*

$$\left|a_{p+2} - ca_{p+1}^2\right| + (c-\rho)\left|a_{p+1}\right|^2 \leq \frac{2!p!B_1}{3(p+2)!}, \qquad \rho < c \leq \frac{\sigma}{2}, \tag{3.20}$$

and

$$\left|a_{p+2} - ca_{p+1}^2\right| + (\sigma-c)\left|a_{p+1}\right|^2 \leq \frac{2!p!B_1}{3(p+2)!}, \qquad \frac{\sigma}{2} < c \leq \sigma. \tag{3.21}$$

Proof. Let $f \in \mathcal{R}_p(q, \phi, \varphi)$, then from (3.8) and (3.15) (when $\rho < c < \sigma$), we get if $\rho < c \leq \frac{\sigma}{2}$:

$$\left|a_{p+2} - ca_{p+1}^2\right| + (c - \rho)\left|a_{p+1}\right|^2 \leq \frac{2!p!B_1}{3(p+2)!}\left\{|w_2| - (-1)|w_1|^2\right\}. \tag{3.22}$$

Hence by applying the Lemma 2.1 on (3.22), we obtain the desired estimates (3.20). If $\frac{\sigma}{2} < c \leq \sigma$, then again from (3.8) and (3.15), we have

$$\left|a_{p+2} - ca_{p+1}^2\right| + (\sigma - c)\left|a_{p+1}\right|^2 \leq \frac{2!p!B_1}{3(p+2)!}\left\{|w_2| - (-1)|w_1|^2\right\}. \tag{3.23}$$

Hence by using Lemma 2.1 on (3.23), we obtain the desired estimates (3.21). ∎

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