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# Hybrid Steepest Descent Method for Solving the Split Fixed Point Problem in Banach Spaces

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**Abstract** In this paper we introduce two algorithms based on the hybrid steepest descent method which converge to a solution of the split fixed point problem for $\lambda$-strictly pseudo-contractive mappings in uniformly convex and 2-uniformly smooth Banach spaces. Our results improve and extend the results of Q. H. Ansari et al. (2016), Y. Yao et al. (2013), and those of J. S. Jung (2016).

**MSC:** 47H09; 47H10; 58C30
**Keywords:** split fixed point problem; hybrid steepest descent method; $\lambda$-stricly pseudo-contractive mapping; $k$-Lipschitzian mapping; $\eta$-strongly monotone operator mappings

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## 1. Introduction

Let $H_1$ and $H_2$ be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. For nonlinear operators $T : H_1 \longrightarrow H_1$ and $U : H_2 \longrightarrow H_2$, the split fixed point problem (SFPP) is to find a point

$$x \in Fix(T) \quad \text{such that} \quad Ax \in Fix(U). \tag{1.1}$$

In particular, if $T = P_C$ and $U = P_Q$, then the SFPP reduces to the split feasibility problem (SFP); that is, to find $x \in C$ such that $Ax \in Q$, where $C$ and $Q$ are nonempty closed convex subsets in $H_1$ and $H_2$, respectively, and $P_C$, $P_Q$ are the respective metric projections.
The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise in phase retrievals and in medical image reconstruction [2]. Various iterative algorithms have been proposed to solve the SFP or related problems in Hilbert spaces, as well as in Banach spaces, see for instance [1, 3–9] and the references therein.

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In the Hilbert space setting, the SFPP has been studied by several authors; see, for instance, [10–12]. In [13], Censor and Segal introduced the iterative scheme:

$$x_{n+1} = U(I - \rho_n A^*(I - T)A)x_n \tag{1.2}$$

which solves the problem (1.1) for directed operators. This algorithm was then extended to the case of quasi-nonexpansive mappings [14], as well as to the case of demicontractive mappings [15].

On the other hand, the hybrid steepest descent method is an algorithmic solution to the variational inequality problem over the fixed point set of a nonlinear mapping. We know that the hybrid steepest descent method is applicable to a broad spectrum of convexly constrained nonlinear inverse problems in real Hilbert spaces.
In [16] Yamada introduced the following hybrid steepest descent method for solving the variational inequality for nonexpansive mappings:

$$x_{n+1} = (1 - \alpha_n \mu F)Sx_n,$$

where $F : H \to H$ is a $k$-Lipschitzian and $\eta$-strongly monotone operator with constants $k > 0$ and $\eta > 0$; and $\mu \in (0, \frac{2\eta}{k^2})$. He proved that if $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality related to $F$, of which the constraint set is the fixed point set $Fix(S)$ of $S$.
Recently, Jung [7] has presented some iterative algorithms based on Yamada's hybrid steepest descent method for solving the SFP. We should mention that some split type feasibility problems have been studied because of their applications in science, engineering, medical sciences, and so on. In [11], Ansari et al. introduced an implicit and an explicit algorithm for the SFPP for firmly nonexpansive mappings and for nonexpansive mappings in a Hilbert space.
Now, the following question arises:

**Question :** Does the hybrid steepest descent method work for $\lambda$-strictly pseudo-contractive mappings in spaces beyond Hilbert spaces?

Our aim in this paper is to answer the above question in the affirmative. Motivated by [11] and [7], we present an algorithm based on hybrid steepest descent method for solving the split fixed point problem for $\lambda$-strictly pseudo-contractive mappings in uniformly convex and 2-uniformly smooth Banach spaces. First, we present an implicit algorithm. Next, by discretizing the continuous implicit algorithm, we obtain an explicit algorithm. We show that both algorithms converge strongly to a solution of the variational inequality problem over the solution set of SFPP. Our results improve and extend some recent results of the literature including those of Ansari et al [11], Yao et al [8], as well as those of Jung [7].

## 2. Preliminaries

Let $E$ be a real Banach space. A proper function $f : E \to (-\infty, +\infty]$ is said to be convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),$$

for all $x, y \in E$ and $\alpha \in (0, 1)$. The function $f$ is said to be lower semicontinuous if the set $\{x \in E : f(x) \leq r\}$ is closed in $E$, for all $r \in \mathbb{R}$. For a proper lower semicontinuous convex function $f : E \to (-\infty, +\infty]$, the subdifferential $\partial f$ of $f$ is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y) \quad \forall y \in E\}.$$

On the other hand, the normalized duality map $J$ from $E$ into $2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

It is well-known that $J(x)$ is the subdifferential of the function $(\frac{1}{2})\|.\|^2$ at $x$.

**Lemma 2.1.** *Let $E_1$ and $E_2$ be two real Banach spaces, and $J_1$ and $J_2$ be the duality mappings on $E_1$ and $E_2$, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^*$ be the adjoint of $A$. Then, for all $x \in E_1$,*

$$A^* J_2(Ax) \subseteq \|A\|^2 J_1(x).$$

*Proof.* Let $x_1 \in E_1$ and $x^* \in A^* J_2(Ax_1) \subseteq E_1^*$. So, there exists $y^* \in J_2(Ax_1)$ such that $x^* = A^* y^*$. Since $y^* \in J_2(Ax_1)$, by definition of $J_2 = \frac{1}{2}\partial\|.\|^2$, we have

$$\langle z - Ax_1, y^* \rangle \leq \frac{1}{2}\|z\|^2 - \frac{1}{2}\|Ax_1\|^2, \quad \forall z \in E_2.$$

Also, for $x \in E_1$,

$$\langle Ax - Ax_1, y^* \rangle \leq \frac{1}{2}\|Ax\|^2 - \frac{1}{2}\|Ax_1\|^2.$$

Thus, for $x \in E_1$,

$$\langle x - x_1, \frac{x^*}{\|A\|^2} \rangle = \frac{1}{\|A\|^2}\langle x - x_1, x^* \rangle = \frac{1}{\|A\|^2}\langle x - x_1, A^* y^* \rangle \leq \frac{1}{2}\|x\|^2 - \frac{1}{2}\|x_1\|^2.$$

It now follows from the definition of $J_1$ that $\frac{x^*}{\|A\|^2} \in J_1(x_1)$. Hence $x^* \in \|A\|^2 J_1(x_1)$. ∎

A Banach space $E$ is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is well known that if $E_1^*$ and $E_2^*$ are strictly convex, then $J_1$ and $J_2$ are single valued. Therefore, for all $x \in E_1$, $A^* J_2(Ax) = \|A\|^2 J_1(x)$.
A Banach space $E$ is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists a $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ for for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. The Banach space $E$ is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in \{z \in E : \|z\| = 1\}$. The modulus of convexity of $E$ is defined by

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon\}$$

for all $\epsilon \in [0, 2]$. We call $E$ uniformly convex if $\delta_E(0) = 0$, $\delta_E(2) = 1$ and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of $E$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1, \|y\| \leq t\}.$$

A Banach space $E$ is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$ be a fixed real number. Then a Banach space $E$ is said to be $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$. It is well known that every $q$-uniformly smooth Banach space is uniformly smooth.

**Lemma 2.2** ([17])**.** *If $E$ is a 2-uniformly smooth Banach space with the best smoothness constant $m > 0$, then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + 2\|y\|^2, \quad \forall x, y \in E.$$

**Definition 2.3.** Let $E$ be a Banach space.

(1) A mapping $f : E \to E$ is called $k$-contractive if $\|fx - fy\| \le k\|x - y\|$ for some constant $k \in [0, 1)$ and for all $x, y \in E$;
(2) A mapping $V : E \to E$ is called $l$-Lipschitzian if $\|Vx - Vy\| \le l\|x - y\|$ for some constant $l \in [0, \infty)$ and all $x, y \in E$;
(3) A mapping $T : E \to E$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in E$;
(4) A mapping $T : E \to E$ is called averaged if $T = (1-\nu)I + \nu G$, where $\nu \in (0, 1)$, $I$ is the identity, and $G : E \to E$ is a nonexpansive mapping.
(5) A mapping $A : E \to E$ is called monotone if $\langle Ax - Ay, J(x - y)\rangle \ge 0, \quad \forall x, y \in E$;
(6) An operator $F : E \to E$ is called $\eta$-strongly monotone with constants $k > 0$ and $\eta > 0$ if

$$\langle Fx - Fy, J(x - y) \ge \eta\|x - y\|^2, \quad \forall x, y \in E.$$

Let $E$ be a real Banach space and $C$ be a nonempty closed convex subset of $E$. A mapping $T : C \to C$ is called $\alpha$-inverse strongly monotone (or briefly, $\alpha$-ism) with constant $\alpha > 0$ if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \ge \alpha\|Tx - Ty\|^2,$$

where $J$ is the normalized duality mapping from $E$ into the dual space $E^*$. If $\alpha = 1$, $T$ is said to be a firmly nonexpansive mapping. A mapping $T : C \to C$ is said to be $\lambda$-strictly pseudo-contractive ($\lambda < 1$) if, for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2. \tag{2.1}$$

Observe that (2.1) can be rewritten as (see [18])

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \ge \lambda\|(I - T)x - (I - T)y\|^2. \tag{2.2}$$

When $E$ is a 2-uniformly smooth Banach space having the best smoothness constant $m$, $T : C \to C$ is called $\lambda$-strictly pseudo-contractive if for each $x, y \in C$,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + (2m^2 - 2\lambda)\|(I - T)x - (I - T)y\|^2.$$

Browder and Petryshyn [19] introduced the concept of a strict pseudo-contractive mapping. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and $T : C \to C$ be a mapping. $T$ is said to be a $k$-strictly pseudo-contraction, if there exists a $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2, \tag{2.3}$$

for all $x, y \in C$. It is easy to see that (2.3) is equivalent to

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\|(x - Tx) - (y - Ty)\|^2,$$

for all $x, y \in C$.

The following proposition was proved in [20] in a Hilbert space setting. The statement is true in Banach spaces as well. To avoid repetition, we omit the details of the proof.

**Proposition 2.4** ([20]). *Let $T : H \to H$ be an operator. (i) if $T$ is $\nu$-ism, then for $\gamma > 0$, $\gamma T$ is $\frac{\nu}{\gamma}$-ism. (ii) $T$ is averaged if and only if the complement $I - T$ is $\nu$-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1)$, $T$ is $\alpha$-averaged if and only if $(I - T)$ is $\frac{1}{2\alpha}$-ism. (iii) The composition of finitely many averaged mappings is averaged. In particular, if $T_i$ is $\alpha_i$-averaged, where $\alpha_i \in (0, 1)$ for $i = 1, 2$, then the composition $T_1T_2$ is $\alpha$-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$. (iv) If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then $\cap_{i=1}^N F(T_i) = F(T_1 \cdots T_N)$. (iii) In case $E$ is a uniformly convex Banach space, every $\alpha$-averaged mapping is nonexpansive.*

**Lemma 2.5.** *Let $E_1$ and $E_2$ be two real uniformly convex and 2-uniformly smooth Banach spaces with the best smoothness constants n and m, and $J_1$ and $J_2$ be the duality mappings on $E_1$ and $E_2$, respectively. Let $A : E_1 \longrightarrow E_2$ be a bounded linear operator and $A^*$ be the adjoint of $A$. Let $S : E_2 \to E_2$ be a $\beta$-psuedo-contractive mapping. Then $U = I - \gamma J_1^{-1}A^*J_2(I - S)A$ is averaged.*

*Proof.* Since $S$ is $\beta$-strict pseudo-contractive, according to (2.2), $I-S$ is $\beta$-inverse strongly monotone. Therefore, for all $x, y \in E_1$,

$$\begin{aligned}
&\langle J_1(x-y), \gamma J_1^{-1}A^*J_2(I-S)Ax - \gamma J_1^{-1}A^*J_2(I-S)Ay\rangle \\
&= \frac{\gamma}{\|A\|^2}\langle \|A\|^2 J_1(x-y), J_1^{-1}A^*J_2(I-S)Ax - J_1^{-1}A^*J_2(I-S)Ay\rangle \\
&= \frac{\gamma}{\|A\|^2}\langle A^*J_2(A(x-y)), J_1^{-1}A^*J_2(I-S)Ax - J_1^{-1}A^*J_2(I-S)Ay\rangle \\
&= \frac{\gamma}{\|A\|^2}\langle J_2(A(x-y)), AJ_1^{-1}A^*J_2(I-S)Ax - AJ_1^{-1}A^*J_2(I-S)Ay\rangle \\
&= \frac{\gamma}{\|A\|^2}\langle A^*J_2(A(x-y)), \|A\|^2 J_2^{-1}J_2(I-S)Ax - \|A\|^2 J_2^{-1}J_2(I-S)Ay\rangle \\
&\geq \gamma\beta\|(I-S)Ax - (I-S)Ay\|^2 \\
&= \gamma\beta\|J_2^{-1}J_2(I-S)Ax - J_2^{-1}J_2(I-S)Ay\|^2 \\
&\geq \frac{\gamma\beta}{\|A^*\|^2}\|A^{-1}J_2^{-1}J_2(I-S)Ax - A^{-1}J_2^{-1}J_2(I-S)Ay\|^2 \\
&= \frac{\gamma\beta}{\|A^*\|^4}\|A^{-1}AJ_1^{-1}A^*J_2(I-S)Ax - A^{-1}AJ_1^{-1}A^*J_2(I-S)Ay\|^2 \\
&= \frac{\beta}{\gamma\|A^*\|^4}\|\gamma J_1^{-1}A^*J_2(I-S)Ax - \gamma J_1^{-1}A^*J_2(I-S)Ay\|^2.
\end{aligned}$$

Noticing that $\|A\| = \|A^*\|$, we have $\gamma J_1^{-1}A^*J_2(I-S)A$ is $\frac{\beta}{\gamma\|A\|^4}$-ism. Since $\gamma \in (0, \frac{\beta}{\|A\|^4})$, we have $\frac{\beta}{\gamma\|A\|^4} > \frac{1}{2}$. So from Proposition 2.4, $U = I - \gamma J_1^{-1}A^*J_2(I-S)A$ is averaged. ∎

**Lemma 2.6** ([21]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space $E$ and $\{\gamma_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1.$$

*Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n)z_n$, $n \geq 0$, and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\|z_n - x_n\| = 0$.*

Recall that a Banach space $E$ is said to satisfy Opial's condition [22] if whenever $\{x_n\}$ is a sequence in $E$ which converges weakly to $x$ as $n \to \infty$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \text{for all} \quad y \in E, y \neq x.$$

**Remark 2.7.** If $E$ is a real uniformly convex and uniformly smooth Banach space, then $E$ satisfies Opial's condition [22].

**Lemma 2.8.** *(Demiclosedness Principle). Let $C$ be a nonempty, closed and convex subset of a real uniformly convex and uniformly smooth Banach space $E$ and $T : C \to C$ be a nonexpansive operator with $Fix(T) \neq \emptyset$. If the sequence $\{x_n\} \subseteq C$ converges weakly to $p$ and the sequence $\{(I - T)x_n\}$ converges strongly to $y$, then $(I - T)p = y$. In particular, if $y = 0$, then $p \in Fix(T)$.*

*Proof.* Let the sequence $x_n \rightharpoonup p$ and $(I-T)x_n \to y$. We show that $(I-T)p = y$. Suppose $p - Tp \neq y$. From Remark 2.7 and the fact that $T$ is nonexpansive, we have

$$\begin{aligned}
\liminf_{n\to\infty} \|Tx_n - Tp\| &\le \liminf_{n\to\infty} \|x_n - p\| \\
&< \liminf_{n\to\infty} \|x_n - (y + Tp)\| \\
&= \liminf_{n\to\infty} \|x_n - Tx_n - y + Tx_n - Tp)\| \\
&= \liminf_{n\to\infty} \|Tx_n - Tp\|
\end{aligned}$$

which is a contradiction. Therefore, the result follows. ∎

**Lemma 2.9** ([23])**.** *Let $E$ be a real uniformly smooth Banach space with the dual space $E^*$ and $J$ be the duality mapping of $E$, and $C$ be a nonempty closed convex subset of $E$. Assume that the mapping $F : C \to E$ is monotone, single-valued, and hemicontinuous in the sense of Brower (i.e, the restriction of $F$ to any line-segment in $C$ is continuous). Then the variational inequality problem:*

$$\textit{find} \quad x^* \in C \quad \textit{such that} \quad \langle F(x^*), J(x - x^*)\rangle \ge 0, \quad \textit{for all} \quad x \in C,$$

*is equivalent to the dual variational inequality*

$$\textit{find} \quad x^* \in C \quad \textit{such that} \quad \langle F(x), J(x - x^*)\rangle \ge 0, \quad \textit{for all} \quad x \in C.$$

**Lemma 2.10** ([24])**.** *Let $\{\gamma_n\}$ be a sequence in $(0, 1)$ and $\{\delta_n\}$ be a sequence in $\mathbb{R}$ satisfying*

*(1)* $\sum_{n=1}^{\infty} \gamma_n = \infty$,
*(2)* $\limsup_{n\to\infty} \gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

*If $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n \delta_n,$$

*for each $n \ge 0$, then $\lim_{n\to\infty} a_n = 0$.*

The following lemma can be easily proved, and therefore, we omit the proof (see also [16]).

**Lemma 2.11.** *Let $E$ be a 2-uniformly smooth Banach space . Let $F : E \to E$ be a $k$-Lipschitzian and $\eta$-strongly monotone operator with constants $k > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{k^2}$ and $0 < t < \xi \leq 1$. Then $S := \xi I - t\mu F : E \to E$ is a contractive mapping with constant $\xi - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.*

In the following arguments we shall use the following notation: for a mapping $T$ and a number $\alpha \in [0, 1]$,

$$T_\alpha x = (1 - \alpha)x + \alpha T x.$$

## 3. The Main Result

We start this section by proving the main result of this paper.

**Theorem 3.1.** *Let $E_1$ and $E_2$ be two real uniformly convex and 2-uniformly smooth Banach spaces with the best smoothness constants $m$ and $n$, and $J_1$ and $J_2$ be the duality mappings on $E_1$ and $E_2$, respectively. Let $A : E_1 \longrightarrow E_2$ be a bounded linear operator and $A^*$ be the adjoint of $A$. Let $T : E_1 \to E_1$ be a $\xi$-strictly pseudo-contractive mapping with $Fix(T) \neq \emptyset$ and $S : E_2 \to E_2$ be a $\beta$-strictly pseudo- contractive mapping with $Fix(S) \neq \emptyset$. Let $V : E_1 \to E_1$ be $l$-Lipschitzian with constant $l \in [0, \infty)$ and let $F : E_1 \to E_1$ be $k$-Lipschitzian and $\eta$-strongly monotone operator with constants $k > 0$ and $\eta > 0$ such that the constants $\mu, \sigma, l$ and $\tau$ satisfy $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \sigma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Suppose $\Omega = \{x \in Fix(T) : Ax \in Fix(S)\} \neq \emptyset$. For any $t \in (0, \frac{1}{\tau - \sigma l})$, define a net $\{x_t\} \subset E_1$ by*

$$x_t = T_\alpha[I - \gamma J_1^{-1} A^* J_2(I - S)]T_\alpha[t\sigma V x_t + (1 - t\mu F)x_t] \tag{3.1}$$

*where $\gamma \in (0, \frac{2\beta}{\|A\|^4})$ and $\alpha \in (0, \frac{\xi}{2m^2})$. Then the net $\{x_t\}$ converges strongly to $x^* \in \Omega$ which is a solution of the following variational inequality*

$$x^* \in \Omega \quad \text{such that} \quad \langle \sigma V x^* - \mu F x^*, J_1(x - x^*)\rangle \geq 0 \quad \forall x \in \Omega. \tag{3.2}$$

*Proof.* First, we show that $T_\alpha$ is nonexpansive. Form Lemma 2.2, for each $x, y \in C$, we have

$$\begin{aligned}
\|T_\alpha x - T_\alpha y\|^2 &= \|(1 - \alpha)x + \alpha T x - [(1 - \alpha)y + \alpha T y]\|^2 \\
&= \|(x - y) + \alpha[(y - Ty) - (x - Tx)]\|^2 \\
&\leq \|x - y\|^2 + 2\alpha^2 m^2 \|(y - Ty) - (x - Tx)\|^2 \\
&+ 2\alpha\langle J_1(x - y), (y - Ty) - (x - Tx)\rangle \\
&\leq \|x - y\|^2 + 2\alpha^2 m^2 \|(y - Ty) - (x - Tx)\|^2 \\
&- 2\alpha\xi\|(y - Ty) - (x - Tx)\|^2 \\
&= \|x - y\|^2 - 2\alpha(\xi - \alpha m^2)\|(y - Ty) - (x - Tx)\|^2.
\end{aligned}$$

Since $\alpha \in (0, \frac{\xi}{2m^2})$, it follows that

$$\|T_\alpha x - T_\alpha y\| \leq \|x - y\|.$$

Set $U = I - \gamma J_1^{-1} A^* J_2(I - S)A$. From Lemma 2.5, $U$ is an averaged mapping. Since every averaged mapping is nonexpansive, $U$ is nonexpansive. Since composition of nonexpansive mappings is nonexpansive, we conclude that $T_\alpha U T_\alpha$ is nonexpansive. We consider the mapping $w_t = T_\alpha U T_\alpha(t\sigma V + (I - t\mu F))$ on $E_1$. Clearly, $w_t$ is a self-mapping on $E_1$, moreover, for $x, y \in E_1$, we have

$$\begin{aligned}
\|w_t x - w_t y\| &= \|T_\alpha U T_\alpha(t\sigma V + (I - t\mu F))x - T_\alpha U T_\alpha(t\sigma V + (I - t\mu F))y\| \\
&\leq \|t\sigma V x + (I - t\mu F))x - [t\sigma V y + (I - t\mu F)y]\| \\
&\leq t\sigma\|Vx - Vy\| + \|(I - t\mu F)x - (I - t\mu F)y\| \\
&\leq t\sigma l\|x - y\| + (1 - t\tau)\|x - y\| = [1 - t(\tau - \sigma l)]\|x - y\|.
\end{aligned}$$

Therefore, $w_t$ is a contractive mapping when $t \in (0, \frac{1}{\tau - \sigma l})$. By the Banach contraction principle, $w_t$ has a unique fixed point in $E_1$, say $x_t$, that is,

$$x_t = T_\alpha U T_\alpha(t\sigma V + (I - t\mu F))x_t.$$

It is now clear that the net $\{x_t\}$ defined by (3.1) is well-defined. Let $p \in \Omega$. Then, $p \in Fix(T)$ and $Ap \in Fix(S)$. From the definition of $U$ we have $p \in Fix(U)$. It now follows that

$$\begin{aligned}
\|x_t - p\| &= \|T_\alpha U T_\alpha(t\sigma V + (I - t\mu F))x_t - p\| \\
&\leq \|t\sigma(V x_t - Vp)\| + \|(I - t\mu F)x_t - (I - t\mu F)p\| + \|t\sigma Vp - t\mu Fp\| \\
&\leq t\sigma\|x_t - p\| + (1 - t\tau)\|x_t - p\| + t\|\sigma Vp - \mu Fp\| \\
&= [1 - (\tau - \sigma l)t]\|x_t - p\| + t\|Vp - \mu Fp\|.
\end{aligned} \tag{3.3}$$

Hence

$$\|x_t - p\| \leq \frac{1}{\tau - \sigma l}\|\sigma Vp - \mu Fp\|.$$

Therefore $\{x_t\}$ is bounded and so are $\{Vx_t\}$, $\{Ux_t\}$ and $\{Fx_t\}$. From (3.1), we have

$$\begin{aligned}
\|x_t - T_\alpha[I - \gamma J_1^{-1}A^* J_2(I - S)A]T_\alpha x_t\| &= \|T_\alpha U T_\alpha(t\sigma V + (I - t\mu F))x_t - T_\alpha U T_\alpha x_t\| \\
&\leq t\|\sigma V x_t - \mu F x_t\|.
\end{aligned}$$

By the boundedness of $\{Vx_t\}$ and $\{Fx_t\}$, we obtain

$$\lim_{t\to 0}\|x_t - T_\alpha U T_\alpha x_t\| = 0. \tag{3.4}$$

Next, we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. Assume that $\{t_n\} \subset (0, \frac{1}{\tau - \sigma l})$ is such that $t_n \to 0^+$ as $n \to \infty$. In particular from (3.4), we have

$$\lim_{n\to\infty}\|x_{t_n} - T_\alpha U T_\alpha x_{t_n}\| = 0. \tag{3.5}$$

Put $z_t = t\sigma V x_t + (I - t\mu F)x_t$, $y_t = T_\alpha(t\sigma V x_t + (I - t\mu F)x_t) = T_\alpha z_t$, $z_n := z_{t_n}$ and $y_n := y_{t_n} = T_\alpha z_n$. Then we have, for any $p \in \Omega$,

$$\begin{aligned}
y_t - p = T_\alpha z_t - p = (1 - \alpha)z_t + \alpha T z_t - p &= \alpha(Tz_t - z_t) + z_t - p \\
&= \alpha(Tz_t - z_t) + t\sigma(V x_t - Vp) \\
&+ (I - t\mu F)x_t - (I - t\mu F)p + t(\sigma Vp - \mu Fp).
\end{aligned} \tag{3.6}$$

Since $T$ is $\xi$-strictly pseudo-contractive with a fixed point p, for all $x \in E_1$, we have

$$\begin{aligned}\|T_\alpha x - p\|^2 &\leq \|x-p\|^2 - 2\alpha(\xi - \alpha m^2)\|x - Tx\|^2\\ &= \|x - p + T_\alpha x - T_\alpha x\|^2 - 2\alpha(\xi - \alpha m^2)\|x - Tx\|^2\\ &\leq 2m^2\|x - T_\alpha x\|^2 + \|T_\alpha x - p\|^2\\ &+ 2\langle x - T_\alpha x, J_1(T_\alpha x - p)\rangle - 2\alpha(\xi - \alpha m^2)\|x - Tx\|^2\\ &\leq \|T_\alpha x - p\|^2 - 2\alpha(\xi - 2\alpha m^2)\|x - Tx\|^2 + 2\langle x - T_\alpha x, J_1(T_\alpha x - p)\rangle.\end{aligned}$$

So,

$$2\alpha(\xi - 2\alpha m^2)\|x - Tx\|^2 \leq 2\langle x - T_\alpha x, J_1(T_\alpha x - p)\rangle.$$

Since $2\alpha(\xi - 2\alpha m^2) > 0$, we have

$$\langle x - T_\alpha x, J_1(T_\alpha x - p)\rangle \geq 0. \tag{3.7}$$

Combining (3.6) with (3.7) along with Lemma 2.11, we get

$$\begin{aligned}\|y_t - p\|^2 &= \langle y_t - p, J_1(y_t - p)\rangle\\ &= \langle y_t - z_t, J_1(y_t - p)\rangle + t\sigma\langle Vx_t - Vp, J_1(y_t - p)\rangle\\ &+ \langle (I - t\mu F)x_t - (I - t\mu F)p, J_1(y_t - p)\rangle + t\langle \sigma Vp - \mu Fp, J_1(y_t - p)\rangle\\ &\leq t\sigma l\|Vx_t - Vp\|\|y_t - p\|\\ &+ (1 - t\tau)\|x_t - p\|\|y_t - p\| + t\langle \sigma Vp - \mu Fp, J_1(y_t - p)\rangle\\ &\leq [1 - t(\tau - \sigma l)]\|x_t - p\|\|y_t - p\| + t\langle \sigma Vp - \mu Fp, J_1(y_t - p)\rangle\\ &\leq [1 - t(\tau - \sigma l)]\|x_t - p\|^2 + t\langle \sigma Vp - \mu Fp, J_1(y_t - p)\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_t - p\|^2 &= \|T_\alpha U y_t - p\|^2 \leq \|y_t - p\|^2\\ &\leq [1 - t(\tau - \sigma l)]\|x_t - p\|^2 + t\langle \sigma Vp - \mu Fp, J_1(y_t - p)\rangle.\end{aligned}$$

Hence, we obtain

$$\|x_t - p\|^2 \leq \frac{1}{\tau - \sigma l}\langle \sigma Vp - \mu Fp, J_1(y_t - p)\rangle.$$

In particular, we have

$$\|x_n - p\|^2 \leq \frac{1}{\tau - \sigma l}\langle \sigma Vp - \mu Fp, J_1(y_n - p)\rangle \quad \forall p \in \Omega. \tag{3.8}$$

Note that

$$\|x_t - z_t\| = \|x_t - [t\sigma Vx_t + (I - t\mu F)x_t]\| \leq t\|\sigma Vx_t - \mu Fx_t\| \to 0 \quad as \quad t \to 0.$$

So,

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.9}$$

Observe that

$$\begin{aligned}\|z_n - p\| &= \|t_n\sigma Vx_n + (I - t_n\mu F)x_n - p\|\\ &= \|(x_n - p) + t_n(\sigma Vx_n - \mu Fx_n)\|\\ &\leq \|x_n - p\| + t_n\|\sigma Vx_n - \mu Fx_n\|.\end{aligned} \tag{3.10}$$

Then, from (3.10) and the fact that $T_\alpha$ is a $\xi$-psuedo-contractive mapping with a fixed point $p$, we deduce

$$\begin{aligned}\|x_n-p\|^2 &= \|T_\alpha UT_\alpha z_n - T_\alpha UT_\alpha p\|^2 \le \|T_\alpha z_n - p\|^2\\ &\le \|z_n-p\|^2 - 2\alpha(\xi-\alpha m^2)\|z_n-Tz_n\|^2\\ &\le [\|x_n-p\| + t_n\|\sigma Vx_n - \mu Fx_n\|]^2 - 2\alpha(\xi-\alpha m^2)\|z_n-Tz_n\|^2\\ &\le \|x_n-p\|^2 + t_n^2\|\sigma Vx_n-\mu Fx_n\|^2\\ &+ 2t_n\|x_n-p\|\|\sigma Vx_n-\mu Fx_n\| - 2\alpha(\xi-\alpha m^2)\|z_n-Tz_n\|^2\\ &\le \|x_n-p\|^2 + t_nM - 2\alpha(\xi-\alpha m^2)\|z_n-Tz_n\|^2,\end{aligned}$$

where $0 \le M = \sup\{t_n\|\sigma Vx_n-\mu Fx_n\|^2 + 2\|x_n-p\|\|\sigma Vx_n-\mu Fx_n\|\}$ is an appropriate constant. Since $2\alpha(\xi-\alpha m^2)>0$, it follows that

$$\lim_{n\to\infty}\|z_n-Tz_n\| = 0.$$

So, we have

$$\lim_{n\to\infty}\|y_n-z_n\| = \lim_{n\to\infty}\|T_\alpha z_n - z_n\| = \alpha^2\lim_{n\to\infty}\|z_n-Tz_n\| = 0. \tag{3.11}$$

Since $\{x_n\}$ is a bounded sequence in a uniformly convex Banach space, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $x^*$. From Remark 2.7, we conclude that $E_1$ satisfies Opial's condition. Therefore, $x_n \rightharpoonup x^*$. Then by (3.9) and (3.11), $y_n \rightharpoonup x^*$. With regard to (3.5), we can use Lemma 2.8 to get $x^* = T_\alpha UT_\alpha x^*$. By Proposition 2.4 $(iv)$, we have $T_\alpha x^* = x^*$ and $Ux^* = x^*$, and hence $S(Ax^*) = Ax^*$. Thus $x^* \in Fix(T)$ and $Ax^* \in Fix(S)$, that is, $x^* \in \Omega$. Therefore, we can substitute $x^*$ for $p$ in (3.8) to obtain

$$\|x_n-x^*\|^2 \le \frac{1}{\tau-\sigma l}\langle \sigma Vx^* - \mu Fx^*, J_1(y_n-x^*)\rangle.$$

Consequently, $y_n \rightharpoonup x^*$ actually implies that $x_n \to x^*$. This argument proves the relative norm-compactness of the net $\{x_t\}$ as $t\to 0^+$. Letting $n\to\infty$ in (3.8), we have

$$\|x^*-p\|^2 \le \frac{1}{\tau-\sigma l}\langle \sigma Vp - \mu Fp, J_1(x^*-p)\rangle, \quad p\in\Omega.$$

This implies that $x^*\in\Omega$ solves the variational inequality

$$\langle \sigma Vp-\mu Fp, J_1(x^*-p)\rangle \ge 0, \quad p\in\Omega. \tag{3.12}$$

By Lemma 2.9, (3.12) is equivalent to its dual variational inequality

$$\langle \sigma Vx^* - \mu Fx^*, J_1(p-x^*)\rangle \ge 0 \quad p\in\Omega.$$

This is exactly (3.2). By uniqueness of the solution of the variational inequality (3.2), we deduce that each cluster point of $\{x_t\}$ as $t\to 0^+$ equals $x^*$. Therefore $x_t\to x^*$ as $t\to 0^+$. This completes the proof. ∎

**Remark 3.2.** If we consider $T = P_C$, $S = P_Q$, $Fix(T) = C$ and $Fix(S) = Q$, then Theorem 3.1 generalizes Theorem 3.2 and other results obtained by Jung in [7]. Furthermore, if $F$ is a self-adjoint, strongly positive bounded linear operator and $V = I$, then Theorem 3.1 generalizes the results of Yao et al [8].

**Theorem 3.3.** *Let $H_1$ and $H_2$ be two real Hilbert spaces. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator and $A^*$ be the adjoint of $A$. Let $G : H_1 \to H_1$ be a firmly nonexpansive mapping with $Fix(G) \neq \emptyset$ and $S : H_2 \to H_2$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. Let $V : H_1 \to H_1$ be $l$-Lipschitzian with constant $l \in [0, \infty)$ and let $F : H_1 \to H_1$ be $k$-Lipschitzian and $\eta$- strongly monotone operator with constants $k > 0$ and $\eta > 0$ such that $\mu, \sigma, l$ and $\tau$ satisfy $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \sigma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Suppose $\Omega = \{x \in Fix(G) : Ax \in Fix(S)\} \neq \emptyset$. For any $t \in (0, \frac{1}{\tau - \sigma l})$, define a net $\{x_t\} \subset H_1$ by*

$$x_t = G[I - \gamma A^*(I - S)]G[t\sigma V x_t + (1 - t\mu F)x_t] \tag{3.13}$$

*where $\gamma \in (0, \frac{2\beta}{\|A\|^2})$ and $\alpha \in (0, 1)$. Then the net $\{x_t\}$ converges strongly to $x^* \in \Omega$ which a solution of the following variational inequality*

$$x^* \in \Omega \quad \text{such that} \quad \langle \sigma V x^* - \mu F x^*, x - x^* \rangle \geq 0 \quad \forall x \in \Omega.$$

*Proof.* A simple calculation shows that every firmly nonexpansive mapping is a nonexpansive mapping, and every nonexpansive mapping is a $\frac{1}{2}$-strictly pseudo-contractive mapping. Therefore every firmly nonexpansive mapping is a $\frac{1}{2}$-strictly pseudo-contractive mapping. Set $h = \frac{1}{\alpha}G + (1 - \frac{1}{\alpha})I$. Then

$$\begin{aligned}\|(I-h)x-(I-h)y\|^2 = \frac{1}{\alpha^2}\|(I-G)x-(I-G)y\|^2 &\leq \frac{2}{\alpha^2}\langle (I-G)x-(I-G)y, x-y\rangle \\ &= \frac{1}{2\alpha}\langle (I-h)x-(I-h)y, x-y\rangle,\end{aligned}$$

from which it follows that $h$ is an $\frac{\alpha}{2}$-strictly pseudo-contractive mapping. Now, putting $T = h$ in Theorem 3.1, the result follows. ■

**Remark 3.4.** Theorem 3.3 generalizes Theorem 3.5 and its follwing results already obtained by Ansari et al [11] based on the hybrid steepest decent method. Our result also generalizes the results of Ansari et al regarding variational and equilibrum problems.

**Theorem 3.5.** *Let $E_1$ and $E_2$ be two real uniformly convex and 2-uniformly smooth Banach spaces with the best smoothness constants $m$ and $n$, and $J_1$ and $J_2$ be the duality mappings on $E_1$ and $E_2$, respectively. Let $A : E_1 \longrightarrow E_2$ be a bounded linear operator and $A^*$ be the adjoint of $A$. Let $T : E_1 \to E_1$ be a $\xi$-strictly pseudo-contractive mapping with $Fix(T) \neq \emptyset$ and $S : E_2 \to E_2$ be a $\beta$-strictly pseudo-contractive mapping with $Fix(S) \neq \emptyset$. Let $V : E_1 \to E_1$ be $l$-Lipschitzian with constant $l \in [0, \infty)$ and let $F : E_1 \to E_1$ be a $k$-Lipschitzian and $\eta$-strongly monotone operator with constants $k > 0$ and $\eta > 0$ such that $\mu, \sigma, l$ and $\tau$ satisfy $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \sigma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Suppose*

$$\Omega = \{x \in Fix(T) : Ax \in Fix(S)\} \neq \emptyset.$$

*For any $t \in (0, \frac{1}{\tau - \sigma l})$, define a sequence $\{x_n\} \subset E_1$ by*

$$x_{n+1} = T_\alpha[I - \gamma J_1^{-1} A^* J_2(I - S)]T_\alpha[\beta_n \sigma V x_n + (1 - \beta_n \mu F)x_n] \tag{3.14}$$

*where $x_1 \in E_1$ is arbitrary, $\gamma \in (0, \frac{2\beta}{\|A\|^4})$, $\alpha \in (0, \frac{\xi}{2m^2})$ and the sequence $\{\beta_n\}$ satisfies the following conditions:*

*(1)* $\{\beta_n\} \subset [0, 1], \lim_{n\to\infty} \beta_n = 0$,
*(2)* $\sum_{n=0}^{\infty} \beta_n = \infty$.

*Then the sequence* $\{x_n\}$ *converges strongly to* $x^* \in \Omega$ *which a solution of the following variational inequality*

$$x^* \in \Omega \quad \textit{such that} \quad \langle \sigma V x^* - \mu F x^*, J_1(x - x^*)\rangle \geq 0 \quad \forall x \in \Omega.$$

*Proof.* Let $p \in \Omega$ and $U = I - \gamma J_1^{-1} A^* J_2(I - S)A$. Then (3.14) becomes

$$x_{n+1} = T_\alpha U T_\alpha(\beta_n \sigma V x_n + (I - \beta_n \mu F)x_n) \quad n \geq 0.$$

We divide the proof into five steps as follows.

**Step 1.** We show that $\{x_n\}$ is bounded. In fact, from (3.14) we deduce that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|T_\alpha U T_\alpha[\beta_n \sigma V x_n + (I - \beta_n \mu F)x_n] - T_\alpha U T_\alpha p\| \\
&\leq \|\beta_n \sigma V x_n + (I - \beta_n \mu F)x_n - p\| \\
&\leq \beta_n \sigma \|V x_n - V p\| + \|(I - \beta_n \mu F)x_n - (I - \beta_n \mu F)p\| \\
&+ \beta_n \|\sigma V p - \mu F p\| \\
&\leq \beta_n \sigma l \|x_n - p\| + (1 - \beta_n \tau)\|x_n - p\| + \beta_n \|\sigma V p - \mu F p\| \\
&= [1 - (\tau - \sigma l)\beta_n]\|x_n - p\| + (\tau - \sigma l)\beta_n \frac{\|\sigma V p - \mu F p\|}{\tau - \sigma l} \\
&\leq \max\{\|x_n - p\|, \frac{\|\sigma V p - \mu F p\|}{\tau - \sigma l}\}.
\end{aligned}$$

It now follows by induction that

$$\|x_{n+1} - p\| \leq \max\{\|x_0 - p\|, \frac{\|\sigma V p - \mu F p\|}{\tau - \sigma l}\}.$$

This means that $\{x_n\}$ is bounded. It is easy to see that $\{V x_n\}$, $\{U x_n\}$ and $\{F x_n\}$ are bounded too.

**Step 2.** We show that $\lim_{n\to\infty} \|T_\alpha U T_\alpha z_n - z_n\| = 0$. To this end, set

$$y_n := T_\alpha[\beta_n \sigma V x_n + (I - \beta_n \mu F)x_n],$$

and

$$z_n := \beta_n \sigma V x_n + (I - \beta_n \mu F)x_n.$$

Since $U$ is averaged by Lemma 2.5, and since every nonexpansive mapping is averaged, it follows that $T_\alpha$ is averaged. Since the composition of finitely many averaged mappings is averaged by Proposition 2.4 (*iv*), $T_\alpha U$ is averaged. Hence, there exists a positive constant $\lambda_1 \in (0, 1)$ such that $T_\alpha U = (1 - \lambda_1)I + \lambda_1 G_1$, where $G_1$ is a nonexpansive mapping. Since $T_\alpha$ is averaged, there exists $\lambda_2 \in (0, 1)$ such that $T_\alpha = (1 - \lambda_2)I + \lambda_2 G_2$, where $G_2$ is a nonexpansive mapping. It follows that

$$\begin{aligned}
y_n &= T_\alpha z_n = ((1 - \lambda_2)I + \lambda_2 G_2)z_n \\
&= ((1 - \lambda_2)I + \lambda_2 G_2)(\beta_n \sigma V x_n + (I - \beta_n \mu F)x_n) \\
&= (1 - \lambda_2)(\beta_n \sigma V x_n + (I - \beta_n \mu F)x_n) + \lambda_2 G_2 z_n \\
&= (1 - \lambda_2)(x_n + \beta_n(\sigma V x_n - \mu F x_n) + \lambda_2 G_2 z_n \qquad (3.15) \\
&= (1 - \lambda_2)x_n + \lambda_2[\frac{(1 - \lambda_2)}{\lambda_2}\beta_n(\sigma V x_n - \mu F x_n) + G_2 z_n] \\
&= (1 - \lambda_2)x_n + \lambda_2 q_n,
\end{aligned}$$

where

$$q_n = \frac{1-\lambda_2}{\lambda_2}\beta_n(\sigma V x_n - \mu F x_n) + G_2 z_n.$$

Moreover, we get

$$\begin{aligned}
\|q_{n+1} - q_n\| &= \|\frac{(1-\lambda_2)}{\lambda_2}\beta_{n+1}(\sigma V x_{n+1} - \mu F x_{n+1}) + G_2 z_{n+1} \\
&\quad - \frac{(1-\lambda_2)}{\lambda_2}\beta_n(\sigma V x_n - \mu F x_n) + G_2 z_n\| \\
&\leq \|G_2 z_{n+1} - G_2 z_n\| \\
&\quad + \frac{(1-\lambda_2)}{\lambda_2}[\beta_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| \\
&\quad + \beta_n\|\sigma V x_n - \mu F x_n\|] \\
&\leq \|z_{n+1} - z_n\| + \frac{(1-\lambda_2)}{\lambda_2}[\beta_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| \\
&\quad + \beta_n\|\sigma V x_n - \mu F x_n\|].
\end{aligned} \tag{3.16}$$

In view of (3.14) and (3.15), we have

$$\begin{aligned}
x_{n+1} &= T_\alpha U y_n \\
&= ((1-\lambda_1)I + \lambda_1 G_1)y_n \\
&= (1-\lambda_1)y_n + \lambda_1 G_1 y_n \\
&= (1-\lambda_1)[(1-\lambda_2)x_n + \lambda_2 q_n] + \lambda_1 G_1 y_n \\
&= (1-\lambda_1)(1-\lambda_2)x_n + (1-\lambda_1)\lambda_2 q_n + \lambda_1 G_1 y_n \\
&= (1-(\lambda_1+\lambda_2-\lambda_1\lambda_2))x_n + (1-\lambda_1)\lambda_2 q_n + \lambda_1 G_1 y_n \\
&= (1-\lambda_3)x_n + \lambda_3[\frac{(1-\lambda_1)\lambda_2}{\lambda_3}q_n + \frac{\lambda_1}{\lambda_3}G_1 y_n] \\
&= (1-\lambda_3)x_n + \lambda_3 p_n,
\end{aligned} \tag{3.17}$$

where

$$\lambda_3 = \lambda_1 + \lambda_2 - \lambda_1\lambda_2$$

and

$$p_n = \frac{(1-\lambda_1)\lambda_2}{\lambda_3}q_n + \frac{\lambda_1}{\lambda_3}G_1 y_n.$$

Thus, from (3.16), we derive that

$$\begin{aligned}
\|p_{n+1} - p_n\| &= \|\frac{(1-\lambda_1)\lambda_2}{\lambda_3}q_{n+1} + \frac{\lambda_1}{\lambda_3}G_1 y_{n+1} - \frac{(1-\lambda_1)\lambda_2}{\lambda_3}q_n - \frac{\lambda_1}{\lambda_3}G_1 y_n\| \\
&\leq \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|q_{n+1} - q_n\| + \frac{\lambda_1}{\lambda_3}\|G_1 y_{n+1} - G_1 y_n\| \\
&\leq \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|q_{n+1} - q_n\| + \frac{\lambda_1}{\lambda_3}\|y_{n+1} - y_n\| \\
&\leq \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|q_{n+1} - q_n\| + \frac{\lambda_1}{\lambda_3}\|z_{n+1} - z_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-\lambda_1)\lambda_2}{\lambda_3}\|z_{n+1}-z_n\| + \frac{(1-\lambda_1)(1-\lambda_2)}{\lambda_3}[\beta_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| \\
&+ \beta_n\|\sigma V x_n - \mu F x_n\|] + \frac{\lambda_1}{\lambda_3}\|z_{n+1}-z_n\| \\
&= \frac{(\lambda_1+\lambda_2-\lambda_1\lambda_2)}{\lambda_3}\|z_{n+1}-z_n\| \\
&+ \frac{(1-\lambda_3)}{\lambda_3}[\beta_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \beta_n\|\sigma V x_n - \mu F x_n\|] \\
&= \|\beta_{n+1}\sigma V x_{n+1} + (I-\beta_{n+1}\mu F)x_{n+1} - \beta_n \sigma V x_n + (I-\beta_n\mu F)x_n\| \\
&+ \frac{(1-\lambda_3)}{\lambda_3}[\beta_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \beta_n\|\sigma V x_n - \mu F x_n\|] \\
&\leq \|x_{n+1}-x_n\| + \beta_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \beta_n\|\sigma V x_n - \mu F x_n\| \\
&+ \frac{(1-\lambda_3)}{\lambda_3}[\beta_n\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \beta_n\|\sigma V x_n - \mu F x_n\|].
\end{aligned} \tag{3.18}$$

This implies that

$$\begin{aligned}
\|p_{n+1}-p_n\| \leq \|x_{n+1}-x_n\| &+ \beta_{n+1}\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \beta_n\|\sigma V x_n - \mu F x_n\| \\
&+ \frac{(1-\lambda_3)}{\lambda_3}[\beta_n\|\sigma V x_{n+1} - \mu F x_{n+1}\| + \beta_n\|\sigma V x_n - \mu F x_n\|],
\end{aligned}$$

and

$$\limsup_{n\to\infty}(\|p_{n+1}-p_n\| - \|x_{n+1}-x_n\|) \leq 0. \tag{3.19}$$

Thus, from (3.17), (3.19) and Lemma 2.6, we have

$$\lim_{n\to\infty}\|p_n - x_n\| = 0. \tag{3.20}$$

Also, by (3.17) and (3.20), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lambda_3 \lim_{n\to\infty}\|p_n - x_n\| = 0, \tag{3.21}$$

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\beta_n\|\sigma V x_n - \mu F x_n\| = 0. \tag{3.22}$$

Therefore, from (3.21) and (3.22), we have

$$\|T_\alpha U T_\alpha z_n - z_n\| = \|x_{n+1} - z_n\| \to 0 \quad as \quad n\to\infty.$$

**Step 3.** We show that $\lim_{n\to\infty}\|T_\alpha z_n - z_n\| = \lim_{n\to\infty}\|y_n - z_n\| = 0$. To this end, let $p \in \Omega$. Then we have

$$\|T_\alpha U T_\alpha z_n - p\| - \|z_n - p\| \leq \|T_\alpha U T_\alpha z_n - z_n\|.$$

By taking limit from both sides, and using Step 2, we obtain

$$\lim_{n\to\infty}(\|T_\alpha U T_\alpha z_n - p\| - \|z_n - p\|) = 0. \tag{3.23}$$

By nonexpansiveness of $T_\alpha U$ and $T_\alpha$, we get

$$\|T_\alpha U T_\alpha z_n - p\| \leq \|T_\alpha z_n - p\| \leq \|z_n - p\|,$$

and so,

$$\|T_\alpha U T_\alpha z_n - p\| - \|z_n - p\| \leq \|T_\alpha z_n - p\| - \|z_n - p\| \leq 0.$$

Thus, from (3.23), we deduce that

$$\lim_{n\to\infty}(\|T_\alpha z_n - p\| - \|z_n - p\|) = 0.$$

Since $\|T_\alpha z_n - p\|^2 \le \|z_n - p\|^2 - 2\alpha(\xi - \alpha m^2)\|Tz_n - z_n\|^2$, $\alpha \in (0, \frac{\xi}{2m^2})$ and the sequences $\{Tz_n\}$ and $\{z_n\}$ are bounded, we have

$$\lim_{n}\|T_\alpha z_n - z_n\| = \alpha^2 \lim_{n\to\infty}\|Tz_n - z_n\| = \alpha^2 \lim_{n\to\infty}\|y_n - z_n\| = 0.$$

**Step 4.** We show that $\limsup_{n\to\infty}\langle\sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle \le 0$, where $x^*$ is the unique solution of the variational inequality (3.2). Indeed, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\sigma Vx^* - \mu Fx^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle\sigma Vx^* - \mu Fx^*, x_{n_i} - x^*\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ which converges weakly to a point $p$. Without loss of generality, we may assume that $\{x_{n_i}\}$ converges weakly to $p$. Therefore, from Step 2, (3.22) and Lemma 2.8, we have $x_{n_i} \to p \in Fix(T_\alpha U T_\alpha)$. Since $T_\alpha$ and $U$ are averaged, by Proposition 2.4 $(iv)$, we have $p \in Fix(T_\alpha)$ and $p \in Fix(U)$, and hence $Ap \in Fix(S)$. Thus $p \in \Omega$. Therefore we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\sigma Vx^* - \mu Fx^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle\sigma Vx^* - \mu Fx^*, x_{n_i} - x^*\rangle \\ &= \langle\sigma Vx^* - \mu Fx^*, p - x^*\rangle \le 0.\end{aligned}$$

This together with (3.22) and Step 3 implies that

$$\limsup_{n\to\infty}\langle\sigma Vx^* - \mu Fx^*, Tz_n - x^*\rangle \le 0.$$

**Step 5.** We show that $\lim_{n\to\infty} x_n = x^*$, where $x^*$ is the unique solution of the variational inequality (3.2). We observe that

$$\|T_\alpha z_n - x^*\|^2 = \langle T_\alpha z_n - z_n, J_1(T_\alpha z_n - x^*)\rangle + \langle z_n - x^*, J_1(T_\alpha z_n - x^*)\rangle.$$

Since $\langle T_\alpha z_n - z_n, J_1(T_\alpha z_n - x^*)\rangle \le 0$, we have

$$\begin{aligned}\|T_\alpha z_n - x^*\|^2 &\le \langle z_n - x^*, J_1(T_\alpha z_n - x^*)\rangle \\ &= \langle\beta_n\sigma(Vx_n - Vx^*) \\ &+ (I - \beta_n\mu F)x_n - (I - \beta_n\mu F)x^*, J_1(T_\alpha z_n - x^*)\rangle \\ &+ \beta_n\langle\sigma Vx^* - \mu Fx^*, J_1(T_\alpha z_n - x^*)\rangle \\ &\le (\beta_n\sigma l\|x_n - x^*\| + (1 - \beta_n\tau)\|x_n - x^*\|)\|T_\alpha z_n - x^*\| \\ &+ \beta_n\langle\sigma Vx^* - \mu Fx^*, J_1(T_\alpha z_n - x^*)\rangle \\ &= (1 - \beta_n(\tau - \sigma l))\|x_n - x^*\|\|T_\alpha z_n - x^*\| \\ &+ \beta_n\langle\sigma Vx^* - \mu Fx^*, J_1(T_\alpha z_n - x^*)\rangle \\ &\le \frac{1 - \beta_n(\tau - \sigma l)}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|T_\alpha z_n - x^*\|^2 \\ &+ \beta_n\langle\sigma Vx^* - \mu Fx^*, J_1(T_\alpha z_n - x^*)\rangle.\end{aligned}$$

It follows that

$$\|T_\alpha z_n - x^*\|^2 \le (1 - \beta_n(\tau - \sigma l))\|x_n - x^*\|^2 \tag{3.24}$$
$$+ \beta_n\langle\sigma Vx^* - \mu Fx^*, J_1(T_\alpha z_n - x^*)\rangle. \tag{3.25}$$

From (3.14) and (3.24), we have

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &= \|T_\alpha U T_\alpha z_n - x^*\|^2 \le \|T_\alpha z_n - x^*\|^2 \\ &\le (1-\beta_n(\tau-\sigma l))\|x_n-x^*\|^2 \\ &+ \beta_n(\tau-\sigma l)\frac{2}{\tau-\sigma l}\langle \sigma V x^* - \mu F x^*, J_1(T_\alpha z_n - x^*)\rangle. \end{aligned} \tag{3.26}$$

Put $\lambda_n = \beta_n(\tau-\sigma l)$ and $\delta_n = \frac{2}{\tau-\sigma l}\langle \sigma V x^* - \mu F x^*, J_1(T_\alpha z_n - x^*)\rangle$. It is easily seen from Step 4 and the conditions (1) and (2) that $\lambda_n \to 0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Since (3.26) reduces to

$$\|x_{n+1}-x^*\|^2 \le (1-\lambda_n)\|x_n-x^*\|^2 + \lambda_n\delta_n,$$

by Lemma 2.10, we conclude that $\lim_{n\to\infty}\|x_n-x^*\| = 0$. This completes the proof. ∎

**Remark 3.6.** If we consider $T = P_C$, $S = P_Q$, $Fix(T) = C$ and $Fix(S) = Q$, then Theorem 3.5 generalizes Theorem 3.5 of Jung in [7]. Furthermore, if $F$ is a self-adjoint, strongly positive bounded linear operator and $V = I$, then Theorem 3.5 generalizes the results of Yao et al [8].

**Theorem 3.7.** *Let $H_1$ and $H_2$ be two Hilbert spaces. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator and $A^*$ be the adjoint of $A$. Let $G : H_1 \to H_1$ be a firmly nonexpansive mapping with $Fix(G) \neq \emptyset$ and $S : H_2 \to H_2$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. Let $V : H_1 \to H_1$ be $l$-Lipschitzian with constant $l \in [0,\infty)$ and let $F : H_1 \to H_1$ be a $k$-Lipschitzian and $\eta$-strongly monotone operator with constants $k > 0$ and $\eta > 0$ such that $\mu, \sigma, l$ and $\tau$ satisfy $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \sigma l < \tau = 1 - \sqrt{1-\mu(2\eta-\mu k^2)}$. Suppose $\Omega = \{x \in Fix(G) : Ax \in Fix(S)\} \neq \emptyset$. For any $t \in (0, \frac{1}{\tau-\sigma l})$, define a sequence $\{x_n\} \subset H_1$ by*

$$x_{n+1} = G[I-\gamma A^*(I-S)]G[\beta_n\sigma V x_n + (1-\beta_n\mu F)x_n] \tag{3.27}$$

*where $x_1 \in H_1$ is arbitrary, $\gamma \in (0, \frac{2\beta}{\|A\|^4})$ and $\alpha \in (0, \frac{\xi}{2m^2})$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which a solution of the following variational inequality*

$$x^* \in \Omega \quad \text{such that} \quad \langle \sigma V x^* - \mu F x^*, x - x^*\rangle \ge 0 \quad \forall x \in \Omega.$$

*Proof.* By a similar argument as in the proof of Theorem 3.3, the result follows. ∎

**Remark 3.8.** Theorem 3.5 generalizes both Theorem 3.7 of Ansari et al [11] based on hybrid steepest decent method, and the results of Ansari et al regarding variational and equilibrum problems.

In the next example we compare our method (the hybrid steepest descent method) with the viscosity iterative method [3, 25, 26]:

$$x_{n+1} = \beta_n\sigma V(x_n) + (I-\beta_n F)(x_n + \gamma A^*(S-I)Ax_n + \frac{1-\beta}{2}(T-I)(x_n+\gamma A^*(S-I)Ax_n))$$

and the other iterative method presented in [13–15], (we call it CSM method, for Censor, Segal, and Moudafi):

$$x_{n+1} = T(x_n - \gamma A^*(I-S)Ax_n).$$

**Example 3.9.** Let $E_2$ be the real Hilbert space $l^2$, and let $S : l_2 \to l_2$ be a mapping defined by

$$S(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots).$$

Then, $Fix(S) = \{0\}$ and

$$\begin{aligned}\|Sx - Sy\|^2 &= \sum_{i=2}^{\infty} |x_i - y_i|^2 \leq \sum_{i=2}^{\infty} |x_i - y_i|^2 \\ &= \|x - y\|^2 \leq \|x - y\|^2 + \beta\|(x - Sx) - (y - Sy)\|^2.\end{aligned}$$

Therefore, each $S$ is a $\beta$-strictly pseudo-contractive mapping. Let $E_1$ be the set of real numbers $\mathbb{R}$, and $T : \mathbb{R} \to \mathbb{R}$ be a mapping defined by:

$$T(x) = \begin{cases} x & x < 0 \\ \dfrac{-x}{2} & x \geq 0 \end{cases}$$

Then $Fix(T) = (-\infty, 0]$ and:
If $x > 0$ and $y \leq 0$, then we have $Tx = x$ and $Ty = \frac{-y}{2}$ and so

$$\begin{aligned}|Tx - Ty|^2 &= |x + \frac{y}{2}|^2 = x^2 + xy + \frac{y^2}{4} \\ &\leq x^2 + y^2 - 2xy + 3xy + \frac{y^2}{4} \\ &\leq (x - y)^2 + \frac{1}{9}\frac{9y^2}{4} = \|x - y\|^2 + \frac{1}{9}\|(x - Tx) - (y - Ty)\|^2.\end{aligned}$$

It is easily seen that if $x, y < 0$ or $x, y \geq 0$, then $T$ is $\frac{1}{9}$-strictly pseudo-contractive. Therefore, $T$ is $\frac{1}{9}$-strictly pseudo-contractive. Let $A : \mathbb{R} \to l_2$ be the linear operator defined by

$$A(x) = (\frac{x}{2}, \frac{x}{4}, \frac{x}{8}, \cdots), \quad x \in \mathbb{R}.$$

Then, $A$ is bounded and $\|A\| = \frac{1}{3}$. It now follows that

$$A^* : l_2 \to \mathbb{R}, \quad A^*(x_1, x_2, \cdots) = \sum_{i=1}^{\infty} \frac{x_i}{2^i}.$$

We define $Vx = \frac{1}{2}x$ and $F = I$. It is claimed that the mapping $V$ is Lipschitzian with constant $l = \frac{1}{2}$ and $F$ is Lipschizian and a strongly monotone operator with constants $k = \eta = 1$. On the other hand, we can take $\mu = 1$ and $\sigma = \frac{1}{2}$ which satisfy $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \sigma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$, respectively. We now put, for $n \in \mathbb{N}$, $\beta_n = \frac{1}{\sqrt{n}}$, $\gamma = \frac{1}{2}$ and $\alpha = \frac{1}{4}$. Furthermore, we have

$$\Omega = \{x \in F(T) : Ax \in F(S)\} = \{0\}.$$

Now, all the assumptions in Theorem 3.5 are satisfied. Let us consider the following numerical algorithm:

$$z_n = \beta_n \sigma V x_n + (1 - \beta_n \mu F) x_n = \frac{1}{4\sqrt{n}} x_n + (1 - \frac{1}{\sqrt{n}}) x_n = (1 - \frac{3}{4\sqrt{n}}) x_n,$$

$$y_n = T_{\frac{1}{4}}(z_n) = \begin{cases} z_n & x_n < 0 \\ \frac{5}{8} z_n & x_n \geq 0, \end{cases}$$

$$(I - S)Ay_n = (\frac{y_n}{4}, \frac{y_n}{8}, \frac{y_n}{16}, \cdots), \quad I - \gamma A^*(I - S)Ay_n = \frac{11}{12} y_n,$$

$$x_{n+1} = \begin{cases} \frac{11}{12}(1 - \frac{3}{4\sqrt{n}}) x_n & x_n < 0 \\ \frac{275}{768}(1 - \frac{3}{4\sqrt{n}}) x_n & x_n \geq 0. \end{cases}$$

If we choose $x_0 < 0$, then $x_{n+1} = \frac{11}{12}(1 - \frac{3}{4\sqrt{n}})x_n$. If we choose $x_0 \geq 0$, then $x_{n+1} = \frac{275}{768}(1 - \frac{3}{4\sqrt{n}})x_n$. By Theorem 3.5, the sequence $\{x_n\}$ converges to an element of $\Omega$. By the viscosity iterative method, we obtain

$$x_{n+1} = \begin{cases} \frac{1}{12}(11 - \frac{8}{\sqrt{n}}) x_n & x_n < 0 \\ \frac{1}{36}(11 - \frac{2}{\sqrt{n}}) x_n & x_n \geq 0. \end{cases}$$

By the CSM method, we have

$$x_{n+1} = \begin{cases} \frac{11}{12} x_n & x_n < 0 \\ \frac{-11}{24} x_n & x_n \geq 0. \end{cases}$$

We have displayed the convergence behavior of $x_n$ for $x_0 = 2$ and $n = 20$ (some steps have been skipped) with respect to the three algorithms in Table 1. It is seen that the sequence generaed by our algorithm vanishes to the fixed point 0 faster than the other two algorithms. In general, we cannot claim that our algorithm is the best one, this indeed requires more work, but al least in some instances our method works better. As the above example shows, the hybrid steepest descent method (HSDM) converges to zero faster than the viscosity iteration method (VIM) and the CSM method.

[Table 1]

| $n$ | $x_n$-VIM | $x_n$-CSM method | $x_n$-HSDM |
|---|---|---|---|
| 0 | 2 | 2 | 2 |
| 1 | $5 \times 10^{-1}$ | $-9.1 \times 10^{-1}$ | $1.7 \times 10^{-1}$ |
| 2 | $1.33 \times 10^{-1}$ | $-8.3 \times 10^{-1}$ | $3.01 \times 10^{-2}$ |
| 3 | $3.6 \times 10^{-2}$ | $-7.6 \times 10^{-1}$ | $6.11 \times 10^{-3}$ |
| 4 | $1.01 \times 10^{-2}$ | $-6.9 \times 10^{-1}$ | $1.36 \times 10^{-3}$ |
| 5 | $2.8 \times 10^{-3}$ | $-6.2 \times 10^{-1}$ | $3.2 \times 10^{-4}$ |
| 6 | $8.03 \times 10^{-4}$ | $-5.7 \times 10^{-1}$ | $8.08 \times 10^{-5}$ |
| 7 | $2.28 \times 10^{-4}$ | $-5.2 \times 10^{-1}$ | $2.07 \times 10^{-5}$ |
| 8 | $6.53 \times 10^{-5}$ | $-4.7 \times 10^{-1}$ | $5.46 \times 10^{-6}$ |
| 17 | $3.2 \times 10^{-9}$ | $-2.0 \times 10^{-2}$ | $2 \times 10^{-10}$ |
| 18 | $9 \times 10^{-10}$ | $-1.8 \times 10^{-2}$ | $1 \times 10^{-10}$ |
| 20 | $1 \times 10^{-10}$ | $-1.6 \times 10^{-2}$ | 0 |

## References


[1] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8 (1994) 221–239.

[2] C. Byrne, Iterative oblique projection onto convex subsets and the split feasibility problem, Inverse Problems 18 (2002) 441–453.

[3] A. Abkar, E. Shahrosvand, The split common fixed point problem of two infinite families of demicontractive mappings and the split common null point problem, Filomat 31 (12) (2017) 3859–3874.

[4] A. Abkar, E. Shahrosvand, Split equality common null point problem for Bregman quasi-nonexpansive mappings, Filomat 32 (11) (2018) 3917–3932.

[5] A. Abkar, E. Shahrosvand, A. Azizi, The split common fixed point problem for a family of multivalued and totally strictly pseudocontractive mappings in Banach spaces, Mathematics 5 (1) (2017) Article no. 11.

[6] Y. Dang, Y. Gao, The strong convergence of a KM-CQ-like algorithm for a split feasibility problem, Inverse Problems 27 (2011) Article ID 015007.

[7] J.S. Jung, Iterative algorithms based on the hybrid steepest descent method for the split feasibility problem, J. Nonlinear Sci. Appl. 9 (2016) 4214–4225.

[8] Y. Yao, P.X. Yaga, M. Kang, Composite projection algorithms for the split feasibility problem, Math. Comput. Model. 57 (2013) 693–700.

[9] F. Wang, H.K. Xu, Approximating curve and strong convergence of the CQ algorithms for the split feasibility problem, J. Inequal. Appl. 2010 (2010) Article ID 102085.

[10] S.M. Alsulami, A. Latif, W. Takahashi, Strong convergence theorems by hybrid methods for the split feasibility problem in Banach spaces, Linear Nonlinear Anal. 1 (2015) 1–11.

[11] Q.H. Ansari, A. Rehan, C.F. Wen, Implicit and explicit algorithms for split common fixed point problems, J. Nonlinear Convex Anal. 17 (7) (2016) 1281–1397.

[12] C. Byrne, Y. Censor, A. Gibali, S. Reich, The split common null point problem, J. Nonlinear Convex Anal. 13 (2012) 759–775.

[13] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009) 587–600.

[14] A. Moudafi, The split common fixed point problem for demicontractive mappings, Inverse Probl. 26 (2010) doi:10.1088/0266-5611/26/5/055007.

[15] A. Moudafi, M. Thera, Proximal and dynamical approaches to equilibrium problems, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, New York (1999), 187–201.

[16] I. Yamada, The hybrid steepest descent for the variational inequality problems over the intersection of fixed points sets of nonexpansive mappings, D. Butnariu, Y. Censor, S. Reich (Eds.), Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, North-Holland, Amsterdam, Holland (2001), 473–504.

[17] H.K. Xu, Inequalities in Banach spaces with applications, J. Nonlinear Anal. 16 (1991) 1127–1138.

[18] E. Zeidler, Nonlinear Functional Analysis and Its Applications III, Springer, New York, USA, 1985.

[19] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197–228.

[20] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20 (2004) 103–120.

[21] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integral, J. Math. Anal. Appl. 305 (2005) 227–239.

[22] O. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1976) 591–597.

[23] G.J. Minty, On the generalization of a direct method of the calculus of variations, Bull. Amer. Math. Soc. 73 (1967) 315–321.

[24] H.K. Xu, Iterative algorithm for nonlinear operators, J. Lond. Math. Soc. 66 (2002) 1–17.

[25] M. Eslamian, General algorithms for split common fixed point problem of demicontractive mappings, Optimization 65 (2016) 443–465.

[26] J. Zhao, S. He, Strong convergence of the viscosity approximation process for the split common fixed-point problem of quasi-nonexpansive mappings, Journal of Applied Mathematics 2012 (2012) Article ID 438023.