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Suzuki-Wardowski Type Theorems for Generalized α-η-GF-Contractions

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Abstract The notion of a generalized α-η-GF-contraction is introduced and Suzuki-Wardowski type theorems for generalized α-η-GF-contractions are proved. These results generalize and improve the main results of [D. Wardowski, N.V. Dung, Fixed points of F-weak contractions on complete metric spaces, Demonstratio Math. 47 (1) (2014) 146–155, N.V. Dung, V.T.L. Hang, A fixed point theorem for generalized F-contractions on complete metric spaces, Vietnam J. Math. 43 (4) (2015) 743–753, N. Hussain, P. Salimi, Suzuki-Wardowski type fixed point theorems for α-GF-contractions, Taiwanese J. Math. 18 (6) (2014) 879–1895]. An application to the Urysohn integral equation is obtained. Examples are also given to illustrate obtained results.

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1. Introduction and Preliminaries

In recent times the metric fixed point theory was attracted by many authors [1]. There were many types of contractions to ensure the existence and uniqueness of the fixed point of maps on metric spaces [2–5]. In 2012 Wardowski [6] introduced the notion of an F-contraction. By using F-contractions Wardowski proved a fixed point theorem which generalizes Banach contraction principle in a different way than in the known results from the literature. Later, Piri and Kumam [7] proved Wardowski type fixed point theorems in metric space by using a modified generalized F-contraction maps.

After that Wardowski and Dung [8] introduced the notion of an F-weak contraction and proved a fixed point theorem for F-weak contractions. Note that F-weak contractions were considered in [9] and [10] under the name *F-generalized contractions* [10, Definition 2.3.(3)]. Recently Dung and Hang [11] generalized an F-weak contraction to a generalized F-contraction and proved a fixed point theorem for generalized F-contraction.

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In 2014 Hussain and Salimi [12] introduced the notion of an α-η-GF-contraction and stated fixed point theorems for α-η-GF-contractions. In 2006, Padcharoen et al [13] introduced the notion of α-type F-contraction in the setting of modular metric spaces and also Janwised and Kitkuanestablish [14] improved the notion of $\alpha-\varphi-$Geraghty contraction type mappings some common fixed point theorems for the mappings satisfying this conditions. On the other hand, Piri and Kumam [15] establish some new fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces. In 2007, Khammahawong, et al. [16] show the existence of best proximity points for multi-valued Suzuki αF-proximal contraction mappings in complete metric spaces.

In this paper we introduce the notion of a generalized α-η-GF-contraction to generalize both an α-η-GF-contraction and a generalized F-contraction. Then we prove Suzuki-Wardowski type theorems for generalized α-η-GF-contractions. These results generalize and improve the main results of [8], [11] and [12]. An application to the Urysohn integral equation is obtained. Also, examples are given to illustrate the obtained results.

The following notions will be needed throughout the paper.

Definition 1.1 ([6], Definition 2.1)**.** Let $\mathcal{F}$ be the family of all functions $F:(0,\infty)\longrightarrow\mathbb{R}$ such that

(F1) F is strictly increasing, that is, for all $\alpha,\beta\in(0,\infty)$ if $\alpha<\beta$, then $F(\alpha)<F(\beta)$.
(F2) For each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to\infty}\alpha_n=0 \text{ if and only if } \lim_{n\to\infty}F(\alpha_n)=-\infty.$$

(F3) There exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\big(\alpha^k F(\alpha)\big)=0$.

Let (X,d) be a metric space and $T:X\longrightarrow X$ be a map. T is called an *F-contraction* on (X,d) if there exist $F\in\mathcal{F}$ and $\tau>0$ such that for all $x,y\in X$,

$$d(Tx,Ty)>0\Rightarrow \tau+F\big(d(Tx,Ty)\big)\le F\big(d(x,y)\big). \tag{1.1}$$

Definition 1.2 ([8], Definition 2.1)**.** Let (X,d) be a metric space and $T:X\longrightarrow X$ be a map. T is called an *F-weak contraction* on (X,d) if there exist $F\in\mathcal{F}$ and $\tau>0$ such that for all $x,y\in X$,

$$\begin{aligned} & d(Tx,Ty)>0 \\ \Rightarrow\quad & \tau+F\big(d(Tx,Ty)\big)\le F\Big(\max\Big\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\Big\}\Big).\end{aligned} \tag{1.2}$$

Definition 1.3 ([11], Definition 7)**.** Let (X,d) be a metric space and $T:X\longrightarrow X$ be a map. T is called a *generalized F-contraction* on (X,d) if there exist $F\in\mathcal{F}$ and $\tau>0$ such that for all $x,y\in X$,

$$\begin{aligned} & d(Tx,Ty)>0 \\ \Rightarrow\quad & \tau+F\big(d(Tx,Ty)\big)\le F\Big(\max\Big\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2},d(T^2x,Tx),d(T^2x,y),d(T^2x,Ty)\Big\}\Big).\end{aligned} \tag{1.3}$$

Definition 1.4 ([12], Definition 2.1)**.** Let (X,d) be a metric space, $T:X\longrightarrow X$ be a map and $\alpha,\eta:X\times X\longrightarrow[0,\infty)$ be two functions. T is called an *α-η-GF-contraction* if there exist $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that for $x, y \in X$ with $\alpha(x,y) \geq \eta(x,Tx)$ and $d(Tx,Ty) > 0$,

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big). \tag{1.4}$$

Definition 1.5. Let $T : X \longrightarrow X$ be a map and $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be functions.

(1) [17, Definition 2.2] T is called an α-*admissible* map if for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1.$$

(2) [18, Definition 2.1] T is called an α-*admissible map with respect to* η if for all $x, y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow \alpha(Tx,Ty) \geq \eta(Tx,Ty).$$

(3) [19, Definition 7] T is called an α-η-*continuous map* if for all $x \in X$,

$$\lim_{n\to\infty} x_n = x, \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} Tx_n = Tx.$$

(4) [12, page 2] T is called *orbitally continuous at* $p \in X$ if $\lim_{n\to\infty} T^n x = p$ implies that $\lim_{n\to\infty} TT^n x = Tp$. T is called *orbitally continuous* on X if T is orbitally continuous for all $p \in X$.

(5) [20, page 2] T is called to have *property* (P) if for every $n \in \mathbb{N}$, $\mathrm{Fix}(T^n) = \mathrm{Fix}(T)$, where $\mathrm{Fix}(T)$ is the set of all fixed points of T.

Remark 1.6 ([19], Remark 6)**.** Let $T : X \longrightarrow X$ be a map, X be an orbitally T-complete metric space and $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be defined by

$$\alpha(x,y) = \begin{cases} 3 & \text{if } x, y \in O(w) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x,y) = 1 \text{ for all } x, y \in X$$

where

$$O(w) = \{w, Tw, T^2 w, \dots, T^n w, \dots\}$$

is called an *orbit* of a point $w \in X$. If $T : X \longrightarrow X$ is an orbitally continuous map on (X,d), then T is α-η-continuous on (X,d).

Let $\mathcal{G}$ be the family of all functions $G : [0,\infty)^4 \longrightarrow [0,\infty)$ such that for all $t_1, t_2, t_3, t_4 \in [0,\infty)$ with $t_1 t_2 t_3 t_4 = 0$, there exists $\tau > 0$ such that $G(t_1,t_2,t_3,t_4) = \tau$. Some examples of elements of $\mathcal{G}$ were given in [12].

Example 1.7 ([12])**.** (1) If $G(t_1,t_2,t_3,t_4) = L\min\{t_1,t_2,t_3,t_4\} + \tau$, for all $t_1,t_2,t_3,t_4 \in [0,\infty)$, where $L \in [0,\infty)$ and $\tau > 0$, then $G \in \mathcal{G}$.

(2) If $G(t_1,t_2,t_3,t_4) = \tau e^{L\min\{t_1,t_2,t_3,t_4\}}$ for all $t_1,t_2,t_3,t_4 \in [0,\infty)$, where $L \in [0,\infty)$ and $\tau > 0$, then $G \in \mathcal{G}$.

(3) If $G(t_1,t_2,t_3,t_4) = L\ln(\min\{t_1,t_2,t_3,t_4\})$ for all $t_1,t_2,t_3,t_4 \in [0,\infty)$, where $L \in [0,\infty)$ and $\tau > 0$, then $G \in \mathcal{G}$.

The paper is presented in three sections. Suzuki-Wardowski type fixed point theorems for generalized α-η-GF-contractions are presented in Section 2. Examples that illustrate the results and applications are presented in Section 3.

2. Main Results

First we introduce the notion of a generalized α-η-GF-contraction on a metric space.

Definition 2.1. Let (X,d) be a metric space, $T : X \longrightarrow X$ be a map, and $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be two functions. T is called a *generalized α-η-GF-contraction* if there exist $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that for all $x, y \in X$ with $\alpha(x,y) \geq \eta(x,Tx)$ and $d(Tx,Ty) > 0$,

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(M(x,y)\big) \tag{2.1}$$

where

$$\begin{aligned} M(x,y) \ = \ & \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\}. \end{aligned}$$

Remark 2.2. By definitions following implications are easy to see.

$$\begin{array}{ccccc} F\text{-contraction} & \Longrightarrow & F\text{-weak contraction} \Longrightarrow & & \text{generalized } F\text{-contraction} \\ \Downarrow & & & & \Downarrow \\ \alpha\text{-}\eta\text{-}GF\text{-contraction} & & \Longrightarrow & & \begin{array}{c}\text{generalized} \\ \alpha\text{-}\eta\text{-}GF\text{-contraction}\end{array} \end{array}$$

The following examples show that the inversions of above implications do not hold.

Example 2.3. (1) There exists an F-weak contraction which is not an F-contraction.
(2) There exists a generalized F-contraction which is not an F-weak contraction.
(3) There exists an α-η-GF-contraction which is not an F-contraction.

Proof. (1). See [8, Example 2.3].
(2). See [11, Example 9].
(3). See [12, Example 2.4]. ■

Example 2.4. There exist a complete metric space (X,d), a map $T : X \longrightarrow X$, two functions α, $\eta : X \times X \longrightarrow [0,\infty)$ and $G \in \mathcal{G}$, $F \in \mathcal{F}$ such that T is a generalized α-η-GF-contraction which is neither an α-η-GF-contraction nor a generalized F-contraction.

Proof. Let $X = [0,\infty)$ with the usual metric, $T : X \longrightarrow X$, $\alpha, \eta : X \times X \longrightarrow [0,\infty)$, $G : [0,\infty)^4 \longrightarrow [0,\infty)$ and $F : [0,\infty) \longrightarrow \mathbb{R}$ be defined as follows for some $\tau > 0$

$$\begin{aligned} Tx &= \begin{cases} \frac{1}{2}e^{-\tau}x^2 & \text{if } x \in [0,1] \\ 3x & \text{if } x \in (1,\infty) \end{cases} \\ \alpha(x,y) &= \begin{cases} \frac{1}{2} & \text{if } x \in [0,1] \cup [e^{\tau},\infty), y \in [0,1] \\ \frac{1}{9} & \text{otherwise} \end{cases} \\ \eta(x,y) &= \frac{1}{4} \\ G(t_1,t_2,t_3,t_4) &= \tau \\ F(r) &= \ln r. \end{aligned}$$

First we prove that T is a generalized α-η-GF-contraction. Let $\alpha(x,y) \geq \eta(x,Tx)$ and $d(Tx,Ty) > 0$. Then $x \in [0,1] \cup [e^{\tau},\infty), y \in [0,1]$. We consider the following two cases.

Case 1. $x, y \in [0,1]$. Then T is an α-η-GF-contraction by the proof of [12, Example 2.4]. Therefore, T is a generalized α-η-GF-contraction.

Case 2. $x \in [e^\tau, \infty), y \in [0,1]$. Then

$$\begin{aligned} d(Tx,Ty) = (3x - \frac{1}{2}e^{-\tau}y^2) = e^{-\tau}(e^\tau 3x - \frac{1}{2}y^2) \\ \leq (e^\tau 9x - y) \leq (9x^2 - y) = d(T^2x, y) \leq M(x,y). \end{aligned}$$

Therefore,

$$\tau + F\big(d(Tx,Ty)\big) = \tau + \ln(e^{-\tau}(e^\tau 3x - \frac{1}{2}y^2)) = \ln(e^\tau 3x - \frac{1}{2}y^2) \leq \ln(M(x,y)),$$

that is, T is a generalized α-η-GF-contraction.

Now we show that T is not an α-η-GF-contraction. We see that

$$d(Te^\tau, T0) = 3e^\tau > 0 \text{ and } \alpha(e^\tau, 0) = \frac{1}{2} > \frac{1}{4} = \eta(e^\tau, Te^\tau).$$

However,

$$\tau + F\big(d(Te^\tau, T0)\big) = \tau + \ln(3e^\tau) > \ln(e^\tau) = F\big(d(e^\tau, 0)\big).$$

So, T is not an α-η-GF-contraction.

Finally we show that T is not a generalized F-contraction. We see that $d(T0, T2) = 6 > 0$ and

$$\begin{aligned} F\big(d(T0,T2)\big) &= \ln 6 \\ F\big(M(0,2)\big) &= F\Big(\Big\{d(0,2), d(0,T0), d(2,T2), \frac{d(0,T2)+d(2,T0)}{2}, \\ &\quad \frac{d(T^20,0)+d(T^20,T2)}{2}, d(T^20,T0), d(T^20,2), d(T^20,T2)\Big\}\Big) \\ &= \ln 6. \end{aligned}$$

Hence $\tau + F\big(d(T0,T2)\big) > F\big(M(0,2)\big)$, that is, T is not a generalized F-contraction. ■

The first result of the paper is as follows.

Theorem 2.5. *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be two functions, and $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that*

(1) T is α-admissible with respect to η.
(2) T is a generalized α-η-GF-contraction.
(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$.
(4) (a) Either T is α-η-continuous or
(b) F and G are continuous, and if $\lim_{n\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Then

(1) T has a fixed point x^*.
(2) If $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} T^n x_0 = x^*$.
(3) If $\alpha(x,y) \geq \eta(x,x)$ for all $x, y \in \mathrm{Fix}(T)$, then the fixed point of T is unique.

Proof. (1). Define the sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Then

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1).$$

Since T is an α-admissible map with respect to η,

$$\alpha(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq \eta(Tx_0, T^2x_0) = \eta(x_1, x_2).$$

Continuing this process, for all $n \in \mathbb{N}$ we get

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}). \tag{2.2}$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. So we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then $d(Tx_{n-1}, Tx_n) > 0$ for all $n \geq 1$. Since T is a generalized α-η-GF-contraction,

$$G\big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\big) + F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\big(M(x_{n-1}, x_n)\big).$$

That is

$$G\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\big) + F\big(d(x_n, x_{n+1})\big) \leq F\big(M(x_{n-1}, x_n)\big) \tag{2.3}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) \qquad (2.4) \\ = \max\Big\{&d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2}, \\ &\frac{d(T^2x_{n-1}, x_{n-1}) + d(T^2x_{n-1}, Tx_n)}{2}, d(T^2x_{n-1}, Tx_{n-1}), d(T^2x_{n-1}, x_n), \\ &d(T^2x_{n-1}, Tx_n)\Big\} \\ = \max\Big\{&d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}, \frac{d(x_{n+1}, x_{n-1})}{2}, \\ &d(x_{n+1}, x_n), d(x_{n+1}, x_n), d(x_{n+1}, x_{n+1})\Big\} \\ = \max\Big\{&d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\Big\} \\ = \max\big\{&d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}. \end{aligned}$$

Since $G \in \mathcal{G}$ and $d(x_{n-1}, x_n) \cdot d(x_n, x_{n+1}) \cdot d(x_{n-1}, x_{n+1}) \cdot 0 = 0$, there exists $\tau > 0$ such that

$$G\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\big) = \tau.$$

By using (2.3) we get

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(M(x_{n-1}, x_n)\big) - \tau. \tag{2.5}$$

If there exists some $n \geq 1$ such that

$$\max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\} = d(x_n, x_{n+1})$$

then (2.5) becomes

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_n, x_{n+1})\big) - \tau.$$

It is a contradiction. Hence, for all $n \geq 1$,

$$\max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\} = d(x_{n-1}, x_n).$$

Then (2.5) becomes

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau.$$

It implies that

$$F\big(d(x_n,x_{n+1})\big) \le F\big(d(x_{n-1},x_n)\big)-\tau \le F\big(d(x_{n-2},x_{n-1})\big)-2\tau \le \ldots \le F\big(d(x_0,x_1)\big)-n\tau. \tag{2.6}$$

Taking the limit as $n \to \infty$ in (2.6) we get

$$\lim_{n\to\infty} F\big(d(x_n,x_{n+1})\big) = -\infty.$$

By (F2) we get

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.7}$$

By (F3), there exists $0 < k < 1$ such that

$$\lim_{n\to\infty} [d(x_n,x_{n+1})]^k F\big(d(x_n,x_{n+1})\big) = 0. \tag{2.8}$$

Using (2.6) we have

$$[d(x_n,x_{n+1})]^k [F\big(d(x_n,x_{n+1})\big) - F\big(d(x_0,x_1)\big)] \le -n\tau[d(x_n,x_{n+0})]^k \le 0. \tag{2.9}$$

By taking the limit as $n \to \infty$ in (2.9), and using (2.7) and (2.8),

$$\lim_{n\to\infty} n[d(x_n,x_{n+1})]^k = 0. \tag{2.10}$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$, $n[d(x_n,x_{n+1})]^k \le 1$. That is $d(x_n,x_{n+1}) \le \dfrac{1}{n^{\frac{1}{k}}}$ for all $n > n_1$. Then, for all $m > n > n_1$,

$$d(x_n,x_m) \le \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{k}}}. \tag{2.11}$$

Since $0 < k < 1$, the series $\sum\limits_{i=1}^{\infty} \dfrac{1}{i^{\frac{1}{k}}}$ converges. Therefore, by taking the limit as $n, m \to \infty$ in (2.11) we get $\lim\limits_{n,m\to\infty} d(x_n,x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.12}$$

We will show that x^* is a fixed point of T by the following two cases.

Case 1. *The condition* (4a) *holds.* From (2.2), (2.12) and since T is an α-η-continuous, $\lim\limits_{n\to\infty} Tx_n = Tx^*$. Then

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*.$$

This proves that x^* is a fixed point of T.

Case 2. *The condition* (4b) *holds.* We consider the following two subcases.

Subcase 2.1. For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_n+1} = Tx^*$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then

$$x^* = \lim_{n\to\infty} x_{i_n+1} = \lim_{n\to\infty} Tx^* = Tx^*.$$

This proves that x^* is a fixed point of T.

Subcase 2.2. There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$ for all $n \geq n_0$. That is $d(Tx_n, Tx^*) > 0$ for all $n \geq n_0$. From (2.2) and (2.12) we have $\alpha(x_n, x^*) \geq \eta(x_n, x_{n+1})$. For all $n \geq n_0$, by using (2.1),

$$\begin{aligned} & G\big(d(x_n,Tx_n), d(x^*,Tx^*), d(x_n,Tx^*), d(x^*,Tx_n)\big) + F\big(d(Tx_n,Tx^*)\big) \\ \leq\ & F\big(M(x_n,x^*)\big) \\ =\ & F\Big(\max\Big\{d(x_n,x^*), d(x_n,Tx_n), d(x^*,Tx^*), \frac{d(x_n,Tx^*)+d(x^*,Tx_n)}{2}, \\ & \frac{d(T^2x_n,x_n)+d(T^2x_n,Tx^*)}{2}, d(T^2x_n,Tx_n), d(T^2x_n,x^*), d(T^2x_n,Tx^*)\Big\}\Big) \\ =\ & F\Big(\max\Big\{d(x_n,x^*), d(x_n,x_{n+1}), d(x^*,Tx^*), \frac{d(x_n,Tx^*)+d(x^*,x_{n+1})}{2}, \\ & \frac{d(x_{n+2},x_n)+d(x_{n+2},Tx^*)}{2}, d(x_{n+2},x_{n+1}), d(x_{n+2},x^*), d(x_{n+2},Tx^*)\Big\}\Big). \end{aligned}$$

It implies that

$$\begin{aligned} & G\big(d(x_n,Tx_n), d(x^*,Tx^*), d(x_n,Tx^*), d(x^*,Tx_n)\big) + F\big(d(Tx_n,Tx^*)\big) \\ \leq\ & F\Big(\max\Big\{d(x_n,x^*), d(x_n,x_{n+1}), d(x^*,Tx^*), \frac{d(x_n,Tx^*)+d(x^*,x_{n+1})}{2}, \\ & \frac{d(x_{n+2},x_n)+d(x_{n+2},Tx^*)}{2}, d(x_{n+2},x_{n+1}), d(x_{n+2},x^*), d(x_{n+2},Tx^*)\Big\}\Big). \end{aligned} \tag{2.13}$$

If $d(x^*, Tx^*) > 0$, then by (2.12) and the fact that F and G are continuous and by taking the limit as $n \to \infty$ in (2.13) we obtain

$$\tau + F\big(d(x^*,Tx^*)\big) \leq F\big(d(x^*,Tx^*)\big).$$

It is a contradiction. Therefore, $d(x^*, Tx^*) = 0$, that is, x^* is a fixed point of T.

By two above cases, T has a fixed point x^*.

(2). It is proved by (2.12).

(3). Let x, y be two fixed points of T. Suppose to the contrary that $x \neq y$. Then $Tx \neq Ty$. Note that $\alpha(x,y) \geq \eta(x,x) = \eta(x,Tx)$. Following (2.1) we have

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(M(x,y)\big),$$

where

$$\begin{aligned} M(x,y) &= \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ &\qquad \frac{d(T^2x,x)+T^2x,Ty)}{2}, d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\} \\ &= d(x,y). \end{aligned}$$

It implies that

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(x,y)\big) \leq F\big(d(x,y)\big).$$

Since $G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) = G\big(0,0,d(x,y),d(y,x)\big) = \tau$ for some $\tau > 0$,

$$\tau + F\big(d(x,y)\big) \leq F\big(d(x,y)\big).$$

It is a contradiction. Therefore $x = y$, that is, T has a unique fixed point. ∎

Form Theorem 2.5 and Remark 2.2 we have the following corollaries.

Corollary 2.6 ([12], Theorem 2.1)**.** *Let (X, d) be a complete metric space, $T : X \longrightarrow X$ be a map, $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be two functions, and $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that*

(1) T is α-admissible with respect to η.
(2) T is an α-η-GF-contraction.
(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$.
(4) T is an α-η-continuous.

Then

(1) T has a fixed point x^*.
(2) If $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in \mathrm{Fix}(T)$, then the fixed point of T is unique.

Proof. It is a direct consequence of Theorem 2.5 since every α-η-GF-contraction is a generalized α-η-GF-contraction. ■

Corollary 2.7 ([11], Theorem 10)**.** *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a generalized F-contraction. If T or F is continuous, Then*

(1) T has a unique fixed point $x^* \in X$.
(2) If $T^{n+1}x \neq T^n x$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} T^n x = x^*$.

Proof. Since T is a generalized F-contraction, T is a generalized α-η-GF-contraction where $G(t_1, t_2, t_3, t_4) = \tau$ for all $t_1, t_2, t_3, t_4 \in [0, \infty)$ and $\alpha(x, y) = \eta(x, y) = 0$ for all $x, y \in X$. Other assumptions of Theorem 2.5 are easy to check. Then the conclusions hold by Theorem 2.5. ■

By using [8, Remark 2.8] we get the following corollaries.

Corollary 2.8. *Let (X, d) be a complete metric space, $T : X \longrightarrow X$ be a map, $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be two functions such that*

(1) T is α-admissible with respect to η.
(2) For all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$ and $d(Tx, Ty) > 0$,

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(T^2x, Tx) + fd(T^2x, y) + gd(T^2x, Ty)$$

where $a, b, c, e, f, g \geq 0$ with $a + b + c + e + f + g < 1$.
(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$.
(4) Either T is α-η-continuous, or if $\lim_{n\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Then

(1) T has a fixed point x^*.
(2) If $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} T^n x_0 = x^*$.
(3) If $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in \mathrm{Fix}(T)$, then the fixed point of T is unique.

Proof. For all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$ and $d(Tx, Ty) > 0$,

$$\begin{aligned} d(Tx, Ty) &\leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(T^2x, Tx) + fd(T^2x, y) + gd(T^2x, Ty) \\ &\leq k \max\left\{d(x, y), d(x, Tx), d(y, Ty), d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\right\} \end{aligned}$$

where $k = a + b + c + e + f + g \in [0, 1)$. As same as the argument in [8, Remark 2.8], T satisfies the generalized α-η-GF-contraction with $F(\alpha) = \ln \alpha$ for all $\alpha > 0$ and $G(t_1, t_2, t_3, t_4) = \ln \frac{1}{k}$ for all $t_1, t_2, t_3, t_4 \in [0, \infty)$. Then the conclusions hold by Theorem 2.5. ■

Corollary 2.9. *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, $\alpha,\eta : X\times X \longrightarrow [0,\infty)$ be two functions such that*

(1) T is α-admissible with respect to η.

(2) For all $x,y\in X$ with $\alpha(x,y)\geq \eta(x,Tx)$ and $d(Tx,Ty)>0$, and for some $k\in[0,1)$,

$$d(Tx,Ty)\leq k\max\Big\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2},\\ \frac{d(T^2x,x)+d(T^2x,Ty)}{2},d(T^2x,Tx),d(T^2x,y),d(T^2x,Ty)\Big\}.$$

(3) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$.

(4) Either T is α-η-continuous, or if $\lim\limits_{n\to\infty} x_n = x$ such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}$, then $\alpha(x_n,x)\geq\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}$.

Then

(1) T has a fixed point x^*.

(2) If $T^{n+1}x_0\neq T^nx_0$ for all $n\in\mathbb{N}$, then $\lim\limits_{n\to\infty}T^nx_0=x^*$.

(3) If $\alpha(x,y)\geq\eta(x,x)$ for all $x,y\in\mathrm{Fix}(T)$, then the fixed point of T is unique.

Proof. As same as the argument in [8, Remark 2.8], T satisfies the generalized α-η-GF-contraction with $F(\alpha)=\ln\alpha$ for all $\alpha>0$ and $G(t_1,t_2,t_3,t_4)=\ln\dfrac{1}{k}$ all $t_1,t_2,t_3,t_4\in[0,\infty)$. Then the conclusions hold by Theorem 2.5. ■

Corollary 2.10. *Let (X,d) be a complete metric space, $T : X\longrightarrow X$ be a map, and $F\in\mathcal{F}$, $G\in\mathcal{G}$ such that*

(1) T is continuous.

(2) For all $x,y\in X$ with $d(x,Tx)\leq d(x,y)$ and $d(Tx,Ty)>0$,

$$G\big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big)+F\big(d(Tx,Ty)\big)\leq F\big(M(x,y)\big) \tag{2.14}$$

where and $M(x,y)$ is defined as in Definition 2.1.

Then

(1) T has a unique fixed point x^*.

(2) For all $x\in X$, if $T^{n+1}x\neq T^nx$ for all $n\in\mathbb{N}$, then $\lim\limits_{n\to\infty}T^nx=x^*$.

Proof. Let $\alpha,\eta : X\times X\longrightarrow[0,\infty)$ be defined by $\alpha(x,y)=\eta(x,y)=d(x,y)$ for all $x,y\in X$,. Then $\eta(x,y)\geq\alpha(x,y)$ for all $x,y\in X$.

Since T is continuous, T is α-η-continuous. Since $d(x,y)\geq d(x,Tx)$ and $d(Tx,Ty)>0$, $\alpha(x,y)\geq\eta(x,Tx)$ and $d(Tx,Ty)>0$. By (2.14), T is a generalized α-η-GF-contraction. Then the conclusions hold by Theorem 2.5. ■

Corollary 2.11 ([12], Theorem 3.1)**.** *Let (X,d) be a complete metric space, $T : X\longrightarrow X$ be a map, and $F\in\mathcal{F}$, $G\in\mathcal{G}$ such that*

(1) T is continuous.

(2) For all $x,y\in X$ with $d(x,Tx)\leq d(x,y)$ and $d(Tx,Ty)>0$,

$$G\big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big)+F\big(d(Tx,Ty)\big)\leq F\big(d(x,y)\big).$$

Then T has a unique fixed point.

Proof. The conclusions hold by replacing $M(x,y)$ in Corollary 2.10 by $d(x,Tx)$. ∎

Corollary 2.12. *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, and $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that*

(1) For all $x,y \in O(w)$ with $d(Tx,Ty) > 0$,

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \le F\big(M(x,y)\big)$$

where $M(x,y)$ is defined as in Definition 2.1 and $O(w)$ is an orbit of a point $w \in X$.

(2) T is orbitally continuous.

Then

(1) T has a fixed point x^*.

(2) If $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} T^n x_0 = x^*$.

(3) If $\mathrm{Fix}(T) \subseteq O(w)$, then the fixed point of T is unique.

Proof. Let $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be defined by

$$\alpha(x,y) = \begin{cases} 3 & \text{if } x,y \in O(w) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x,y) = 1.$$

From Remark 1.6, T is an α-η-continuous map. Let $\alpha(x,y) \ge \eta(x,y)$, then $x,y \in O(w)$. It implies that $Tx, Ty \in O(w)$, and then $\alpha(Tx,Ty) \ge \eta(Tx,Ty)$. Therefore, T is an α-admissible map with respect to η. Since $w, Tw \in O(w)$, we have $\eta(w,Tw) \le \alpha(w,Tw)$.

Let $\alpha(x,y) \ge \eta(x,Tx)$ and $d(Tx,Ty) > 0$. Then $x,y \in O(w)$ and $d(Tx,Ty) > 0$. By using the hypothesis (1) we have

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \le F\big(M(x,y)\big),$$

that is, T is a generalized α-η-GF-contraction.

By using Theorem 2.5, the conclusions (1) and (2) hold. If $\mathrm{Fix}(T) \subseteq O(w)$, then $\eta(x,x) \le \alpha(x,y)$ for all $x,y \in \mathrm{Fix}(T)$. By using Theorem 2.5 again, the conclusion (3) also holds. ∎

The second result of the paper is as follows.

Theorem 2.13. *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be two functions, and $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that*

(1) T is α-admissible with respect to η.

(2) T is a generalized α-η-GF-contraction.

(3) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$.

(4) F is continuous.

(5) If $\lim\limits_{n\to\infty} x_n = x$ such that $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$, then either $\eta(Tx_n,T^2x_n) \le \alpha(Tx_n,x)$ or $\eta(T^2x_n,T^3x_n) \le \alpha(T^2x_n,x)$ for all $n \in \mathbb{N}$.

Then

(1) T has a fixed point x^*.

(2) If $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} T^n x_0 = x^*$.

(3) If $\alpha(x,y) \ge \eta(x,x)$ for all $x,y \in \mathrm{Fix}(T)$, then the fixed point of T is unique.

Proof. (1). Define the sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. As in the proof of Theorem 2.5, if there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. So we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. We also have

$$\lim_{n\to\infty} x_n = x^* \tag{2.15}$$

and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

From the hypothesis (5) we have either $\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x^*)$ or $\eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x^*)$ for all $n \in \mathbb{N}$, that is, $\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, x^*)$ or $\eta(x_{n+2}, x_{n+3}) \leq \alpha(x_{n+2}, x^*)$ for all $n \in \mathbb{N}$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that for all $k \in \mathbb{N}$,

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x^*).$$

So, from (2.1), we get

$$G\big(d(x_{n_k},Tx_{n_k}), d(x^*,Tx^*), d(x_{n_k},Tx^*), d(x^*,Tx_{n_k})\big)+F\big(d(Tx_{n_k},Tx^*)\big) \leq F\big(M(x_{n_k},x^*)\big) \tag{2.16}$$

where

$$\begin{aligned}
&M(x_{n_k},x^*)\\
&= \max\Big\{d(x_{n_k},x^*), d(x_{n_k},Tx_{n_k}), d(x^*,Tx^*), \frac{d(x_{n_k},Tx^*)+d(x^*,Tx_{n_k})}{2}, d(Tx_{n_k},Tx^*),\\
&\qquad \frac{d(T^2x_{n_k},x_{n_k})+d(T^2x_{n_k},Tx^*)}{2}, d(T^2x_{n_k},Tx_{n_k}), d(T^2x_{n_k},x^*), d(T^2x_{n_k},Tx^*)\Big\}\\
&= \max\Big\{d(x_{n_k},x^*), d(x_{n_k},x_{n_k+1}), d(x^*,Tx^*), \frac{d(x_{n_k},Tx^*)+d(x^*,x_{n_k+1})}{2}, d(x_{n_k+1},Tx^*),\\
&\qquad \frac{d(x_{n_k+2},x_{n_k})+d(x_{n_k+2},Tx^*)}{2}, d(x_{n_k+2},x_{n_k+1}), d(x_{n_k+2},x^*), d(x_{n_k+2},Tx^*)\Big\}.
\end{aligned}$$

Suppose to the contrary that $x^* \neq Tx^*$. Since G and F are continuous, taking the limit as $n \to \infty$ in (2.16) and using (2.15), we have

$$\tau + F\big(d(x^*,Tx^*)\big) \leq F\big(\max\{d(x^*,Tx^*), d(x^*,x^*)\}\big) = F\big(d(x^*,Tx^*)\big).$$

It is a contradiction. Therefore, $d(x^*,Tx^*) = 0$, that is, x^* is a fixed point of T.

(2) and (3). As in the proofs of Theorem 2.5.(2) and Theorem 2.5.(3). ∎

Corollary 2.14 ([12], Theorem 2.2). *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be two functions, and $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that*

(1) T is α-admissible with respect to η.
(2) T is an α-η-GF-contraction.
(3) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$.
(4) F is continuous.
(5) If $\lim_{n\to\infty} x_n = x$ such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$, then either $\eta(Tx_n,T^2x_n) \leq \alpha(Tx_n,x)$ or $\eta(T^2x_n,T^3x_n) \leq \alpha(T^2x_n,x)$ for all $n \in \mathbb{N}$.

Then

(1) T has a fixed point x^*.
(2) If $\alpha(x,y) \geq \eta(x,x)$ for all $x,y \in \mathrm{Fix}(T)$, then the fixed point of T is unique.

Proof. It is a direct consequence of Theorem 2.13 since every α-η-GF-contraction is a generalized α-η-GF-contraction. ∎

Corollary 2.15. *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, and $F \in \mathcal{F}$ such that*

$$\frac{1}{2(1+\tau)}d(x,Tx) \le d(x,y) \quad \Rightarrow \quad \tau + F\big(d(Tx,Ty)\big) \le F\big(M(x,y)\big) \tag{2.17}$$

for some $\tau > 0$ and all $x, y \in X$, where $M(x,y)$ is defined as in Definition 2.1. If F is continuous, then

(1) T has a unique fixed point x^*.

(2) For all $x \in X$, if $T^{n+1}x \neq T^n x$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} T^n x = x^*$.

Proof. Let $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be defined by

$$\alpha(x,y) = d(x,y), \eta(x,y) = \frac{1}{2(1+\tau)}d(x,y) \text{ for all } x,y \in X.$$

Since $\frac{1}{2(1+\tau)}d(x,y) \le d(x,y)$, $\eta(x,y) \le \alpha(x,y)$. Also, $\alpha(x,Tx) \ge \eta(x,Tx)$ for all $x \in X$. Since $\alpha(Tx,Ty) \ge \eta(Tx,Ty)$, T is α-admissible with respect to η. Moreover, by (2.17), T is an α-η-GF-contraction where $G(t_1,t_2,t_3,t_4) = \tau$ for all $t_1,t_2,t_3,t_4 \in [0,\infty)$.

Let $\lim_{n\to\infty} x_n = x$. If $d(Tx_n, T^2x_n) = 0$ for some n, then Tx_n is a fixed point of T. Now we suppose that $Tx_n \neq T^2x_n$ for all $n \in \mathbb{N}$. Then $\frac{1}{2(1+\tau)}d(Tx_n,T^2x_n) \le d(Tx_n,T^2x_n)$ for all $n \in \mathbb{N}$. From (2.17) we have

$$F\big(d(T^2x_n,T^3x_n)\big) \le \tau + F\big(d(T^2x_n,T^3x_n)\big) \le F\big(d(Tx_n,T^2x_n)\big).$$

Combining with(F1) to give

$$d(T^2x_n,T^3x_n) \le d(Tx_n,T^2x_n). \tag{2.18}$$

Suppose to the contrary that there exists $n_0 \in \mathbb{N}$ such that

$$\eta(Tx_{n_0},T^2x_{n_0}) > \alpha(Tx_{n_0},x) \quad \text{and} \quad \eta(T^2x_{n_0},T^3x_{n_0}) > \alpha(T^2x_{n_0},x).$$

It implies

$$\frac{1}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0}) > d(Tx_{n_0},x) \quad \text{and} \quad \frac{1}{2(1+\tau)}d(T^2x_{n_0},T^3x_{n_0}) > d(T^2x_{n_0},x).$$

By using (2.18) we have

$$\begin{aligned} d(Tx_{n_0},T^2x_{n_0}) &\le d(Tx_{n_0},x) + d(T^2x_{n_0},x) \\ &< \frac{1}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0}) + \frac{1}{2(1+\tau)}d(T^2x_{n_0},T^3x_{n_0}) \\ &\le \frac{1}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0}) + \frac{1}{2(1+\tau)}d(Tx_{n_0},T^2x_{n_0}) \\ &= \frac{1}{1+\tau}d(Tx_{n_0},T^2x_{n_0}) \\ &\le d(Tx_{n_0},T^2x_{n_0}). \end{aligned}$$

It is a contradiction. Then, for all $n \in \mathbb{N}$ we have either $\eta(Tx_n,T^2x_n) \le \alpha(Tx_n,x)$ or $\eta(T^2x_n,T^3x_n) \le \alpha(T^2x_n,x)$.

By the above and applying Theorem 2.13, the conclusions hold. ∎

Theorem 2.16. *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, $\alpha : X \times X \longrightarrow [0,\infty)$ be a function and $F \in \mathcal{F}$ such that*

(1) T is α-continuous and α-admissible.

(2) There exists $\tau > 0$ such that

$$\tau + F\big(d(Tx,T^2x)\big) \leq F\big(M(x,Tx)\big) \tag{2.19}$$

for all $x \in X$ with $d(Tx,T^2x) > 0$, where $M(x,Tx)$ is defined as in Definition 2.1.

(3) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$.

Then T has the property (P).

Proof. For every $n \in \mathbb{N}$, put $x_n = T^n x_0 = Tx_n$. Since T is α-admissible, we have

$$\alpha(x_1,x_2) = \alpha(x_1,Tx_1) \geq 1.$$

As same as the argument in the proof of Theorem 2.5, $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x^*$. Since T is α-continuous and $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$,

$$x^* = \lim_{n\to\infty} x_{n+1} \lim_{n\to\infty} Tx_n = Tx^*$$

that is, x^* is a fixed point of T. Hence T has a fixed point and $F(T^n) = F(T)$ for $n = 1$.

For each $n > 1$, it is clear that $\mathrm{Fix}(T) \subset \mathrm{Fix}(T^n)$. Suppose to the contrary that there exists $w \in \mathrm{Fix}(T^n)$ but $w \notin \mathrm{Fix}(T)$. Since $w \notin \mathrm{Fix}(T)$, $d(w,Tw) > 0$. Then

$$d(T^n w,T^{n+1}w) = d(w,Tw) > 0.$$

It implies that $d(T^k w,T^{k+1}w) > 0$ for all $k = 1,\ldots,n$. Then by (2.19) we get

$$\begin{aligned} F\big(d(w,Tw)\big) &= F\big(d(T^n w,T^{n+1}w)\big) \\ &= F\big(d(TT^{n-1}w,T^2T^{n-1}w)\big) \\ &\leq F\big(d(T^{n-1}w,T^n w)\big) - \tau \\ &\leq F\big(d(T^{n-2}w,T^{n-1}w)\big) - 2\tau \\ &\leq \ldots \\ &\leq F\big(d(w,Tw)\big) - n\tau. \end{aligned} \tag{2.20}$$

It is a contradiction. Then $\mathrm{Fix}(T^n) \subset \mathrm{Fix}(T)$. It implies that $\mathrm{Fix}(T^n) = \mathrm{Fix}(T)$. Therefore, T has the property (P). ∎

Corollary 2.17 ([12], Theorem 2.3)**.** *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, $\alpha : X \times X \longrightarrow [0,\infty)$ be a function and $F \in \mathcal{F}$ such that*

(1) T is α-continuous and α-admissible.

(2) There exists $\tau > 0$ such that

$$\tau + F\big(d(Tx,T^2x)\big) \leq F\big(d(x,Tx)\big)$$

for all $x \in X$ with $d(Tx,T^2x) > 0$.

(3) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$.

Then T has the property (P).

Proof. The conclusion holds by replacing $M(x,Tx)$ in Theorem 2.16 by $d(x,Tx)$. ∎

Corollary 2.18. *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, and $F \in \mathcal{F}$ such that*

(1) For $x \in X$ with $d(Tx,T^2x) > 0$,

$$\tau + F\big(d(Tx,T^2x)\big) \le F\big(M(x,Tx)\big)$$

where $M(x,Tx)$ is defined as in Definition 2.1.

(2) T is orbitally continuous.

Then T has the property (P).

Proof. Let $\alpha,\eta : X \times X \longrightarrow [0,\infty)$ be defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in O(w) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x,y) = 1 \text{ for all } x,y \in X.$$

If $\alpha(x,y) \ge 1$, then $x,y \in O(w)$ and $Tx,Ty \in O(w)$. Therefore, $\alpha(Tx,Ty) \ge 1$ and so T is α-admissible. Since $w,Tw \in O(w)$, $\alpha(w,Tw) \ge 1$. By using Remark 1.6, T is an α-continuous map. For each $x \in X$ with $d(Tx,T^2x) > 0$, from(1) we have

$$\tau + F\big(d(Tx,T^2x)\big) \le F\big(M(x,Tx)\big).$$

Then, applying Theorem 2.16, T has the property (P). ∎

Remark 2.19. By choosing suitable functions α and η as in [12, Section 4], generalizations of [12, Theorems 4.2-4.6] are easily obtained.

3. Examples and Applications

In this section we give examples to illustrate results presented in Section 2 and give an application to the Urysohn integral equation.

The following example shows that Theorem 2.5 is a proper extension of Corollary 2.6 and Corollary 2.7.

Example 3.1. Let (X,d) be the space metric and T,G,F,α,η be defined as in Example 2.4. Then

(1) Corollary 2.6 and Corollary 2.7 are not applicable to T,G,F,α,η.

(2) Theorem 2.5 is applicable to T,G,F,α,η.

Proof. (1). By Example 2.4, T is neither a generalized F-contraction nor an α-η-GF-contraction. Then Corollary 2.6 and Corollary 2.7 are not applicable to T,G,F,α,η.

(2). Also, from Example 2.4, T is a generalized α-η-GF-contraction. We show that other assumptions of Theorem 2.5 are also satisfied.

Let $\alpha(x,y) \ge \eta(x,y)$. Then $x \in [0,1] \cup [e^\tau,\infty), y \in [0,1]$. Therefore, $Tx \in [0,\infty)$ and $Ty \in [0,1]$. It implies that $\alpha(Tx,Ty) \ge \eta(Tx,Ty)$. Hence, T is an α-admissible map with respect to η.

Let $\lim\limits_{n\to\infty} x_n = x$ and $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Then, $x_n \in [0,1]$ for all $n \in \mathbb{N}$. It implies that $x \in [0,1]$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} d(Tx_n,Tx) &= \lim_{n\to\infty} \frac{1}{2}e^{-\tau} \mid x_n^2 - x^2 \mid \\ &= \lim_{n\to\infty} \frac{1}{2}e^{-\tau} \mid x_n - x \mid\mid x_n + x \mid \\ &= 0. \end{aligned}$$

Hence, T is α-η-continuous. Note that $\alpha(0, T0) \geq \eta(0, T0)$.

By the above, all assumptions of Theorem 2.5 are fulfilled. So Theorem 2.5 is applicable to T, G, F, α, η. ∎

The following example shows that Theorem 2.13 is a proper extension of Corollary 2.14.

Example 3.2. Let $X = \mathbb{R}$ with the usual metric, $a > 1$, $T : X \longrightarrow X$ be defined by $Tx = ax$ for all $x \in \mathbb{R}$, and

$$\begin{aligned} \alpha(x,y) &= \begin{cases} x - y + 1 & \text{if } x \geq y \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ \eta(x,y) &= 1 \text{ for all } x, y \in \mathbb{R} \\ Ft &= \ln t \text{ for all } t > 0 \\ G(t_1, t_2, t_3, t_4) &= \ln a \text{ for all } t_1, t_2, t_3, t_4 \geq 0. \end{aligned}$$

Then

(1) T is not an α-η-GF-contraction, and thus Corollary 2.14 is not applicable to T, G, F, α, η.
(2) Theorem 2.13 is applicable to T, G, F, α, η.

Proof. (1). For $x = 1$ and $y = 0$,

$$d(Tx, Ty) = |a - 0| = a > d(x, y) = |1 - 0| = 1.$$

It implies that $F\big(d(T1, T0)\big) > F\big(d(1, 0)\big)$. Then (1.4) does not hold. So T is not an α-η-GF-contraction. Then Corollary 2.14 is not applicable to T, G, F, α, η.

(2). For all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$ and $d(Tx, Ty) > 0$ we have $x > y \geq 0$. Then

$$\begin{aligned} M(x,y) = \max\Big\{&d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, \\ &d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\} \\ = \max\Big\{&|x-y|, |x-ax|, |y-ay|, \frac{|x-ay|+|y-ax|}{2}, \frac{|a^2x-ax|+|a^2x-ay|}{2}, \\ &|a^2x-ax|, |a^2x-y|, |a^2x-ay|\Big\} \\ \geq{}& |a^2x - ay| \\ ={}& a^2x - ay \\ \geq{}& a^2(x-y) \\ ={}& ad(Tx,Ty). \end{aligned}$$

It implies that $\ln a + \ln d(Tx, Ty) \leq \ln M(x, y)$. Then (2.1) holds. So T is a generalized α-η-GF-contraction. We show that other assumptions of Theorem 2.13 are also satisfied.

Let $\alpha(x, y) \geq \eta(x, y)$. Then $x \geq y \geq 0$ and thus $Tx = ax \geq Ty = ay \geq 0$. It implies that $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. So T is α-admissible with respect to η.

Let $\lim\limits_{n\to\infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$. For all $n \in \mathbb{N}$, since $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, $x_n \geq x_{n+1}$. It implies that $x_n \geq x \geq 0$ for all $n \in \mathbb{N}$. So $ax_n \geq x \geq 0$ and then $\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n)$ for all $n \in \mathbb{N}$.

Note that $\alpha(0, T0) = 1 \geq \eta(0, T0)$. Then all assumptions of Theorem 2.13 are fulfilled. Therefore, Theorem 2.13 is applicable to T, G, F, α, η. ∎

Finally we apply the result on generalized α-η-GF-contractions to study the existence and uniqueness of solutions to the Urysohn integral equation [21].

Example 3.3. Let $C[0,1]$ be the set of continuous functions on $[0,1]$ with the metric $d(x,y)=\max_{t\in[0,1]}|x(t)-y(t)|$ for all $x\in C[0,1]$ and

(1) $K:[0,1]\times[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a function such that $K(t,s,x(s))=2x(t)+2s$ for all $s,t\in[0,1]$ and $x\in C[0,1]$.
(2) $\alpha,\eta : C[0,1]\times C[0,1]\longrightarrow\mathbb{R}$ be defined by

$$\alpha(x,y)=\begin{cases}1 & \text{if } x(s)\geq y(s)\geq 0 \text{ for all } s\in[0,1]\\ 0 & \text{otherwise}\end{cases}\quad \text{and } \eta(x,y)=1 \text{ for all } x,y\in C[0,1].$$

Then

(1) The integral $\displaystyle\int_0^1 K(t,s,x(s))ds$ exists.
(2) $Tx\in C[0,1]$ for all $x\in C[0,1]$, where $Tx(t)=\displaystyle\int_0^1 K(t,s,x(s))ds$ for all $x\in C[0,1]$ and all $t\in[0,1]$.
(3) There does not exist $k\in[0,1)$ such that for all $x,y\in C[0,1]$ with $\alpha(x,y)\geq\eta(x,Tx)$ and $d(Tx,Ty)>0$, and all $s,t\in[0,1]$,

$$\begin{aligned}|K(s,t,x(t))-&K(s,t,y(t))|\\ &\leq k\max\Big\{|x(s)-y(s)|,|x(s)-Tx(s)|,|y(s)-Ty(s)|,\\ &\qquad\frac{|x(s)-Ty(s)|+|y(s)-Tx(s)|}{2}\Big\}.\end{aligned}\tag{3.1}$$

(4) There exists $k\in[0,1)$ such that for all $x,y\in C[0,1]$ with $\alpha(x,y)\geq\eta(x,Tx)$ and $d(Tx,Ty)>0$, and all $s,t\in[0,1]$,

$$\begin{aligned}&|K(s,t,x(t))-K(s,t,y(t))|\\ &\leq k\max\Big\{|x(s)-y(s)|,|x(s)-Tx(s)|,|y(s)-Ty(s)|,\\ &\frac{|x(s)-Ty(s)|+|y(s)-Tx(s)|}{2},\frac{|T^2x(s)-x(s)|+|T^2x(s)-Ty(s)|}{2},\\ &|T^2x(s)-Tx(s)|,|T^2x(s)-y(s)|,|T^2x(s)-Ty(s)|\Big\}.\end{aligned}\tag{3.2}$$

(5) T is α-admissible with respect to η.
(6) There exists $x_0\in C[0,1]$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$.

Proof. (1). The integral $\displaystyle\int_0^1 K(t,s,x(s))ds$ exists since $K(t,s,x(s))=2x(t)+2s$ is continuous with respect to s on $[0,1]$.

(2). For every $t\in[0,1]$ and $x\in C[0,1]$,

$$Tx(t)=\int_0^1 K(t,s,x(s))ds=\int_0^1\big[2x(t)+2s\big]ds=2x(t)+1$$

which is continuous with respect to t on $[0,1]$. This proves that $Tx\in C[0,1]$ for all $x\in C[0,1]$.

(3). Let $x(s) = 1$ and $y(s) = 0$ for all $s \in [0,1]$. Then $\alpha(x,y) \geq \eta(x,Tx)$ and $d(Tx,Ty) = 2 > 0$. Suppose to the contrary that (3.1) holds. Then

$$\begin{aligned} |K(s,t,x(t)) - K(s,t,y(t))| &= \big|(2+2t) - (0+2t)\big| \\ &= 2 \\ &\leq k \max\Big\{|1-0|, |1-3|, |0-1|, \frac{|1-1|+|0-3|}{2}\Big\} \\ &= 2k. \end{aligned}$$

So $k \geq 1$. It is a contradiction.

(4). Let $\alpha(x,y) \geq \eta(x,Tx)$. Then $x(s) \geq y(s) \geq 0$ for all $s \in [0,1]$. For all $s,t \in [0,1]$,

$$|K(s,t,x(t)) - K(s,t,y(t))| = |2x(s) + 2t - \big(2y(s) + 2t\big)| = 2\big(x(s) - y(s)\big). \quad (3.3)$$

and

$$\begin{aligned} |T^2x(s) - y(s)| &= |2Tx(s) + 1 - y(s)| \\ &= \big|4x(s) + 3 - y(s)\big| \\ &\geq 4\big(x(s) - y(s)\big). \end{aligned} \quad (3.4)$$

From (3.3) and (3.4),

$$\begin{aligned} &|K(s,t,x(t)) - K(s,t,y(t))| \\ &\leq \frac{1}{2}\big|T^2x(s) - y(s)\big| \\ &\leq \frac{1}{2}\max\Big\{|x(s)-y(s)|, |x(s)-Tx(s)|, |y(s)-Ty(s)|, \\ &\quad \frac{|x(s)-Ty(s)|+|y(s)-Tx(s)|}{2}, \frac{|T^2x(s)-x(s)|+|T^2x(s)-Ty(s)|}{2}, \\ &\quad |T^2x(s)-Tx(s)|, |T^2x(s)-y(s)|, |T^2x(s)-Ty(s)|\Big\}. \end{aligned}$$

This proves that (3.2) holds if $k \in \big[\frac{1}{2}, 1\big)$.

(5). Since T is continuous, T is α-admissible with respect to η.

(6). Put $x_0(s) = 0$ for all $s \in [0,1]$. Then $x_0 \in C[0,1]$ and $x_0(s) \geq Tx_0(s) \geq 0$ for all $s \in [0,1]$. It implies that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. ■

The following result is a sufficient condition to guarantee that the Urysohn integral equation has a unique solution. Note that Example 3.3 guarantees for the existence of the function K in the following Theorem 3.4.

Theorem 3.4. *Let $C[0,1]$ be the set of continuous functions on $[0,1]$ with the metric $d(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|$ for all $x \in C[0,1]$, and $K : [0,1] \times [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\alpha, \eta : C[0,1] \times C[0,1] \longrightarrow \mathbb{R}$ be three functions satisfying the following.*

(1) The integral $\int_0^1 K(t,s,x(s))ds$ exists for all $x \in C[0,1]$ and all $s \in [0,1]$.

(2) $Tx \in C[0,1]$ for all $x \in C[0,1]$ where $Tx(t) = \int_0^1 K(t,s,x(s))ds$ for all $x \in C[0,1]$ and all $t \in [0,1]$.

(3) T is α-admissible with respect to η.

(4) There exists $k \in [0,1)$ *such that for all* $x, y \in C[0,1]$ *with* $\alpha(x,y) \geq \eta(x,Tx)$ *and* $d(Tx,Ty) > 0$, *and all* $s, t \in [0,1]$,

$$\begin{aligned} & |K(s,t,x(t)) - K(s,t,y(t))| \\ \leq\ & k \max\Big\{|x(s)-y(s)|, |x(s)-Tx(s)|, |y(s)-Ty(s)|, \\ & \frac{|x(s)-Ty(s)|+|y(s)-Tx(s)|}{2}, \frac{|T^2x(s)-x(s)|+|T^2x(s)-Ty(s)|}{2}, \\ & |T^2x(s)-Tx(s)|, |T^2x(s)-y(s)|, |T^2x(s)-Ty(s)|\Big\}. \end{aligned} \tag{3.5}$$

(5) There exists $x_0 \in C[0,1]$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$.

(6) Either T *is* α-η-*continuous, or if* $\lim_{n\to\infty} x_n = x$ *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$, *then* $\alpha(x_n, x) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$.

Then the Urysohn integral equation

$$x(t) = \int_0^1 K(t,s,x(s))ds \tag{3.6}$$

has a unique solution $x^* \in C[0,1]$.

Proof. By using (3.5), for all $x, y \in C[0,1]$ with $\alpha(x,y) \geq \eta(x,Tx)$ and $d(Tx,Ty) > 0$, and all $s \in [0,1]$,

$$\begin{aligned} |Tx(s) - Ty(s)| &= \Big|\int_0^1 K(s,t,x(t))dt - \int_0^1 K(s,t,y(t))dt\Big| \\ &\leq \int_0^1 \Big|K(s,t,x(t)) - K(s,t,y(t))\Big|dt \\ &\leq \int_0^1 k \max\Big\{|x(s)-y(s)|, |x(s)-Tx(s)|, |y(s)-Ty(s)|, \\ &\quad \frac{|x(s)-Ty(s)|+|y(s)-Tx(s)|}{2}, \frac{|T^2x(s)-x(s)|+|T^2x(s)-Ty(s)|}{2}, \\ &\quad |T^2x(s)-Tx(s)|, |T^2x(s)-y(s)|, |T^2x(s)-Ty(s)|\Big\}dt \\ &\leq k \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ &\quad \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\} \int_0^1 dt \\ &= k \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ &\quad \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\}. \end{aligned}$$

Therefore, for all $x, y \in C[0,1]$ with $\alpha(x,y) \geq \eta(x,Tx)$ and $d(Tx,Ty) > 0$,

$$\begin{aligned} d(Tx,Ty) \leq\ & k \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\}. \end{aligned}$$

Note that $C[0,1]$ is complete and $T : C[0,1] \longrightarrow C[0,1]$. So all assumptions of Corollary 2.9 are fulfilled. Then, by Corollary 2.9, the Urysohn integral equation (3.6) has a unique solution $x^* \in C[0,1]$. ■

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