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Coupled Fixed Point Theorems under Nonlinear Contractive Conditions in S-Metric Spaces

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Abstract The aim of this paper is to prove a number of coupled fixed point theorems under φ-contractions for a mapping $F : X \times X \to X$ in S-metric spaces.

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1. Introduction

In 1922. Banach [1] proposed a theorem, which is well-known as Banach's Fixed Point Theorem (or Banach's Contraction Principle, BCP for short) to establish the existence of solutions for nonlinear operator equations and integral equations. Since then, because of simplicity and usefulness, it has become a very popular tool in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis. Later, a huge amount of literature is witnessed on applications, generalizations and extensions of this theorem. They are carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups. Considering different mappings etc.

Many mathematics problems require one to find a distance between two or more objects which is not easy to measure precisely in general. There exist different approaches to obtaining the appropriate concept of a metric stucture. Due to the need to construct a suitable framework to model several distinguished problems of practical nature, the study of metric spaces has attracted and continues to attract the interest of many authors. Over last few decades, a numbers of generalizations of metric spaces have thus appeared in several papers, such as 2-metric spaces, G-metric spaces, D^*-metric spaces, partial metric spaces and cone metric spaces. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [2–8]. Sedghi et al [9] have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space and a D^*-metric space. Also, they have

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proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

The Banach contraction principle is the most powerful tool in the history of fixed point theory. Boyd and Wong [10] extended the Banach contraction principle to the nonlinear contraction mappings. The notion of coupled fixed point was initiated by Gnana Bhaskar and Lakshmikantham [11] in 2006. After this many authors worked on coupled fixed point theorems.

2. Preliminaries

2.1. Instructions to Authors

We begin by briefly recalling some basic definitions and results for S-metric spaces that will be needed in the sequel.

Definition 2.1 ([9])**.** Let X be a nonempty set. An S-metric on X is a function $S : X \times X \times X \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an *S-metric space.*

Immediate examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an S-metric on X.
(2) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, then $S(x, y, z) = ||x - z|| + ||y - z||$ is an S-metric on X.
(3) Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, y) + d(y, z)$ is an S-metric on X.

Lemma 2.2 ([9])**.** *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Definition 2.3 ([9])**.** Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, we define the *open ball* $B_S(x, r)$ and *closed ball* $B_S\ [x, r]$ with center x and radius r as follows respectively:

$$\begin{aligned} B_s(x, r) &= \{y \in X : S(y, y, x) < r\}, \\ B_s[x, r] &= \{y \in X : S(y, y, x) \leq r\}. \end{aligned}$$

Example 2.4 ([9])**.** Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y+z-2x|+|y-z|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_s(1, 2) &= \{y \in \mathbb{R} : S(y, y, 1) < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} \\ &= (0, 2). \end{aligned}$$

Definition 2.5 ([9])**.** Let (X, S) be an S-metric space and $A \subset X$.

(1) If for every $x \in A$ there exists $r > 0$ such that $B_S(x, r) \subset A$, then the subset A is called *open subset of X*.

(2) Subset A of X is said to be *S-bounded* if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X *converges to* x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \Longrightarrow S(x_n, x_n, x) < \varepsilon,$$

and we denote by $\lim_{n \longrightarrow \infty} x_n = x$.

(4) Sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.

(5) The S-metric spaces (X, S) is said to be *complete* if every Cauchy sequence is convergent.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

Definition 2.6 ([12]). Let (X, S) and (X', S') be two S-metric spaces, and let $f : (X, S) \to (X', S')$ be a function. Then f is said to be *continuous at a point* $a \in X$ if and only if for every sequence x_n in X, $S(x_n, x_n, a) \to 0$ implies $S'(f(x_n), f(x_n), f(a)) \to 0$. A function f is *continuous* at X if and only if it is continuous at all $a \in X$.

Definition 2.7 ([11]). An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $F : X \times X \longrightarrow X$ if

$$F(x, y) = x \quad \textit{and} \quad F(y, x) = y.$$

Definition 2.8 ([11]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ be a mapping. F is said to *have the mixed monotone property* if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1 \leq x_2 \Longrightarrow F(x_1, y) \leq F(x_2, y), \text{ for } x_1, x_2 \in X$$

and

$$y_1 \leq y_2 \Longrightarrow F(x, y_2) \leq F(x, y_1), \text{ for } y_1, y_2 \in X.$$

By following Matkowski [13], we let Φ be the set of all nondecreasing functions $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $\lim_{n \to +\infty} \varphi^n(t) = 0$ for all $t > 0$. Then, it is an easy matter to show that

(1) $\phi(t) < t$ for all $t > 0$,

(2) $\phi(0) = 0$.

In this paper, some coupled fixed point theorems are proved for a mapping $F : X \times X \longrightarrow X$ satisfying a condition based on some $\varphi \in \Phi$.

3. Main Results

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and (X, S) a compact S-metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping such that F has the mixed monotone property. Assume that there exists $\varphi \in \Phi$ such that*

$$S(F(x, y), F(x, y), F(u, v)) \leq \varphi[\max(S(x, x, u), S(y, y, v))], \tag{1}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

Proof. Suppose $x_0, y_0 \in X$ are such that $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(y_0, x_0)$. Define

$$x_1 = F(x_0, y_0) \quad and \quad y_1 = F(y_0, x_0). \tag{2}$$

Then, $x_0 \le x_1$, $y_0 \ge y_1$. Again, define $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Since F has the mixed monotone property, we have $x_0 \le x_1 \le x_2$ and $y_2 \le y_1 \le y_0$.

Continuing like this, we can construct two sequences x_n and y_n in X such that

$$\begin{aligned} x_n = F(x_{n-1}, y_{n-1}) \le x_{n+1} = F(x_n, y_n) \quad and \\ y_{n+1} = F(y_n, x_n) \le y_n = F(y_{n-1}, x_{n-1}). \end{aligned} \tag{3}$$

If, for some integer n, we have

$$(x_{n+1}, y_{n+1}) = (x_n, y_n),$$

then

$$x_n = F(x_n, y_n) \quad and \quad y_n = F(y_n, x_n)$$

that is, (x_n, y_n) is a coupled fixed point of F. Thus, we suppose that $(x_{n+1}, y_{n+1}) \ne (x_n, y_n)$ for all $n \in N$; that is, we assume that either $x_{n+1} \ne x_n$ or $y_{n+1} \ne y_n$. For any $n \in N$, we have

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_n) &= S(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\le \varphi[\max(S(x_n, x_n, x_{n-1}), S(y_n, y_n, y_{n-1}))]. \end{aligned} \tag{4}$$

$$\begin{aligned} S(y_n, y_n, y_{n+1}) &= S(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\le \varphi[\max(S(y_{n-1}, y_{n-1}, y_n), S(x_{n-1}, x_{n-1}, x_n))]. \end{aligned} \tag{5}$$

From eq. (4), (5) we get that

$$\begin{aligned} &\max(S(x_{n+1}, x_{n+1}, x_n), S(y_n, y_n, y_{n+1})) \\ &\qquad \le \varphi[\max(S(x_n, x_n, x_{n-1}), S(y_{n-1}, y_{n-1}, y_n))]. \end{aligned} \tag{6}$$

By continuing the process of eq. (6) we get

$$\begin{aligned} \max(S(x_{n+1}, x_{n+1}, x_n), S(y_n, y_n, y_{n+1})) \\ &\le \varphi[\max(S(x_n, x_n, x_{n-1}), S(y_{n-1}, y_{n-1}, y_n))] \\ &\le \varphi^2[\max(S(x_{n-1}, x_{n-1}, x_{n-2}), S(y_{n-2}, y_{n-2}, y_{n-1}))] \\ &\vdots \\ &\le \varphi^n[\max(S(x_1, x_1, x_0), S(y_0, y_0, y_1))]. \end{aligned} \tag{7}$$

Now, we will show that x_n and y_n are Cauchy sequences in X. Let $\varepsilon > 0$. Since

$$\lim_{n\to\infty} \varphi^n[\max(S(x_1, x_1, x_0), S(y_0, y_0, y_1))] = 0, \tag{8}$$

and, there exist $n_0 \in N$ such that

$$\varphi^n[\max(S(x_1, x_1, x_0), S(y_0, y_0, y_1))] < \varepsilon \qquad for\ all \quad n = n_0. \tag{9}$$

This implies that

$$\max(S(x_{n+1}, x_{n+1}, x_n), S(y_n, y_n, y_{n+1})) < \varepsilon, \qquad for\ all \quad n = n_0. \tag{10}$$

For $m, n \in N$, we will prove that

$$\max(S(x_n, x_n, x_m), S(y_n, y_n, y_m)) < \varepsilon, \qquad for\ all \quad m \ge n \ge n_0. \tag{11}$$

By definition (2.1)(3), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\leq \cdots \leq 2\sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m) \\ &\leq 2\sum_{i=n}^{m-1} S(x_i, x_i, x_{i+1}) \\ &= 2\sum_{i=n}^{m-1} S(x_{i+1}, x_{i+1}, x_i) \\ &\leq 2\sum_{i=n}^{m-1} S((F(x_i, y_i), F(x_i, y_i), F(x_{i-1}, y_{i-1})) \\ &\leq 2\sum_{i=n}^{m-1} \varphi[\max(S(x_i, x_i, x_{i-1}), S(y_i, y_i, y_{i-1}))] \\ &\leq \cdots \leq 2\sum_{i=n}^{m-1} \varphi^n[\max(S(x_1, x_1, x_0), S(y_1, y_1, y_0))] \longrightarrow 0. \end{aligned} \tag{12}$$

Similarly, we show that

$$S(y_n, y_n, y_m) < \varepsilon. \tag{13}$$

Hence, we have

$$\max(S(x_n, x_n, x_m), S(y_n, y_n, y_m)) < \varepsilon. \tag{14}$$

Thus,(11) holds for all $m \geq n \geq n_0$. Hence, x_n and y_n are Cauchy sequences in X. Since X is a complete S-metric space, there exists x and $y \in X$ such that x_n and y_n converge to x and y respectively. Finally, we show that (x, y) is a coupled fixed point of F. Since F is continuous and $(x_n, y_n) \longrightarrow (x, y)$. We have

$$x_{n+1} = F(x_{n+1}, y_{n+1}) \longrightarrow F(x, y).$$

By the uniqueness of limit, we get that $x = F(x, y)$. Similarly, we show that $y = F(y, x)$. So, (x, y) is a coupled fixed point of F. ∎

By taking $\varphi(t) = kt$, where $k \in (0, 1]$, in Theorem 3.1, we have the following.

Corollary 3.2. *Let $(X, \leq)$ be a partially ordered set and (X, S) a compact S-metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping such that F has the mixed monotone property. Assume that there exists $k \in [0, 1)$ such that*

$$(F(x, y), F(x, y), F(u, v)) \leq k[\max(S(x, x, u), S(y, y, v))], \tag{15}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

As a consequence of Corollary 3.2, we have the following.

Corollary 3.3. *Let $(X, \leq)$ be a partially ordered set and (X, S) a compact S-metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping such that F has the mixed monotone property. Assume that there exists $a_1, a_2 \in [0, 1)$ such that*

$$(F(x, y), F(x, y), F(u, v)) \leq a_1(S(x, x, u) + a_2 S(y, y, v)), \tag{16}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

Theorem 3.4. *Let $(X, \leq)$ be a partially ordered set and (X, S) a compact S-metric space. Let $F : X \times X \longrightarrow X$ be a mapping having mixed monotone property. Assume that there exists $\varphi \in \Phi$ such that*

$$S(F(x, y), F(x, y), F(u, v) \leq \varphi[\max(S(x, x, u), S(y, y, v))], \tag{17}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Assume also that X has the following properties:

(i) If a nondecreasing sequence $x_n \longrightarrow x$, then $x_n \leq x$ for all $n \in N$,

(ii) If a nondecreasing sequence $y_n \longrightarrow y$, then $y_n \geq y$ for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

Proof. By following the same process in Theorem 3.1, we construct two Cauchy sequences x_n and y_n in X with

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \quad \text{and} \quad y_1 \geq y_2 \geq \cdots \geq y_n \geq \cdots, \tag{18}$$

such that $x_n \longrightarrow x \in X$ and $y_n \longrightarrow y \in X$. By the hypotheses on X, we have $x_n \leq x$ and $y_n \geq y$ for all $n \in N$. From (17), we have

$$\begin{aligned} S(F(x, y), F(x, y), x_{n+1}) &= S(F(x, y), F(x, y), F(x_n, y_n)) \\ &\leq \varphi[\max(S(x, x, x_n), S(y, y, y_n))]. \end{aligned} \tag{19}$$

$$\begin{aligned} S(y_{n+1}, y_{n+1}, F(y, x)) &= S(F(y_n, x_n), F(y_n, x_n), F(y, x)) \\ &\leq \varphi[\max(S(y_n, y_n, y), S(x_n, x_n, x))]. \end{aligned} \tag{20}$$

From (19), (20) we get,

$$\begin{aligned} &\max[S(F(x, y), F(x, y), x_{n+1}), S(y_{n+1}, y_{n+1}, F(y, x)) \\ \leq\ &\varphi[\max(S(x, x, x_n), S(y, y, y_n), S(y_n, y_n, y), S(x_n, x_n, x))]. \end{aligned} \tag{21}$$

Letting $n \longrightarrow \infty$ in (21), it follows that $x = F(x, y)$ and $y = F(y, x)$. Hence (x, y) is a coupled fixed point of F. ■

By taking $\varphi(t) = kt$, where $k \in (0, 1)$, in Theorem 3.4, we have the following result.

Corollary 3.5. *Let $(X, \leq)$ be a partially ordered set and (X, S) a compact S-metric space. Let $F : X \times X \longrightarrow X$ be a mapping having mixed monotone property. Assume that there exists $k \in [0, 1)$ such that*

$$S(F(x, y), F(x, y), F(u, v)) \leq k[\max(S(x, x, u), S(y, y, v))], \tag{22}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Assume also that X has the following properties:

(i) If a nondecreasing sequence $x_n \longrightarrow x$, then $x_n \leq x$ for all $n \in N$,

(ii) If a nondecreasing sequence $y_n \longrightarrow y$, then $y_n \geq y$ for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

As a consequence of Corollary 3.5, we have the following.

Corollary 3.6. *Let $(X, \leq)$ be a partially ordered set and (X, S) a compact S-metric space. Let $F : X \times X \longrightarrow X$ be a mapping having mixed monotone property. Assume that there exists $a_1, a_2 \in [0, 1)$ such that*

$$S(F(x,y), F(x,y), F(u,v)) \leq a_1(S(x,x,u) + a_2 S(y,y,v)), \tag{23}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Assume also that X has the following properties:

(i) If a nondecreasing sequence $x_n \longrightarrow x$, then $x_n \leq x$ for all $n \in N$,

(ii) If a nondecreasing sequence $y_n \longrightarrow y$, then $y_n \geq y$ for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

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