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On Approximation of the Combination of Variational Inequality Problem and Equilibrium Problem for Nonlinear Mappings

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Abstract Using the concept of the combination of equilibrium problem, we introduce the combination of variational inequality problem for a finite family of inverse-strongly monotone mappings. Under some control conditions, we prove the strong convergence theorem for these nonlinear problems and fixed points of nonspreading mapping. Finally, we give numerical examples in two-dimensional space of real numbers in order to compare numerical results between the combination of variational inequality problem and the variational inequality problem.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. The fixed point problem for the mapping $T : C \to H$ is to find $x \in C$ such that

$$x = Tx. \tag{1.1}$$

We denote the set of solutions of (1.1) by $Fix(T)$. It is well known that $Fix(T)$ is closed and convex and $P_{Fix(T)}$ is well-defined.

In 2000, Moudafi [1] introduced the viscosity approximation method for nonexpansive mapping S as follows:
Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping such that $Fix(S)$ is nonempty. Let $f : C \to C$ be a contraction,

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that is, there exists $\alpha \in (0,1)$ such that $\|fx - fy\| \leq \alpha \|x - y\|, \forall x, y \in C$, and let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 \in C \text{ arbitrary chosen}, \\ x_{n+1} = \frac{1}{1+\epsilon_n} S x_n + \frac{\epsilon_n}{1+\epsilon_n} f(x_n), \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\varepsilon_n\} \subset (0,1)$ satisfies certain conditions. Then the sequence $\{x_n\}$ converges strongly to $z \in Fix(S)$, where $z = P_{Fix(S)} f(z)$ and $P_{Fix(S)}$ is the metric projection of H onto $Fix(S)$.

There are many researchers investigating the viscosity approximation method for other types of nonlinear mappings, see, for instance, [2–4].

In 2006, using the concept of the viscosity approximation method (1.2), Marino and Xu [5] introduced the general iterative method and obtained the strong convergence theorem. Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $f : H \to H$ be a contractive mapping on H and let $\{x_n\}$ be generated by

$$\begin{cases} x_0 \in H \text{ arbitrary chosen}, \\ x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n \gamma f(x_n), n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the appropriate conditions. Then $\{x_n\}$ converges strongly to a fixed point $\tilde{x}$ of T which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z \rangle \leq 0, z \in Fix(T).$$

Observe that if $A \equiv I$ and $\gamma = 1$, then the general iterative method (1.3) reduces to the viscosity approximation method (1.2).

Let $F : C \times C \to \mathbb{R}$ be bifunction. *The classical equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C, \tag{1.4}$$

which was first considered and investigated by Blum and Oettli [6] in 1994. The set of solutions of (1.4) is denoted by $EP(F)$.

The equilibrium problem provides a general framework to study a wide class of problems arising in economics, finance, network analysis, transportation, elasticity and optimization. The theory of equilibrium problems has become an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences, see [6–8].

If we take $F(x, y) = \langle y - x, Ax \rangle$, where $A : C \to H$ is a nonlinear mapping, then the classical equilibrium problem is equivalent to finding an element $x \in C$ such that

$$\langle y - x, Ax \rangle \geq 0, \forall y \in C, \tag{1.5}$$

which is well-known as *the classical variational inequality problem*. The solution set of the problem (1.5) is denoted by $VI(C, A)$.

Variational inequalities were introduced and investigated by Stampacchia [9] in 1964. It is now well known that variational inequalities cover as diverse disciplines as optimal control, optimization, mathematical programming, mechanics and finance, see [10, 11]. There are several techniques to analyze various iterative methods for solving variational inequality problem and the related optimization problems, see [12–20] and the references therein.

In 2013, Suwannaut and Kangtunyakarn [21] introduced *the combination of equilibrium problem* which is to find $x \in C$ such that

$$\sum_{i=1}^{N} a_i F_i (x, y) \geq 0, \forall y \in C, \tag{1.6}$$

where $F_i : C \times C \to \mathbb{R}$ be bifunctions and $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, \dots, N$. The set of solution (1.6) is denoted by

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \left\{x \in C : \left(\sum_{i=1}^{N} a_i F_i\right)(x, y) \geq 0, \forall y \in C\right\}.$$

Remark 1.1. Very recently, in the work of Suwannaut and Kangtunyakarn [22], Khuangsatung and Kangtunyakarn [23] and Bnouhachem [8], they give the numerical examples for main theorems and show that their iteration for the combination of equilibrium problem converges faster than their iteration for the classical equilibrium problem.

Next, we consider the special cases of this problem.

(i) If $F_i = F$, for all $i = 1, 2, \dots, N$, then the combination of equilibrium problem (1.6) reduces to the classical equilibrium problem (1.4).

(ii) If we put $F_i(x, y) = \langle y - x, A_i x\rangle$, where $A_i : C \to H$ is a nonlinear mapping and $i = 1, 2, \dots, N$, then the combination of equilibrium problem is equivalent to finding an element $x \in C$ such that

$$\sum_{i=1}^{N} a_i F_i(x, y) = \sum_{i=1}^{N} a_i \langle y - x, A_i x\rangle = \left\langle y - x, \sum_{i=1}^{N} a_i A_i x \right\rangle \geq 0, \forall y \in C, \tag{1.7}$$

In this paper, we introduced *the combination of variational inequality problem* which is to find $x \in C$ such that

$$\left\langle y - x, \sum_{i=1}^{N} a_i A_i x \right\rangle \geq 0, \forall y \in C, \tag{1.8}$$

where $A_i : C \to H$ is a nonlinear operator and $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, \dots, N$. The set of solution (1.8) is denoted by

$$VI\left(C, \sum_{i=1}^{N} a_i A_i\right) = \left\{u \in C : \left\langle y - x, \sum_{i=1}^{N} a_i A_i x \right\rangle \geq 0, \forall y \in C\right\}.$$

If $A_i = A$, for all $i = 1, 2, \dots, N$, then the combination of variational inequality problem(1.8) becomes the classical variational inequality problem (1.5).

In 2012, Zegeye and Shahzad [24] introduced an iterative method and proved that if C is a nonempty closed convex subset of a real Hilbert space H, $T_1 : C \to C$ is a pseudo-contractive mapping and $T_2 : C \to H$ is a continuous monotone mapping such that $F := F(T_1) \cap VI(C, T_2) \neq \emptyset$. For $\{r_n\} \subset (0, \infty)$ defined $T_{r_n}, F_{r_n} : H \to C$ by the following: for $x \in H$ and $\{r_n\} \subset (0, \infty)$ define

$$T_{r_n} x := \left\{z \in C : \langle y - z, T_1 z\rangle - \frac{1}{r_n} \langle y - z, (1 + r_n) z - x\rangle \leq 0, y \in C\right\}. \tag{1.9}$$

$$F_{r_n} x := \left\{z \in C : \langle y - z, T_2 z\rangle + \frac{1}{r_n} \langle y - z, z - x\rangle \geq 0, y \in C\right\}. \tag{1.10}$$

Then the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} := \alpha_n f(x_n) + (1-\alpha_n) T_{r_n} F_{r_n} x_n, n \geq 1, \tag{1.11}$$

where $f : C \to C$ is a contraction mapping and $\{\alpha_n\} \subset [0,1]$ and $\{r_n\}$ satisfy certain conditions, converges strongly to $z \in F$, where $z = P_F f(z)$.

Later, by modifying (1.11), Wangkeeree and Nammanee [25] introduced the general iterative method as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} := \alpha_n \gamma f(x_n) + (I - \alpha_n A) T_{r_n} F_{r_n} x_n, n \geq 1, \end{cases} \tag{1.12}$$

where $T_1, T_2 : C \to H$ be a continuous pseudo-contractive mapping and a continuous monotone mapping, respectively, $T_{r_n}, F_{r_n} : H \to C$ is defined by (1.9) and (1.10), A is a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator and f is a contraction mapping on H. Then they proved that if $\{\alpha_n\}$ and $\{r_n\}$ satisfy some control conditions, then $\{x_n\}$ generated by $x_1 \in C$ and (1.12) converges strongly to $z = P_{F(T_1)\cap VI(C,T_2)}(I - A + \gamma f)(z)$.

Numerous mathematicians studied iterative methods for pseudo-contractive mapping, monotone mapping and some related mappings and proved the strong convergence theorems, see, for examples, [4, 26–28].

Question A: Does the iterative method for the combination of variational inequality problem converges faster than the iterative method for the classical variational inequality problem?

In this article, motivated by the related research described above, we introduced the new iterative method modified from (1.11) and (1.12). Then, under some appropriate conditions, we prove the strong convergence theorem for the combination of equilibrium problem, the combination of variational inequality problem and a fixed point set of non-spreading mapping. In the last section, we give a numerical example for our main result in two-dimensional space of real numbers to compare the numerical results between the combination of variational inequality problem and the classical variational inequality problem and provide an answer to Question A.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations " $\rightharpoonup$" and " $\to$", respectively. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

Such an operator P_C is called the metric projection of H onto C.
We now recall the following definition and well-known lemmas.

Definition 2.1. Let the mapping $T : H \to H$. Then T is called

(i) a strongly positive operator on H if there exists a constant $\bar{\gamma} > 0$ with property

$$\langle Tx, x\rangle \geq \bar{\gamma} \|x\|^2, \forall x \in H.$$

(ii) ξ-inverse-strongly monotone if there exists a positive real number ξ such that

$$\langle x - y, Tx - Ty\rangle \geq \xi \|Tx - Ty\|^2, \forall x, y \in H.$$

(iii) contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in H.$$

(iv) a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in H.$$

(v) quasi-nonexpansive if

$$\|Tx - p\| \leq \|x - p\|, \text{ for every } x \in H \text{ and } p \in Fix(T).$$

(vi) a nonspreading mapping if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in H.$$

Lemma 2.2 ([29]). *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

Lemma 2.3 ([30]). *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.4 ([31]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([32]). *Let H be a real Hilbert space. Then the following results hold:*

(i) For all $x, y \in H$ and $\alpha \in [0,1]$,

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2,$$

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$, *for each $x, y \in H$.*

Lemma 2.6 ([33]). *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequences of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} Fix(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{i=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$S(x) = \sum_{i=1}^{\infty} \lambda_n T_n x,$$

for $x \in C$ is well defined, nonexpansive and $Fix(S) = \bigcap_{n=1}^{\infty} Fix(T_n)$ hold.

Lemma 2.7 ([21]). *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \dots, N$, let A_i be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\dots,N} \gamma_i$ and $\bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$. Let $\{a_i\}_{i=1}^{N} \subseteq (0,1)$ with $\sum_{i=1}^{N} a_i = 1$. Then, the following properties hold:*

(i) $\left\| I - \rho \sum_{i=1}^{N} a_i A_i \right\| \leq 1 - \rho\bar{\gamma}$ and $I - \rho \sum_{i=1}^{N} a_i A_i$ is a nonexpansive mapping, for every $0 < \rho < \|A_i\|^{-1} (i = 1, 2, \dots, N)$.

(ii) $VI\left(C, \sum_{i=1}^{N} a_i A_i\right) = \bigcap_{i=1}^{N} VI(C, A_i)$.

Lemma 2.8 ([34]). *Let C be a nonempty closed convex subset of H. Then a mapping $T : C \to C$ is nonspreading if and only if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \text{ for all } x, y \in C.$$

Remark 2.9 ([22]). If T is nonspreading with $Fix(T) \neq \emptyset$, then T is quasi-nonexpansive.

Example 2.10. Let an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\langle x, y\rangle = x \cdot y = x_1 y_1 + x_2 y_2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be given by $\|x\| = \sqrt{x_1^2 + x_2^2}$, for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Let $I = [1, 2]$ and let $T : I^2 \to I^2$ be defined by

$$Tu = \left(\frac{x_1 + 2}{3}, \frac{6x_2 + 1}{7} \right), \text{ for all } x = (x_1, x_2) \in I^2.$$

Let $K = [0, 2]$ and let $S : K^2 \to K^2$ be defined by

$$Sx = \begin{cases} \left(\frac{x_1+2}{2}, \frac{x_2+2}{2}\right) & \text{if } x \in (1, 2] \times (1, 2], \\ \left(\frac{x_1}{2}, \frac{x_2}{2}\right) & \text{if } x \in [0, 1] \times [0, 1]. \end{cases}$$

Observe that T is nonspreading and quasi-nonexpansive. Furthermore, S is quasi–nonexpansive, but S is not nonspreading. Therefore, the converse of Remark 2.9 does not hold.

Using Remark 2.9, we obtain the following results. Therefore, the proof is omitted.

Remark 2.11. Let $T : H \to H$ be a nonspreading mapping with $Fix(T) \neq \emptyset$. Define $S : H \to H$ by $Sx := \lambda Tx + (1 - \lambda)x$, where $\lambda \in (0, 1)$. Then, there hold the following statement:

(i) $Fix(T) = Fix(S)$;

(ii) S is a quasi-nonexpansive mapping, that is,

$$\|Sx - y\| \leq \|x - y\|, \text{ for every } x \in H \text{ and } y \in Fix(T).$$

Remark 2.12. Let $T : H \to H$ be a nonspreading mapping and let $S : H \to H$ be a nonexpansive mapping with $Fix(T) \cap Fix(S) \neq \emptyset$. Define $Q : H \to H$ by $Qx := \alpha Tx + (1 - \alpha)Sx$, where $\alpha \in (0, 1)$. Then $Fix(Q) = Fix(T) \cap Fix(S)$.

Assumption 2.13 ([6]). Let $F : C \times C \to \mathbb{R}$ be a bifunction and F satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) For each $x, y, z \in C$,

$$\lim_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.14 ([21]). *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \dots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ with $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Then,*

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0, 1)$ for every $i = 1, 2, \dots, N$ and $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.15 ([6]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.16 ([7]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(iii) $Fix(T_r) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Remark 2.17 ([21]). Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \dots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ with $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Then $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4) and

$$Fix(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i), \text{ for all } r > 0,$$

where $a_i \in (0, 1)$, for each $i = 1, 2, \dots, N$, and $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.18 ([35]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be α-inverse strongly monotone. If $0 < \lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.*

Lemma 2.19. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \dots, N$, let $G_i : C \to H$ be an α_i-inverse strongly monotone mapping with $\alpha_i > 0$, $\bar{\alpha} = \min_{1 \leq i \leq N}\{\alpha_i\}$ and $\bigcap_{i=1}^{N} VI(C, G_i) \neq \emptyset$. Then, we have the following statement hold.*

(i) $VI\left(C, \sum_{i=1}^{N} b_i G_i\right) = \bigcap_{i=1}^{N} VI(C, G_i)$,

(ii) if $0 < \gamma \leq 2\bar{\alpha}$, we have $I - \gamma \sum_{i=1}^{N} b_i G_i$ is a nonexpansive mapping,

where $b_i \in (0, 1)$ for every $i = 1, 2, \dots, N$ and $\sum_{i=1}^{N} b_i = 1$.

Proof. To show (i), let $x, y \in C$. For every $i = 1, 2, \ldots, N$, put $F_i(x, y) = \langle y - x, G_i x\rangle$ and it is clear that F_i satisfies the conditions $(A1) - (A4)$. Applying Lemma 2.14, we can conclude that

$$VI\left(C, \sum_{i=1}^{N} b_i G_i\right) = \bigcap_{i=1}^{N} VI\left(C, G_i\right).$$

To prove (ii), it is obvious that $\sum_{i=1}^{N} b_i G_i$ is an $\bar{\alpha}$-inverse strongly monotone mapping. Since $0 < \gamma \leq 2\bar{\alpha}$, by Lemma 2.18, hence we get $I - \gamma \sum_{i=1}^{N} b_i G_i$ is a nonexpansive mapping of C into H. ∎

Lemma 2.20 ([24]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.21 ([24]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping. For $r > 0$ and $x \in H$, define a mapping $F_r : H \to C$ as follows:*

$$F_r x := \left\{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then the following hold:

(i) F_r is single-valued;

(ii) F_r is a firmly nonexpansive type mapping, i.e., for all $x, y \in H$,

$$\|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x - y\rangle;$$

(iii) $Fix\left(F_r\right) = VI(C, A)$;

(iv) $VI(C, A)$ is closed and convex.

Remark 2.22. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $G_i : C \to H$ be α_i-inverse strongly monotone mapping with $\alpha_i > 0$, $\bar{\alpha} = \min_{1\leq i\leq N}\{\alpha_i\}$ and $\bigcap_{i=1}^{N} VI\left(C, G_i\right) \neq \emptyset$. Then, we obtain $\sum_{i=1}^{N} b_i G_i$ is $\bar{\alpha}$-inverse strongly monotone. From Lemma 2.19 and Lemma 2.21, we have

$$Fix(F_r) = VI\left(C, \sum_{i=1}^{N} b_i G_i\right) = \bigcap_{i=1}^{N} VI\left(C, G_i\right), \text{ for all } r > 0,$$

where $b_i \in (0, 1)$, for each $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} b_i = 1$.

3. Strong Convergence Theorem

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $k = 1, 2, \ldots, \bar{N}$, define $\bar{A} : H \to H$ by $\bar{A}x = \sum_{k=1}^{\bar{N}} c_k A_k x$, where A_k is a strongly positive linear bounded operator on H with coefficient $\gamma_k > 0$, $\bar{\gamma} = \min_{k=1,2,\ldots,\bar{N}} \gamma_k$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Define $S_n : H \to H$ by $S_n x := \lambda_n T x + (1 - \lambda_n) x$, where $T : H \to H$ is a nonspreading mapping. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let $G_i : C \to H$ be an α_i-inverse strongly monotone mapping with $\bar{\alpha} =$*

$\min_{i=1,2,\ldots,N}\{\alpha_i\}$. *Suppose that* $\Omega := Fix(T)\cap\bigcap_{i=1}^{N} EP(F_i)\cap\bigcap_{i=1}^{N} VI(C,G_i)\neq\emptyset$. *Let* $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *be sequences generated by* $x_1\in H$ *and*

$$\begin{cases} \sum_{i=1}^{N} a_iF_i(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall y\in C,\\ \left\langle y-v_n,\sum_{i=1}^{N} b_iG_iv_n\right\rangle+\frac{1}{s_n}\langle y-v_n,v_n-x_n\rangle\geq 0,\forall y\in C,\\ y_n=\beta_nx_n+\delta_nu_n+\mu_nv_n,\\ x_{n+1}=\alpha_n\gamma f(y_n)+\left(I-\alpha_n\bar{A}\right)S_ny_n,\forall n\in\mathbb{N}, \end{cases}\tag{3.1}$$

where f *is an* α*-contractive mapping on* H, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\mu_n\}$, $\{r_n\}$, $\{s_n\}$, $\{\lambda_n\}\subset[0,1]$, $\beta_n+\delta_n+\mu_n=1$, $0<a_i<1$, $0<b_i<1$, *for all* $i=1,2,\ldots,N$, $0<c_k<1$, *for all* $k=1,2,\ldots,\bar{N}$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $0<\tau\leq\beta_n,\delta_n,\mu_n\leq\upsilon<1$, *for some* $\tau,\upsilon>0$;

(iii) $\sum_{n=1}^{\infty}\lambda_n<\infty$;

(iv) $0<\epsilon\leq r_n\leq\eta<\infty$, *for some* $\epsilon,\eta>0$;

(v) $0<\theta\leq s_n\leq\varphi<\infty$, *for some* $\theta,\varphi>0$;

(vi) $\sum_{i=1}^{N}a_i=\sum_{i=1}^{N}b_i=1$ *and* $\sum_{k=1}^{\bar{N}}c_k=1$;

(vii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$, $\sum_{n=1}^{\infty}|\mu_{n+1}-\mu_n|<\infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$, $\sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* $x^*=P_\Omega(I-\bar{A}+\gamma f)x^*$.

Proof. The proof of this theorem will be divided into five steps.

Step 1. We show that $\{x_n\}$ is bounded.
Since $\alpha_n\to 0$ as $n\to\infty$, without loss of generality, we may assume that $\alpha_n<\frac{1}{\|A_i\|}$, for all $n\in\mathbb{N}$ and $i=1,2,\ldots,N$. Since $\sum_{i=1}^{N}a_iF_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^{N} a_iF_i(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall y\in C,$$

by Lemma 2.16 and Remark 2.17, we have $u_n=T_{r_n}x_n$ and $Fix(T_{r_n})=\bigcap_{i=1}^{N}EP(F_i)$. Since $\sum_{i=1}^{N}b_iG_i$ is $\bar{\alpha}$-inverse strongly monotone and

$$\left\langle y-v_n,\sum_{i=1}^{N} b_iG_iv_n\right\rangle+\frac{1}{s_n}\langle y-v_n,v_n-x_n\rangle\geq 0,\forall y\in C,$$

by Lemma 2.21 and Remark 2.22, we have $v_n = T_{s_n} x_n$ and $Fix(F_{s_n}) = \bigcap_{i=1}^{N} VI(C, G_i)$. Let $z \in \Omega$. By nonexpansiveness of T_{r_n} and F_{s_n}, we have

$$\begin{aligned} \|y_n - z\| &\le \beta_n \|x_n - z\| + \delta_n \|T_{r_n} x_n - z\| + \mu_n \|F_{s_n} x_n - z\| \\ &\le \|x_n - z\|. \end{aligned} \tag{3.2}$$

From Lemma 2.7, Remark 2.11(ii) and (3.2), we obtain

$$\begin{aligned} &\|x_{n+1} - z\| \\ &\le \alpha_n \left\|\gamma f(y_n) - \bar{A}z\right\| + \left\|I - \alpha_n \bar{A}\right\| \|S_n y_n - z\| \\ &\le \alpha_n \gamma \|f(y_n) - f(z)\| + \alpha_n \left\|\gamma f(z) - \bar{A}z\right\| + (1 - \alpha_n \bar{\gamma}) \|y_n - z\| \\ &= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \|x_n - z\| + \alpha_n \left\|\gamma f(z) - \bar{A}z\right\| \\ &\le \max\left\{ \|x_1 - z\|, \frac{\left\|\gamma f(z) - \bar{A}z\right\|}{\bar{\gamma} - \gamma\alpha} \right\}. \end{aligned}$$

By induction, we get $\|x_n - z\| \le \max\left\{ \|x_1 - z\|, \frac{\|\gamma f(z) - \bar{A}z\|}{\bar{\gamma} - \gamma\alpha} \right\}, \forall n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{v_n\}$ and $\{y_n\}$.

Step 2. Claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
By the definition of x_n and Lemma 2.7, we obtain

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ \le& \alpha_n \gamma \|f(y_n) - f(y_{n-1})\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(y_{n-1})\| \\ &+ \left\|I - \alpha_n \bar{A}\right\| \|S_n y_n - S_{n-1} y_{n-1}\| \\ &+ \left\|\left(I - \alpha_n \bar{A}\right) S_{n-1} y_{n-1} - \left(I - \alpha_{n-1} \bar{A}\right) S_{n-1} y_{n-1}\right\| \\ \le& \alpha_n \gamma \alpha \|y_n - y_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(y_{n-1})\| \\ &+ (1 - \alpha_n \bar{\gamma}) \|(y_n - \lambda_n (I - T) y_n) - (y_{n-1} - \lambda_{n-1}(I - T) y_{n-1})\| \\ &+ |\alpha_n - \alpha_{n-1}| \left\|\bar{A} S_{n-1} y_{n-1}\right\| \\ \le& (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \|y_n - y_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(y_{n-1})\| \\ &+ (1 - \alpha_n \bar{\gamma}) \Big[\lambda_n \|(I - T) y_n - (I - T) y_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|(I - T) y_{n-1}\|\Big] \\ &+ |\alpha_n - \alpha_{n-1}| \left\|\bar{A} S_{n-1} y_{n-1}\right\| \\ \le& (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \Big[\beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \delta_n \|u_n - u_{n-1}\| \\ &+ |\delta_n - \delta_{n-1}| \|u_{n-1}\| + \mu_n \|v_n - v_{n-1}\| + |\mu_n - \mu_{n-1}| \|v_{n-1}\|\Big] \\ &+ \gamma |\alpha_n - \alpha_{n-1}| \|f(y_{n-1})\| + (1 - \alpha_n \bar{\gamma}) \Big[\lambda_n \|(I - T) y_n - (I - T) y_{n-1}\| \\ &+ |\lambda_n - \lambda_{n-1}| \|(I - T) y_{n-1}\|\Big] + |\alpha_n - \alpha_{n-1}| \left\|\bar{A} S_{n-1} y_{n-1}\right\|. \end{aligned} \tag{3.3}$$

By using the same proof as in Step 2 of Theorem 3.1 in [21], we obtain

$$\|u_n - u_{n-1}\| \le \|x_n - x_{n-1}\| + \frac{1}{\epsilon} |r_n - r_{n-1}| \|u_n - x_n\|. \tag{3.4}$$

By applying the similar proof of Theorem 3.1 in [25], we get

$$\|v_n - v_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{\theta} |s_n - s_{n-1}| \|v_n - x_n\| . \tag{3.5}$$

Substitute (3.4) and (3.5) into (3.3) to get

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\leq (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| K + \frac{1}{\epsilon} |r_n - r_{n-1}| K \\ &\quad + |\delta_n - \delta_{n-1}| K + \frac{1}{\theta} |s_n - s_{n-1}| K + |\mu_n - \mu_{n-1}| K \\ &\quad + (1 + \gamma) |\alpha_n - \alpha_{n-1}| K + 2\lambda_n K + |\lambda_n - \lambda_{n-1}| K, \end{aligned} \tag{3.6}$$

where $K = \max_{n\in\mathbb{N}}\{\|x_n\|, \|u_n\|, \|v_n\| \|f(y_n)\|, \|u_n - x_n\|, \|v_n - x_n\|,$ $\|(I - T)y_n\|, \|\bar{A}S_n y_n\|\}$. From (3.6), the condition (i), (vi) and Lemma 2.4, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

Step 3. Prove that $\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|v_n - x_n\| = \lim_{n\to\infty} \|S_n x_n - x_n\| = 0.$ To claim this, let $z \in \Omega$. By Lemma 2.7, Remark 2.11(ii) and (3.2), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \left\|\alpha_n \left(\gamma f(y_n) - \bar{A}S_n y_n\right) + (S_n y_n - z)\right\|^2 \\ &\leq \|S_n y_n - z\|^2 + 2\alpha_n \left\langle \gamma f(y_n) - \bar{A}S_n y_n, x_{n+1} - z\right\rangle \\ &\leq \|y_n - z\|^2 + 2\alpha_n \left\|\gamma f(y_n) - \bar{A}S_n y_n\right\| \|x_{n+1} - z\| \\ &\leq \|x_n - z\|^2 - \beta_n\delta_n \|x_n - u_n\|^2 - \beta_n\mu_n \|x_n - v_n\|^2 \\ &\quad + 2\alpha_n \left\|\gamma f(y_n) - \bar{A}S_n y_n\right\| \|x_{n+1} - z\|, \end{aligned}$$

which follows that

$$\begin{aligned} \beta_n\delta_n \|x_n - u_n\|^2 \leq{} & (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| \\ & + 2\alpha_n \left\|\gamma f(y_n) - \bar{A}S_n y_n\right\| \|x_{n+1} - z\| . \end{aligned}$$

From (3.7), the condition (i) and (ii), this implies that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.8}$$

By following the same argument as above, we have

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{3.9}$$

Since

$$\|y_n - x_n\| \leq \delta_n \|u_n - x_n\| + \mu_n \|v_n - x_n\|,$$

by (3.8), (3.9) and the condition (ii), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.10}$$

Observe that

$$\|x_n - S_n y_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \left\|\gamma f(y_n) - \bar{A}S_n y_n\right\|,$$

which implies by (3.7) and the condition (i) that

$$\lim_{n\to\infty} \|x_n - S_n y_n\| = 0. \tag{3.11}$$

Since

$$\begin{aligned}\|x_n - S_n x_n\| &\leq \|x_n - S_n y_n\| + \|S_n y_n - S_n x_n\| \\ &= \|x_n - S_n y_n\| + \|y_n - x_n\| + \lambda_n \|(I-T)y_n - (I-T)x_n\|,\end{aligned}$$

by (3.10), (3.11) and the condition (iii), we obtain

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{3.12}$$

Step 4. Show that $\limsup_{n\to\infty} \langle \gamma f(x^*) - \bar{A}x^*, x_n - x^* \rangle \leq 0$, where $x^* = P_\Omega(I - \bar{A} + \gamma f)x^*$. To see this, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - \bar{A}x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle \gamma f(x^*) - \bar{A}x^*, x_{n_k} - x^* \rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$. Define $\Theta_n : H \to H$ by

$$\Theta_n x := \rho_1 S_n x + \rho_2 T_{r_n} x + \rho_3 F_{s_n} x,$$

where $\rho_1, \rho_2, \rho_3 \in (0,1)$ and $\rho_1 + \rho_2 + \rho_3 = 1$. Let $K_n : H \to C$ be given by

$$K_n x := \frac{\rho_2}{1-\rho_1} T_{r_n} x + \frac{\rho_3}{1-\rho_1} F_{s_n} x.$$

Since T_{r_n}, F_{s_n} are nonexpansive, by Lemma 2.6, we get

$$Fix(K_n) = Fix(T_{r_n}) \cap Fix(F_{s_n}). \tag{3.13}$$

From Remark 2.11, 2.12, 2.17, 2.22 and (3.13), we obtain

$$\begin{aligned}Fix(\Theta_n) &= Fix(S_n) \cap Fix(K_n) \\ &= Fix(S_n) \cap Fix(T_{r_n}) \cap Fix(F_{s_n}) \\ &= Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(C, G_i) \\ &= \Omega.\end{aligned}$$

Since

$$\|\Theta_n x_n - x_n\| \leq \rho_1 \|S_n x_n - x_n\| + \rho_2 \|T_{r_n} x_n - x_n\| + \rho_3 \|F_{s_n} x_n - x_n\|,$$

by (3.8), (3.9) and (3.12), we have

$$\lim_{n\to\infty} \|\Theta_n x_n - x_n\| = 0. \tag{3.14}$$

Assume that $\omega \notin \Omega$. Then, we get $\omega \neq \Theta_{n_k}\omega$. By the Opial's condition, we obtain

$$\begin{aligned}&\liminf_{k\to\infty} \|x_{n_k} - \omega\| \\ &< \liminf_{k\to\infty} \|x_{n_k} - \Theta_{n_k}\omega\| \\ &\leq \liminf_{k\to\infty} \Big(\|x_{n_k} - \Theta_{n_k} x_{n_k}\| + \|\Theta_{n_k} x_{n_k} - \Theta_{n_k}\omega\| \Big) \\ &\leq \liminf_{k\to\infty} \Big(\|x_{n_k} - \Theta_{n_k} x_{n_k}\| + \|x_{n_k} - \omega\| + \rho_1 \lambda_{n_k} \|(I-T)x_{n_k} - (I-T)\omega\| \\ &= \liminf_{k\to\infty} \|x_{n_k} - \omega\|.\end{aligned}$$

This is a contradiction. Therefore, we have $\omega \in \Omega$. Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$, by Lemma 2.2, we can conclude that

$$\begin{aligned}\limsup_{n\to\infty} \left\langle \gamma f(x^*) - \bar{A}x^*, x_n - x^* \right\rangle &= \lim_{k\to\infty} \left\langle \gamma f(x^*) - \bar{A}x^*, x_{n_k} - x^* \right\rangle \\ &= \left\langle \gamma f(x^*) - \bar{A}x^*, \omega - x^* \right\rangle \\ &\leq 0. \end{aligned} \tag{3.15}$$

Step 5. Finally, claim that the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(I - \bar{A} + \gamma f)x^*$. By Lemma 2.5, Lemma 2.7, Remark 2.11(ii) and (3.2), we have

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\leq \left\| \left(I - \alpha_n \bar{A}\right)(S_n y_n - x^*) \right\|^2 + 2\alpha_n \left\langle \gamma f(y_n) - \bar{A}x^*, x_{n+1} - x^* \right\rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|y_n - x^*\|^2 + 2\alpha_n \gamma \|f(y_n) - f(x^*)\| \|x_{n+1} - x^*\| \\ &\quad + 2\alpha_n \left\langle \gamma f(x^*) - \bar{A}x^*, x_{n+1} - x^* \right\rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - x^*\|^2 + \alpha_n \gamma \alpha \left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right) \\ &\quad + 2\alpha_n \left\langle \gamma f(x^*) - \bar{A}x^*, x_{n+1} - x^* \right\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\leq \left(1 - \frac{2\alpha_n (\bar{\gamma} - \gamma\alpha)}{1 - \alpha_n \gamma \alpha}\right) \|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n (\bar{\gamma} - \gamma\alpha)}{1 - \alpha_n \gamma\alpha} \left(\frac{\alpha_n \bar{\gamma}^2}{2(\bar{\gamma} - \gamma\alpha)} \|x_n - x^*\|^2 + \frac{1}{\bar{\gamma} - \gamma\alpha} \left\langle \gamma f(x^*) - \bar{A}x^*, x_{n+1} - x^* \right\rangle \right). \end{aligned}$$

From (3.15), the condition (i) and Lemma 2.4, we can conclude that $\{x_n\}$ converges strongly to $x^* = P_\Omega(I - \bar{A} + \gamma f)x^*$. By (3.8), (3.9) and (3.10), we have $\{u_n\}$, $\{v_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_\Omega(I - \bar{A} + \gamma f)x^*$. This completes the proof. ■

Remark 3.2. By putting $\delta_n, \mu_n = 0, \beta_n, \lambda_n = 1$ and $A_k = A, \forall k = 1, 2, \ldots, \bar{N}$ in Theorem 3.1 and using Remark 2.9, then the iterative algorithm (3.1) becomes

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \forall n \in \mathbb{N},$$

which is a modification of a general iterative method in the results of Marino and Xu [5] and is also a general form of viscosity approximation method defined by Moudafi [1]. By assuming the initial condition $x_1 \in H$ and the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{Fix(T)}(I - A + \gamma f)x^*$, where T is a quasi-nonexpansive mapping on H.

Example 3.3. Let $l^2 = \left\{x = \{x_i\}_{i=1}^{\infty} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\right\}$ with norm defined by $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|\right)^{\frac{1}{2}}$. Let the mappings $A, f, T : l^2 \to l^2$ be defined by

$$\begin{aligned} Ax &= \frac{x}{2}, \\ fx &= \frac{x}{4}, \\ Tx &= \frac{6x+1}{7}, \text{ for all } x = \{x_i\}_{i=1}^{\infty} \in l^2. \end{aligned}$$

Moreover, let $\gamma = \frac{1}{5}$, $\alpha_n = \frac{1}{5n}$, for every $n \in \mathbb{N}$. Suppose that $\{x_n\} \subseteq l^2$ be generated by $x_1 = \{x_i^1\}_{i=1}^{\infty} \in l^2$ and

$$x_{n+1} = \frac{1}{25n}\left(\frac{x_n}{4}\right) + \left(I - \frac{1}{5n}A\right)\frac{6x_n+1}{7}, \forall n \in \mathbb{N}.$$

All parameters and mappings satisfy every conditions in Remark 3.2 and $Fix(T) = \{\mathbf{1}\}$, where $\mathbf{1} = \{1\}_{i=1}^{\infty} \in l^2$. Therefore, the sequence $\{x_n\}$ converges strongly to $\mathbf{1}$.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $k = 1, 2, \ldots, \bar{N}$, define $\bar{A} : H \to H$ by $\bar{A}x = \sum_{k=1}^{\bar{N}} c_k A_k x$, where A_k is a strongly positive linear bounded operator on H with coefficient $\gamma_k > 0$, $\bar{\gamma} = \min_{k=1,2,\ldots,\bar{N}} \gamma_k$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Define $S_n : H \to H$ by $S_n x := \lambda_n T x + (1 - \lambda_n) x$, where $T : H \to H$ is a nonspreading mapping. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let $G : C \to H$ be an α-inverse strongly monotone mapping. Suppose that $\Omega := Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap VI(C, G) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ \langle y - v_n, G v_n\rangle + \frac{1}{s_n}\langle y - v_n, v_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = \beta_n x_n + \delta_n u_n + \mu_n v_n, \\ x_{n+1} = \alpha_n \gamma f(y_n) + \left(I - \alpha_n \bar{A}\right) S_n y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.16}$$

where f is an α-contractive mapping on H, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\mu_n\}$, $\{r_n\}$, $\{s_n\}$, $\{\lambda_n\} \subset [0, 1]$, $\beta_n + \delta_n + \mu_n = 1$, $0 < a_i < 1$, for all $i = 1, 2, \ldots, N$, $0 < c_k < 1$, for all $k = 1, 2, \ldots, \bar{N}$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n, \delta_n, \mu_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(iv) $0 < \epsilon \leq r_n \leq \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(v) $0 < \theta \leq s_n \leq \varphi < \infty$, *for some* $\theta, \varphi > 0$;

(vi) $\sum_{i=1}^{N} a_i = 1$ *and* $\sum_{k=1}^{\bar{N}} c_k = 1$;

(vii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$, $\sum_{n=1}^{\infty}|\mu_{n+1}-\mu_n|<\infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$, $\sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* $x^*=P_\Omega(I-\bar{A}+\gamma f)x^*$.

Proof. Put $G_i=G$, for every $i=1,2,\ldots,N$. Then, by Theorem 3.1, we have the desired conclusion. ∎

Corollary 3.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *For* $k=1,2,\ldots,\bar{N}$, *define* $\bar{A}:H\to H$ *by* $\bar{A}x=\sum_{k=1}^{\bar{N}}c_kA_kx$, *where* A_k *is a strongly positive linear bounded operator on* H *with coefficient* $\gamma_k>0$, $\bar{\gamma}=\min_{k=1,2,\ldots,\bar{N}}\gamma_k$ *and* $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. *Define* $S_n:H\to H$ *by* $S_nx:=\lambda_nTx+(1-\lambda_n)x$, *where* $T:H\to H$ *is a nonspreading mapping. For every* $i=1,2,\ldots,N$, *let* $B_i:C\to H$ *be a* β_i*-inverse strongly monotone mapping with* $\bar{\beta}=\min_{i=1,2,\ldots,N}\{\beta_i\}$ *and let* $G_i:C\to H$ *be an* α_i*-inverse strongly monotone mapping with* $\bar{\alpha}=\min_{i=1,2,\ldots,N}\{\alpha_i\}$. *Suppose that* $\Omega:=Fix(T)\cap\bigcap_{i=1}^{N}VI(C,B_i)\cap\bigcap_{i=1}^{N}VI(C,G_i)\neq\emptyset$. *Let* $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *be sequences generated by* $x_1\in H$ *and*

$$\begin{cases}\left\langle y-u_n,\sum_{i=1}^{N}a_iB_iu_n\right\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall y\in C,\\ \left\langle y-v_n,\sum_{i=1}^{N}b_iG_iv_n\right\rangle+\frac{1}{s_n}\langle y-v_n,v_n-x_n\rangle\geq 0,\forall y\in C,\\ y_n=\beta_nx_n+\delta_nu_n+\mu_nv_n,\\ x_{n+1}=\alpha_n\gamma f(y_n)+\left(I-\alpha_n\bar{A}\right)S_ny_n,\forall n\in\mathbb{N},\end{cases}\tag{3.17}$$

where f *is an* α*-contractive mapping on* H, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\mu_n\}$, $\{r_n\}$, $\{s_n\}$, $\{\lambda_n\}\subset[0,1]$, $\beta_n+\delta_n+\mu_n=1$, $0<a_i<1$, $0<b_i<1$, *for all* $i=1,2,\ldots,N$, $0<c_k<1$, *for all* $k=1,2,\ldots,\bar{N}$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $0<\tau\leq\beta_n,\delta_n,\mu_n\leq\upsilon<1$, *for some* $\tau,\upsilon>0$;

(iii) $\sum_{n=1}^{\infty}\lambda_n<\infty$;

(iv) $0<\epsilon\leq r_n\leq\eta<\infty$, *for some* $\epsilon,\eta>0$;

(v) $0<\theta\leq s_n\leq\varphi<\infty$, *for some* $\theta,\varphi>0$;

(vi) $\sum_{i=1}^{N}a_i=\sum_{i=1}^{N}b_i=1$ *and* $\sum_{k=1}^{\bar{N}}c_k=1$;

(vii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$, $\sum_{n=1}^{\infty}|\mu_{n+1}-\mu_n|<\infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$, $\sum_{n=1}^{\infty}|s_{n+1}-s_n|<\infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* $x^*=P_\Omega(I-\bar{A}+\gamma f)x^*$.

Proof. Put $F_i=\langle y-u_n,B_iu_n\rangle$, for every $i=1,2,\ldots,N$. This follows that $\sum_{i=1}^N a_iB_i$ is $\bar{\beta}$-inverse strongly monotone and F_i satisfies the condition (A1)-(A4). Hence, by Remark 2.22 and Theorem 3.1, the conclusion can be obtained. ■

4. Numerical Examples

Example 4.1. Let $\mathbb{R}^2$ be the two dimensional space of real numbers with an inner product $\langle\cdot,\cdot\rangle:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ defined by $\langle x,y\rangle=x\cdot y=x_1y_1+x_2y_2$ and a usual norm $\|\cdot\|:\mathbb{R}^2\to\mathbb{R}$ given by $\|x\|=\sqrt{x_1^2+x_2^2}$, for all $x=(x_1,x_2)\,,y=(y_1,y_2)\in\mathbb{R}^2$. For $k=1,2,\ldots,\bar{N}$, let $c_k=\frac{9}{10^k}+\frac{1}{\bar{N}10^N}$ and let the mapping $A:\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$$A_kx=\left(\frac{kx_1}{2},\frac{kx_2}{2}\right),\ \text{for every } x=(x_1,x_2)\in\mathbb{R}^2.$$

Let the mappings $f:\mathbb{R}^2\to\mathbb{R}^2$, $T:\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$$fx=\left(1+\frac{x_1}{4},1+\frac{x_2}{4}\right),$$

$$Tx=\begin{cases}\left(-\frac{x_1}{2},-\frac{2x_2}{3}\right), & \text{if } x\in[0,\infty)\times[0,\infty),\\ (x_1,x_2)\,, & \text{if } x\in(-\infty,0)\times(-\infty,0),\end{cases}$$

where $x=(x_1,x_2)\in\mathbb{R}^2$. For every $i=1,2,\ldots,N$ and $I=[-100,0]$, let $F_i:I^2\times I^2\to\mathbb{R}$, $G_i:I^2\to\mathbb{R}^2$ be defined by

$$F_i(x,y)=i(y-x)\cdot(y+3x+\mathbf{4}),$$

$$G_i(x)=\left(\frac{x_1+1}{2i},\frac{x_2+1}{2i}\right),\ \text{for all } x=(x_1,x_2)\,,y=(y_1,y_2)\in I^2,$$

where $\mathbf{4}=(4,4)$. Let $\gamma=\frac{1}{5}$, $\alpha_n=\frac{1}{5n}$, $\beta_n=\frac{n}{2n+3}$, $\delta_n=\frac{2n+5}{6n+9}$, $\mu_n=\frac{n+4}{6n+9}$, $r_n=\frac{3n}{7n+3}$, $s_n=\frac{n}{8n+7}$ and $\lambda_n=\frac{1}{n^2}$ for every $n\in\mathbb{N}$.

Put $a_i=\frac{4}{5^i}+\frac{1}{N5^N}$, $b_i=\frac{7}{8^i}+\frac{1}{N8^N}$, for every $i=1,2,\ldots,N$. Then, by Theorem 3.1, the sequences $u_n=\left(u_n^1,u_n^2\right)$, $v_n=\left(v_n^1,v_n^2\right)$, $y_n=\left(y_n^1,y_n^2\right)$ and $x_n=\left(x_n^1,x_n^2\right)$ converge strongly to $\{\mathbf{-1}\}$, where $\mathbf{-1}=(-1,-1)$.

Solution. It is clear that the sequences $\{\alpha_n\},\{\beta_n\},\{\delta_n\},\{\mu_n\},\{r_n\}$, $\{s_n\}$ and $\{\lambda_n\}$ satisfy all the conditions of Theorem 3.1. It is easy to show that T is a nonspreading mapping and

$$Fix(T)=\begin{cases}\{(0,0)\}\,, & \text{if } x\in[0,\infty)\times[0,\infty),\\ \{x\}\,, & \text{if } x\in(-\infty,0)\times(-\infty,0),\end{cases}$$

It is obvious that $\sum_{i=1}^{N} a_i F_i$ satisfies all conditions in Theorem 3.1 and $EP(\sum_{i=1}^{N} a_i F_i) = \bigcap_{i=1}^{N} EP(F_i) = \{\mathbf{-1}\}$, where $\mathbf{-1} = (-1,-1)$. By the definition of G_i, we obtain $\bigcap_{i=1}^{N} VI(C, G_i) = \{\mathbf{-1}\}$. Then we have

$$Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(C, G_i) = \{\mathbf{-1}\}.$$

Put $\rho = \sum_{i=1}^{N} \left(\frac{2}{3^i} + \frac{1}{N3^N}\right) i$. By using the same method as in Example 5.2 of [22], we obtain

$$u_n = \left(u_n^1, u_n^2\right) = \left(\frac{x_n^1 - 4\rho r_n}{1 + 4\rho r_n}, \frac{x_n^2 - 4\rho r_n}{1 + 4\rho r_n}\right). \tag{4.1}$$

Take $\tau = \sum_{i=1}^{N} \left(\frac{6}{7^i} + \frac{1}{N7^N}\right) \frac{1}{i}$. Consider

$$\begin{aligned}
0 &= \left\langle y - v_n, \sum_{i=1}^{N} b_i G_i v_n \right\rangle + \frac{1}{s_n} \langle y - v_n, v_n - x_n \rangle \\
&= \left(y_1 - v_n^1, y_2 - v_n^2\right) \cdot \tau \left(\frac{v_n^1 + 1}{2}, \frac{v_n^2 + 1}{2}\right) \\
&\quad + \frac{1}{s_n} \left(y_1 - v_n^1, y_2 - v_n^2\right) \cdot \left(v_n^1 - x_n^1, v_n^2 - x_n^2\right) \\
&= \left(\frac{\tau}{2}\left(y_1 - v_n^1\right)\left(v_n^1 + 1\right) + \frac{\tau}{2}\left(y_2 - v_n^2\right)\left(v_n^2 + 1\right)\right) \\
&\quad + \frac{1}{s_n}\left(\left(y_1 - v_n^1\right)\left(v_n^1 - x_n^1\right) + \left(y_2 - v_n^2\right)\left(v_n^2 - x_n^2\right)\right) \\
&= \left(\frac{\tau}{2}\left(y_1 - v_n^1\right)\left(v_n^1 + 1\right) + \frac{1}{s_n}\left(y_1 - v_n^1\right)\left(v_n^1 - x_n^1\right)\right) \\
&\quad + \left(\frac{\tau}{2}\left(y_2 - v_n^2\right)\left(v_n^2 + 1\right) + \frac{1}{s_n}\left(y_2 - v_n^2\right)\left(v_n^2 - x_n^2\right)\right)
\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}
0 &= \left(\tau s_n \left(y_1 - v_n^1\right)\left(v_n^1 + 1\right) + 2\left(y_1 - v_n^1\right)\left(v_n^1 - x_n^1\right)\right) \\
&\quad + \left(\tau s_n \left(y_2 - v_n^2\right)\left(v_n^2 + 1\right) + 2\left(y_2 - v_n^2\right)\left(v_n^2 - x_n^2\right)\right) \\
&= \left(y_1 - v_n^1\right)\left(\tau s_n \left(v_n^1 + 1\right) + 2\left(v_n^1 - x_n^1\right)\right) \\
&\quad + \left(y_2 - v_n^2\right)\left(\tau s_n \left(v_n^2 + 1\right) + 2\left(v_n^2 - x_n^2\right)\right) \\
&= \left(y_1 - v_n^1\right)\left(v_n^1\left(\tau s_n + 2\right) + \tau s_n - 2x_n^1\right) \\
&\quad + \left(y_2 - v_n^2\right)\left(v_n^2\left(\tau s_n + 2\right) + \tau s_n - 2x_n^2\right) \\
&= Q_1(y_1) + Q_2(y_2),
\end{aligned} \tag{4.2}$$

where $Q_1(y_1) = \left(y_1 - v_n^1\right)\left(v_n^1\left(\tau s_n + 2\right) + \tau s_n - 2x_n^1\right)$ and $Q_2(y_2) = \left(y_2 - v_n^2\right)$ $\left(v_n^2\left(\tau s_n + 2\right) + \tau s_n - 2x_n^2\right)$. From (4.2), if $Q_1(y_1) = 0, \forall y_1 \in \mathbb{R}$, thus we have

$$v_n^1 = \frac{2x_n^1 - \tau s_n}{\tau s_n + 2}. \tag{4.3}$$

Similarly, from (4.2), if $Q_2(y_2) = 0, \forall y_2 \in \mathbb{R}$, hence we get

$$v_n^2 = \frac{2x_n^2 - \tau s_n}{\tau s_n + 2}. \tag{4.4}$$

From (4.1), (4.3) and (4.4), the iterative scheme (3.1) becomes

$$\begin{cases} y_n = \frac{n}{2n+3} x_n + \frac{2n+5}{6n+9} u_n + \frac{n+4}{6n+9} v_n, \\ x_{n+1} = \frac{1}{25n} f(x_n) + \left(I - \frac{1}{5n}\bar{A}\right) S_n y_n, \forall n \in \mathbb{N}, \end{cases} \tag{4.5}$$

where $x_n = (x_n^1, x_n^2)$, $y_n = (y_n^1, y_n^2)$, $u_n = (u_n^1, u_n^2) = \left(\frac{x_n^1 - 4\rho r_n}{1+4\rho r_n}, \frac{x_n^2 - 4\rho r_n}{1+4\rho r_n}\right)$ and $v_n = (v_n^1, v_n^2) = \left(\frac{2x_n^1 - \tau s_n}{\tau s_n + 2}, \frac{2x_n^2 - \tau s_n}{\tau s_n + 2}\right)$.

For the iterative scheme (3.1), Table 1 shows the numerical results of sequences $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$.

Example 4.2. In this example, we consider the same mappings and parameters as in Example 4.1 except the following mapping G. Let $I = [-100, 0]$ and let the mapping $G : I^2 \to \mathbb{R}^2$ be defined by

$$G(x) = \left(\frac{x_1 + 1}{2}, \frac{x_2 + 1}{2}\right), \text{ for all } x = (x_1, x_2), y = (y_1, y_2) \in I^2.$$

It is clear that

$$Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap VI(C, G) = \{\textbf{-1}\}.$$

By the definition of G, using the same method as (4.2), we obtain

$$v_n^1 = \frac{2x_n^1 - s_n}{s_n + 2}, \tag{4.6}$$

and

$$v_n^2 = \frac{2x_n^2 - s_n}{s_n + 2}. \tag{4.7}$$

From (4.1), (4.6) and (4.7), we rewrite the iterative method (3.16) as follows:

$$\begin{cases} y_n = \frac{n}{2n+3} x_n + \frac{2n+5}{6n+9} u_n + \frac{n+4}{6n+9} v_n, \\ x_{n+1} = \frac{1}{25n} f(x_n) + \left(I - \frac{1}{5n}\bar{A}\right) S_n y_n, \forall n \in \mathbb{N}, \end{cases} \tag{4.8}$$

where $x_n = (x_n^1, x_n^2)$, $y_n = (y_n^1, y_n^2)$, $u_n = (u_n^1, u_n^2) = \left(\frac{x_n^1 - 4\rho r_n}{1+4\rho r_n}, \frac{x_n^2 - 4\rho r_n}{1+4\rho r_n}\right)$ and $v_n = (v_n^1, v_n^2) = \left(\frac{2x_n^1 - s_n}{s_n + 2}, \frac{2x_n^2 - s_n}{s_n + 2}\right)$.

For the iterative method (3.16), Table 2 shows the numerical results of sequences $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$.

n	$u_n = (u_n^1, u_n^2)$	$v_n = (v_n^1, v_n^2)$	$y_n = (y_n^1, y_n^2)$	$x_n = (x_n^1, x_n^2)$
1	(-1.400000,-1.800000)	(-1.963303,-2.926606)	(-1.707768,-2.415535)	(-2.000000,-3.000000)
2	(-1.179076,-1.409193)	(-1.471657,-2.077745)	(-1.352961,-1.806523)	(-1.495093,-2.131298)
3	(-1.092021,-1.241806)	(-1.250698,-1.658764)	(-1.190673,-1.501036)	(-1.264562,-1.695194)
4	(-1.046742,-1.148906)	(-1.129613,-1.412912)	(-1.099729,-1.317710)	(-1.137210,-1.437112)
5	(-1.020826,-1.092276)	(-1.058382,-1.258676)	(-1.045302,-1.200724)	(-1.061931,-1.274401)
⋮	⋮	⋮	⋮	⋮
15	(-0.985777,-0.988862)	(-0.958918,-0.967829)	(-0.967153,-0.974278)	(-0.956145,-0.965658)
⋮	⋮	⋮	⋮	⋮
26	(-0.991587,-0.991712)	(-0.975542,-0.975905)	(-0.980286,-0.980579)	(-0.973852,-0.974240)
27	(-0.991939,-0.992033)	(-0.976557,-0.976830)	(-0.981096,-0.981316)	(-0.974935,-0.975227)
28	(-0.992265,-0.992336)	(-0.977499,-0.977704)	(-0.981848,-0.982014)	(-0.975940,-0.976160)
29	(-0.992568,-0.992621)	(-0.978374,-0.978528)	(-0.982547,-0.982672)	(-0.976874,-0.977039)
30	(-0.992849,-0.992889)	(-0.979187,-0.979303)	(-0.983198,-0.983291)	(-0.977742,-0.977866)

TABLE 1. The values of $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ with $x_1 = (-2, -3)$ and $n = \bar{N} = N = 30$ of the iterative method (3.1).

n	$u_n = (u_n^1, u_n^2)$	$v_n = (v_n^1, v_n^2)$	$y_n = (y_n^1, y_n^2)$	$x_n = (x_n^1, x_n^2)$
1	(-1.400000,-1.800000)	(-1.967742,-2.935484)	(-1.709247,-2.418495)	(-2.000000,-3.000000)
2	(-1.179557,-1.410155)	(-1.475739,-2.086710)	(-1.354714,-1.810257)	(-1.496423,-2.133958)
3	(-1.092600,-1.243040)	(-1.253938,-1.666489)	(-1.192304,-1.504722)	(-1.266226,-1.698739)
4	(-1.047279,-1.150120)	(-1.132015,-1.419178)	(-1.101096,-1.321002)	(-1.138785,-1.440674)
5	(-1.021274,-1.093356)	(-1.060068,-1.263590)	(-1.046376,-1.203508)	(-1.063263,-1.277610)
⋮	⋮	⋮	⋮	⋮
15	(-0.985727,-0.988903)	(-0.958446,-0.967692)	(-0.966976,-0.974323)	(-0.955992,-0.965784)
⋮	⋮	⋮	⋮	⋮
26	(-0.991532,-0.991663)	(-0.975180,-0.975565)	(-0.980119,-0.980428)	(-0.973679,-0.974088)
27	(-0.991886,-0.991984)	(-0.976210,-0.976499)	(-0.980936,-0.981168)	(-0.974769,-0.975077)
28	(-0.992214,-0.992288)	(-0.977165,-0.977383)	(-0.981695,-0.981869)	(-0.975781,-0.976012)
29	(-0.992519,-0.992575)	(-0.978053,-0.978217)	(-0.982400,-0.982531)	(-0.976721,-0.976895)
30	(-0.992802,-0.992844)	(-0.978878,-0.979002)	(-0.983056,-0.983155)	(-0.977595,-0.977726)

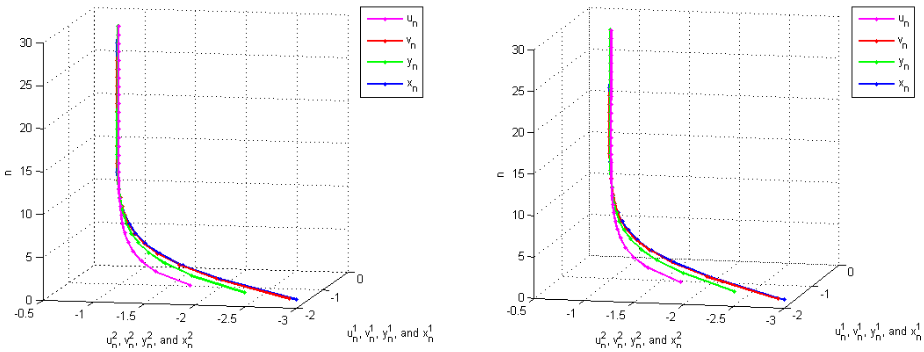
TABLE 2. The values of $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ with $x_1 = (-2, -3)$ and $n = \bar{N} = N = 30$ of the iterative method (3.16).

Remark 4.3. From the above numerical results, we can conclude that

(i) Table 1 shows that the sequences $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ converge to **-1**, where $\{\mathbf{-1}\} = \{(-1,-1)\} = Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(C, G_i)$ and the convergence of $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ can be guaranteed by Theorem 3.1.

(ii) Table 2 shows that the sequences $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ converge to **-1**, where $\{\mathbf{-1}\} = \{(-1,-1)\} = Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap VI(C, G)$ and the convergence of $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ can be guaranteed by Corollary 3.4.

(iii) From Table 1 and Table 2, we have that the iterative method for the combination of variational inequality problem (3.1) converges faster than the iterative method for the classical variational inequality problem (3.16).



(A) The iterative method (3.1) (B) The iterative method (3.16)

FIGURE 1. The convergence of $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ in three-dimensional space with initial value $x_1 = (-2, -3)$ and $n = \bar{N} = N = 30$ of the iterative method (3.1) and (3.16).

5. Conclusion

In this research, we introduce and analyze a general iterative method for approximating a common solution of a combination of equilibrium problems, variational inequality problems and a fixed point of a nonspreading mapping. It can be seen as an improvement and modification of some existing algorithms for solving a variational inequality problem, an equilibrium problem and a fixed point problem of a nonspreading mapping or some related mappings. Some previous research works, for example, [1–5], can be considered as special cases of Theorem 3.1. Moreover, some numerical examples for our main theorem are provided. In these examples, we find that the iterative method for the combination of variational inequality problem and the combination of equilibrium problem (3.1) converges faster than the iterative method for the classical ones (3.16).

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