Thai Journal of Mathematics
Volume 6 (2008) Number 2 : 337–342

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

On Weak McCoy rings

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Abstract : In this paper we prove that all semicommutative rings are weak McCoy, and also we show that $T_n(R)$ is weak McCoy. Then we give an example of weak McCoy ring that is not semicommutative. We also classify how the weak McCoy property behaves under direct product and direct limit.

Keywords : Reduced ring; Semi-commutative ring; McCoy condition.
2000 Mathematics Subject Classification : 16U99, 16S15

1 Introduction

Throughout in this paper, all rings are associative with identity. Recently Nielsen [4] called a ring R is right McCoy if the equation $f(x)g(x) = 0$ where $f(x), g(x) \in R[x]\backslash\{0\}$, implies that there exists $s \in R\backslash\{0\}$ such that $f(x)s = 0$. Left McCoy rings are defined similarly. McCoy rings was chosen because McCoy [3] had noted that every commutative rings satisfies this condition.
We say that a ring R is right weak McCoy whenever, $f(x) = a_0 + a_1x + ... + a_mx^m, g(x) = b_0 + b_1x + ... + b_nx^n \in R[x]\backslash\{0\}$ satisfies $f(x)g(x) = 0$ then $a_is \in Nil(R)$ for some $s \in R\backslash\{0\}$. Left weak McCoy rings are defined similarly. If a ring is both left and right weak McCoy we say that the ring is weak McCoy ring.
We investigate this generalization of McCoy rings and their properties. Clearly, McCoy rings are weak McCoy. Examples are given to show that the converse is not true.
For notation $Nil(R), M_n(R), T_n(R)$, $R[x]$ and $R[x, x^{-1}]$ denote nillradical, the $n \times n$ matrix ring over R, upper triangular matrix ring over R, polynomial ring over R and laurent polynomial ring over R respectively.

2 WEAK McCoy RINGS

Definition 2.1. *Let R be an associative ring with identity. We say that a ring R is right weak McCoy whenever* $f(x) = a_0 + a_1x + ... + a_mx^m, g(x) = b_0 + b_1x + ... + b_nx^n \in R[x]\backslash\{0\}$ *satisfies* $f(x)g(x) = 0$ *then* $a_is \in Nil(R)$ *for some* $s \in R\backslash\{0\}$. *We defined left weak McCoy rings similarly. If a ring is both left and right weak McCoy we say that the ring is a weak McCoy ring.*

Clearly, any McCoy ring is weak McCoy. In the following we will show that the converse is not true.

Proposition 2.2. *Let R be a ring then for all* $n \geq 2$, $T_n(R)$ *is weak McCoy ring.*

Proof. It is clear that for any $A \in T_n(R)$, there exist $B \neq 0$ in $T_n(R)$ such that AB and BA be elements of $Nil(T_n(R))$. □

In Theorem 2.2 [4], it is shown that $T_n(R)$ is not McCoy ring. Thus we have examples of weak McCoy rings which are not McCoy ring.
A ring R is called weak Armendariz ring [6] if $f(x) = a_0 + a_1x + ... + a_nx^n, g(x) = b_0 + b_1x + ... + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in Nil(R)$ for each i,j. It is clear that every weak Armendariz ring is weak McCoy, but the following example shows that weak McCoy rings need not be weak Armendariz.

Example 2.3. *Let R be a ring and let* $S = M_2(R)$, *then S is not weak Armendariz by Example 2.5 [6]. Therefore the ring*

$$R_n := \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in S \right\}$$

is not weak Armendariz, but it is clear that R_n *is weak McCoy ring.*

From Proposition 2.2 we may suspect that if R is weak McCoy then every n-by-n full matrix ring $M_n(R)$ is weak McCoy, when $n \geq 2$. But the following example erase the possibility.
Recall that a ring R is reduced if R has not nilpotent element.

Example 2.4. *Let R be a reduced ring, then R is weak McCoy ring. Let* $S = M_2(R)$,

$$f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x$$

and

$$g(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} x$$

be polynomials in $S[x]$. *Then* $f(x)g(x) = 0$, *but if* $P\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in Nil(R)$ *and* $P\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \in Nil(R)$ *for some* $P \in S\backslash\{0\}$ *then it is obvious that P=0, Thus S is not weak McCoy.*

Given a ring R and a bimodule $_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This ring is isomorphism to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and usual matrix operation are used.

Corollary 2.5. *Let R be a weak McCoy ring, then the trivial extension* $T(R, R)$ *is weak McCoy.*

Proof. It is obvious. □

Proposition 2.6. *Let R be a ring and I an ideal of R such that* R/I *is weak McCoy ring. If* $I \subseteq Nil(R)$, *then R is weak McCoy.*

Proof. Let $f(x) = a_0 + a_1x + ... + a_mx^m, g(x) = b_0 + b_1x + ... + b_nx^n \in R[x]\backslash\{0\}$ be such that $f(x)g(x) = 0$ then $(\sum_{i=0}^{m} \bar{a}_ix^i)(\sum_{j=0}^{n} \bar{b}_jx^j) = 0$ in R/I. Thus there exists n_i such that $(\bar{a}_i\bar{s})^{n_i} = 0$ for some $s \notin I$. Hence $a_is \in Nil(R)$. This means that R is weak McCoy. □

Recall that a ring R is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$[1].

Proposition 2.7. *Let R be a ring which* $Nil(R) \trianglelefteq R$. *Then R is weak McCoy.*

Proof. Let $Nil(R) \trianglelefteq R$, then R is weak Armendariz and so R will be weak McCoy. □

In Lemma 3.1[6] Liu and Zhao proved that if R be a semicommutative then Nil(R) is an ideal of R . Let R be a ring such that $Nil(R) \trianglelefteq R$. By the following example we show that R need not be semicommutative.

Example 2.8. *Let R be a semicommutative ring then* R_n *[Example 2.3] is not semicommutative by Example 1.3 [2]. But* R_n *is a ring which* $Nil(R_n) \trianglelefteq R_n$.

The recent example give an example of weak McCoy ring which is not semicommutative.

3 Basic structure and extension of weak McCoy rings

In this section we study the properties of weak McCoy rings. The notation $\prod$ denotes the direct product.

Proposition 3.1. *finite direct product of weak McCoy rings is weak McCoy.*

Proof. Let $R_1, ..., R_n$ be weak McCoy rings and let $R = \prod_{k=1}^{n} R_k$. Consider $f(x) = a_0 + a_1x + ... + a_mx^m, g(x) = b_0 + b_1x + ... + b_nx^n \in R[x]\backslash\{0\}$ such that $f(x)g(x) = 0$ where $a_i = (a_{i1}, ..., a_{in})$, $b_j = (b_{j1}, ..., b_{jn})$ in R. For each $k = 1, ..., n$ we put $f_k(x) = \Sigma_{i=0}^{m} a_{ik}x^i$, $g_k(x) = \Sigma_{j=0}^{n} b_{jk}x^j$ in $R_k[x]$. Then $f_k(x)g_k(x) = 0$. So by weak McCoy property of R_k there exist $0 \neq s_k \in R_k$ such that $a_iks_k \in Nil(R_k)$ for all i. Take $s = (s_1, ..., s_n)$, thus $a_is \in Nil(R)$ for each i. □

In following proposition we consider the case of direct limit of direct system of weak McCoy rings.

Proposition 3.2. *The direct limit of a direct system of weak McCoy rings is also weak McCoy.*

Proof. Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of weak McCoy rings R_i for $i \in I$ and ring homomorphism $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is directed partially ordered set. Set $R = \underrightarrow{lim} R_i$ be direct limit of D with $l_i : R_i \to R$ and $l_j\alpha_{ij} = l_i$. We will prove that R is weak McCoy ring. Take $x, y \in R$, then $x = l_i(x_i), y = l_j(y_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define $x + y = l_k(\alpha_{ik}(x_i) + \alpha_{jk}(y_j))$ and $xy = l_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j))$, where $\alpha_{ik}(x_i)$ and $\alpha_{jk}(y_j)$ are in R_k. Then R forms a ring with $l_i(0) = 0$ and $l_i(1) = 1$.
Now suppose $f(x)g(x) = 0$ for $f(x) = \Sigma_{s=0}^{m} a_sx^s$, $g(x) = \Sigma_{t=0}^{n} b_tx^t$ in $R[x]\backslash\{0\}$. There are $i_s, j_t, k \in I$ such that $a_s = l_{i_s}(a_{i_s})$, $b_t = l_{j_t}(b_{j_t})$, $i_s \leq k, j_t \leq k$. So $a_sb_t = l_k(\alpha_{i_sk}(a_{i_s})\alpha_{j_tk}(b_{j_t}))$. Thus $f(x)g(x) = (\Sigma_{s=0}^{m} l_k(\alpha_{i_sk}(a_{i_s}))(\Sigma_{t=0}^{n} l_{j_t}(b_{j_t})) = 0$. But R_k is weak McCoy ring and so there exist $0 \neq d \in R_k$ such that $l_k(a_{i_sk}d) \in Nil(R_k)$. Thus $a_sl_k(d) \in Nil(R)$ and R is weak McCoy ring. □

Proposition 3.3. *Let R be a ring and Δ be a multiplicative closed subset of R consisting of central regular element and R be a weak McCoy ring. Then $\Delta^{-1}R$ is weak McCoy.*

Proof. Let R be a weak McCoy ring and $S = \Delta^{-1}R$. Suppose $f(x) = \Sigma_{i=0}^{m} \alpha_ix^i, g(x) = \Sigma_{j=0}^{n} \beta_jx^j$ are in $S[x]\backslash\{0\}$, $(\alpha_i, \beta_j \in S)$, Then we can assume that $\alpha_i = a_iu^{-1}$ and $\beta_j = b_jv^{-1}$ for some $a_i, b_j \in R$, $u, v \in \Delta$ for all i, j. Now suppose that $f(x)g(x) = 0$. Let $f_1(x) = \Sigma_{i=0}^{m} a_ix^i, g_1(x) = \Sigma_{j=0}^{n} b_jx^j$. Thus $f_1(x)g_1(x) = 0$ in R[x]. Thus $a_is \in Nil(R)$ for some $0 \neq s \in R$ $(0 \leq i \leq m)$. So $\alpha_is \in Nil(S)$ for any $0 \leq i \leq m$. Thus $\Delta^{-1}R$ is weak McCoy ring. □

The ring of Laurent polynomials in x, coefficients in a ring R, consists of all formal sums $\Sigma_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers; denotes it by $R[x; x^{-1}]$.

Corollary 3.4. *For a ring R, Let R[x] is weak McCoy ring. then* $R[x, x^{-1}]$ *is weak McCoy.*

Proof. Let $\Delta = \{1, x, x^2, ...\}$. Then clearly Δ is multiplicatively closed subset of $R[x]$. Since $R[x, x^{-1}] = \Delta^{-1}R$, it follows that $R[x, x^{-1}]$ is weak McCoy ring. □

Also we know that if R be a semicommutative ring, then $R[x]$ is weak Armendariz by Theorem 3.8[6]. Thus $R[x]$ is weak McCoy rings.

Acknowledgement(s) : I would like to thank the referee(s) for his comments and suggestions on the manuscript.

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(Received 25 May 2008)

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