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Fixed Point Theorems with Respect to a *c*-Distance in Cone Metric Spaces Endowed with a Graph

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Abstract In this paper we study the existence of the fixed points for mappings with respect to a c-distance in cone metric spaces endowed with a graph. Our results are significant, since we need to use neither the continuity of mapping nor the normality of cone and provide a more general statement of Rahimi and Soleimani Rad [H. Rahimi, G. Soleimani Rad, Note on "Common fixed point results for noncommuting mappings without continuity in cone metric spaces", Thai J. Math. 11 (2013) 589–599] and Kaewkhao et al. [A. Kaewkhao, W. Sintunavarat, P. Kumam, Common fixed point theorems of c-distance on cone metric spaces, J. Nonlinear Anal. Appl. 2012 (2021) Article ID jnaa-00137].

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1. Introduction and Preliminaries

Let E be a real Banach space. Then a subset P of E is called a cone if and only if

(i) P is closed, non-empty and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
(iii) if $x, -x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ ($intP$ is the interior of P). The cone P is named normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies that $\|x\| \leq K\|y\|$. If $intP \neq \emptyset$, then the cone P is called solid. Also, we shall make use of the following properties for all $u, v, w, c \in E$ when the cone P may be non-normal.

(p_1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.

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(p_2) If $\theta \preceq v \ll c$ for each $c \in intP$, then $v = \theta$.
(p_3) If $v \preceq \lambda v$ where $v \in P$ and $0 < \lambda < 1$, then $v = \theta$.
(p_4) Let $a_n \to \theta$ in E, $\theta \preceq a_n$ and $\theta \ll c$. Then there exists positive integer n_0 such that $a_n \ll c$ for each $n > n_0$.

Ordered normed spaces and cones have many applications in applied mathematics. Hence, fixed point theory in K-metric and K-normed spaces was developed in the mid-20th century ([1, 2]). In 2007, Huang and Zhang [3] reintroduced such spaces under the name of cone metric spaces by substituting the set of real numbers by an ordered normed space and proved some fixed point results (see, [4, 5]). On the other, in 1996, Kada et al. [6] introduced the concept of w-distance in metric spaces, where nonconvex minimization problems were treated. Further, Cho et al. [7] and Wang and Guo [8] defined the concept of c-distance which is a cone version of the w-distance. Then some fixed point theorems with respect to a w-distance in metric spaces and a c-distance in cone metric spaces and tvs-cone metric spaces were proved in [9–13] and the references cited in them.

Consistent with Huang and Zhang [3] and Cho et al. [7], the following definitions and lemma will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. A mapping $d : X \times X \to E$ is called a cone metric on X and (X, d) is called a cone metric space if (d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$; (d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$; (d_3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

For notions such as convergence and Cauchy sequences, completeness, continuity and etc. in cone metric spaces and also other properties in a cone, we refer to [3, 4].

Definition 1.2. Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied:

(q_1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
(q_2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;
(q_3) for all $n \geq 1$ and $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Note that, for c-distance q, $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ and $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.

Lemma 1.3. *Let (X, d) be a cone metric space and q be a c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $\{u_n\}$ and $\{v_n\}$ be sequences in cone P converging to θ and $x, y, z \in X$. Then the following hold:*

(c_1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $y = z$. Specifically, if $q(x, y) = \theta$ and $q(x, z) = \theta$, then $y = z$.*
(c_2) *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

The most important graph theory approach to metric fixed point theory which introduced so far is attributed to Jachymski [14]. In this approach, the underlying metric space was equipped with a directed graph and the Banach contraction was obtained. For more details on the theory of graphs, see [14, 15]. Consider a directed graph G with $V(G) = X$ such that the set $E(G)$ consisting of the edges of G contains all loops (that is, $\Delta(X) \subseteq E(G)$, where $\Delta(X) = \{(x,x) \in X \times X : x \in X\}$) and suppose that G has no parallel edges. Then G can be represented by the ordered pair $(V(G), E(G))$.

The purpose of this paper is to study the existence of fixed points for mappings under c-distance in cone metric spaces endowed with a graph. Our results are generalizations of some well-known fixed point theorems given in terms of a c-distance from cone metric spaces equipped with a partial order to cone metric spaces endowed with a graph. Also, our results are interesting since we need to use neither the continuity of mapping nor the normality of cone.

2. Main Results

Following Jachymski [14, Definition 2.4], we define the concept of orbitally G-continuous for self-map f on cone metric spaces.

Definition 2.1. Let (X,d) be a cone metric space endowed with a graph G. A mapping $f : X \to X$ is called orbitally G-continuous on X if for all $x, y \in X$ and all sequences $\{b_n\}$ of positive integers with $(f^{b_n}x, f^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$, the convergence $f^{b_n}x \to y$ implies $f(f^{b_n}x) \to fy$.

Clearly, a continuous mapping on a cone metric space is orbitally G-continuous for all graphs G but the converse is not totally true. The following example will shows that how a graph plays an effective role to imply a weaker type of continuity.

Example 2.2. Let $E = C^1_{\mathbb{R}}[0,1]$ with the norm $\|\varphi\| = \|\varphi\|_\infty + \|\varphi'\|_\infty$, $X = [0,1]$ and consider the non-normal cone $P = \{\varphi \in E : \varphi(t) \geq 0 \text{ on } [0,1]\}$. Also, let a mapping $d : X \times X \to E$ introduced by $d(x,y)(t) = |x-y| \cdot \varphi(t)$ for all $x, y \in X$, where $\varphi(t) = 2^t \in P \subset E$ with $t \in [0,1]$. Then (X,d) is a cone metric space, and the cone P is a non-normal solid cone in the Banach space E. Consider the mapping $f : X \to X$ by $f(1) = 1$ and $f(x) = \frac{x^2}{4}$ for all $x \in X$ with $x \neq 1$. Obviously, f is not continuous at $x = 1$, and in particular, on the whole X. Now assume that X is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x,x) : x \in X\}$; that is, $E(G)$ contains nothing but all loops. Observe that for all $x, y \in X$ such that $(x,y) \in E(G)$, we get $x = y$. If $x, y \in X$ and $\{b_n\}$ is a sequence of positive integers with $(f^{b_n}x, f^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$ such that $f^{b_n}x \to y$, then $\{f^{b_n}x\}$ is necessarily a constant sequence. Thus, $f^{b_n}x = y$ for all $n \geq 1$ and so $f(f^{b_n}x) \to fy$. Hence, f is orbitally G-continuous on X.

In this section, let (X,d) be a cone metric space associated with c-distance q and endowed with a graph G with $V(G) = X$ and $\Delta(X) \subseteq E(G)$. Throughout this section, we denote $X_f = \{x \in X : (x, fx) \in E(G)\}$. Our main result is the following theorem for mappings with respect to a given c-distance in a complete cone metric space endowed with a graph.

Theorem 2.3. *Let (X,d) be a complete cone metric space endowed with a graph G, q be a c-distance on X and $f : X \to X$ be an orbitally G-continuous mapping. Suppose that there exist mappings $\alpha_i : X \to [0,1)$ for $i = 1,2,3,4$ such that the following conditions hold:*

- *(i) $\alpha_i(fx) \leq \alpha_i(x)$ and $(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4)(x) < 1$ for all $x \in X$;*
- *(ii) f preserves the edges of G; that is, $(x,y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$;*

(iii) for all $x, y \in X$ with $(x, y) \in E(G)$,

$$q(fx, fy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, fy) + \alpha_4(x)q(x, fy). \quad (2.1)$$

If $X_f \neq \emptyset$, then f has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Proof. Let $x_0 \in X_f$. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, suppose that $fx_0 \neq x_0$. Since f preserves the edges of G and $(x_0, fx_0) \in E(G)$, then it follows that by induction $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, where $x_n = fx_{n-1} = f^n x_0$. Now, let $x = x_{n-1}$ and $y = x_n$ in (2.1). Since $(x_{n-1}, x_n) \in E(G)$, we have

$$\begin{aligned}
q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\
&\preceq \alpha_1(x_{n-1})q(x_{n-1}, x_n) + \alpha_2(x_{n-1})q(x_{n-1}, x_n) + \alpha_3(x_{n-1})q(x_n, x_{n+1}) \\
&\quad + \alpha_4(x_{n-1})q(x_{n-1}, x_{n+1}) \\
&\preceq (\alpha_1 + \alpha_2 + \alpha_4)(fx_{n-2})q(x_{n-1}, x_n) + (\alpha_3 + \alpha_4)(fx_{n-2})q(x_n, x_{n+1}) \\
&\preceq (\alpha_1 + \alpha_2 + \alpha_4)(x_{n-2})q(x_{n-1}, x_n) + (\alpha_3 + \alpha_4)(x_{n-2})q(x_n, x_{n+1}) \\
&\vdots \\
&\preceq (\alpha_1 + \alpha_2 + \alpha_4)(x_0)q(x_{n-1}, x_n) + (\alpha_3 + \alpha_4)(x_0)q(x_n, x_{n+1}),
\end{aligned}$$

which implies that

$$q(x_n, x_{n+1}) \preceq \frac{(\alpha_1 + \alpha_2 + \alpha_4)(x_0)}{1 - (\alpha_3 + \alpha_4)(x_0)} q(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. By repeating the procedure, we get

$$q(x_n, x_{n+1}) \preceq h^n q(x_0, x_1) \quad (2.2)$$

for all $n \in \mathbb{N}$, where $0 \leq h = \dfrac{(\alpha_1 + \alpha_2 + \alpha_4)(x_0)}{1 - (\alpha_3 + \alpha_4)(x_0)} < 1$ by (i). Now, let $m, n \in \mathbb{N}$ with $m > n$. It follows from $h \in [0, 1)$ and (2.2) that

$$\begin{aligned}
q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\
&\preceq \frac{h^n}{1 - h} q(x_0, x_1).
\end{aligned}$$

Since $\frac{h^n}{1-h} q(x_0, x_1)$ converges to θ, Lemma 1.3 (c_2) implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x' \in X$ such that $x_n = f^n x_0 \to x'$ as $n \to \infty$.

We next show that x' is a fixed point for f. At the first, since f preserves the edges of G, it follows by induction that $f^n x_0 \in X_f$ for all $n \geq 0$. Thus, $(f^n x_0, f^{n+1} x_0) \in E(G)$ for all $n \geq 0$. Now, since f is orbitally G-continuous on X, we have $f^{n+1} x_0 = f(f^n x_0) \to fx'$ as $n \to \infty$ which implies that $fx' = x'$ (because the limit of a convergent sequence is unique). Now, suppose that $fz = z$. Then, from (2.1), we have

$$\begin{aligned}
q(z, z) &= q(fz, fz) \\
&\preceq \alpha_1(z)q(z, z) + \alpha_2(z)q(z, fz) + \alpha_3(z)q(z, fz) + \alpha_4(z)q(z, fz) \\
&= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(z)q(z, z),
\end{aligned}$$

which implies that $q(z, z) = \theta$ by (i) and (p_3). This completes the proof. ■

Example 2.4. Consider E, $\|.\|$, φ, X, P and d as in Example 2.2. Then (X, d) is a cone metric space, and the cone P is a non-normal solid cone in the Banach space E. Define the mapping $f : X \to X$ by $f(\frac{1}{2}) = 0$ and $f(x) = \frac{x^2}{4}$ for all $x \in X$ with $x \neq \frac{1}{2}$. Now assume that X is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, x) : x \in X\} \cup \{(0, \frac{1}{2}), (\frac{1}{2}, 0)\}$. Clearly, f is not continuous on the whole X, but f is orbitally G-continuous on X. Let $q : X \times X \to E$ be a c-distance defined by $q(x, y)(t) = d(x, y)(t)$ for all $x, y \in X$, where $t \in [0, 1]$. Take mappings $\alpha_1(x) = \frac{x+1}{4}$, $\alpha_2(x) = \alpha_4(x) = \frac{x}{8}$ and $\alpha_3(x) = 0$ for all $x \in X$. Observe that:

(1) if $x \in X - \{\frac{1}{2}\}$, then $\alpha_1(f(x)) = \alpha_1(\frac{x^2}{4}) = \frac{1}{4}\big(\frac{x^2}{4} + 1\big) \le \frac{x+1}{4} = \alpha_1(x)$, and if $x = \frac{1}{2}$, then $\alpha_1(f(\frac{1}{2})) = \alpha_1(0) = \frac{0+1}{4} \le \frac{\frac{1}{2}+1}{4} = \alpha_1(\frac{1}{2})$;

(2) if $x \in X - \{\frac{1}{2}\}$, then $\alpha_2(f(x)) = \alpha_4(f(x)) = \frac{x^2}{32} \le \frac{x}{8} = \alpha_2(x) = \alpha_4(x)$, and if $x = \frac{1}{2}$, then $\alpha_2(f(\frac{1}{2})) = \alpha_4(f(\frac{1}{2})) = \frac{0}{8} \le \frac{\frac{1}{2}}{8} = \alpha_2(\frac{1}{2}) = \alpha_4(\frac{1}{2})$;

(3) $\alpha_3(f(x)) = \alpha_3(x) = 0$ for all $x \in X$;

(4) $(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4)(x) = \frac{x+1}{4} + \frac{x}{8} + \frac{x}{4} < 1$ for all $x \in X$;

(5) let $x \in X$. Then

$$q(fx, fx)(t) \preceq \alpha_1(x)q(x, x)(t) + ((\alpha_2 + \alpha_3 + \alpha_4)(x))q(x, fx)(t)$$

or

$$\begin{aligned} q(f0, f\frac{1}{2})(t) &\preceq \alpha_1(0)q(0, \frac{1}{2})(t) + \alpha_2(0)q(0, f0)(t) + \alpha_3(0)q(\frac{1}{2}, f\frac{1}{2})(t) \\ &\quad + \alpha_4(0)q(0, f\frac{1}{2})(t), \\ q(f\frac{1}{2}, f0)(t) &\preceq \alpha_1(\frac{1}{2})q(\frac{1}{2}, 0)(t) + \alpha_2(\frac{1}{2})q(\frac{1}{2}, f\frac{1}{2})(t) + \alpha_3(\frac{1}{2})q(0, f0)(t) \\ &\quad + \alpha_4(\frac{1}{2})q(\frac{1}{2}, f0)(t); \end{aligned}$$

(6) since $(0, f(0)) = (0, 0) \in E(G)$, so $X_f \neq \emptyset$.

Thus, all of the conditions of Theorem 2.3 are established. Obviously, f has a fixed point $x = 0 \in [0, 1]$ and $q(0, 0) = 0$.

Now, let $Fixf$ be the set of fixed points of the mapping f.

Theorem 2.5. *In addition to the hypothesis of Theorem 2.3, if the subgraph of G with the vertex set $Fixf$ is connected, then the restriction of f to X_f is a Picard operator.*

Proof. Let the subgraph of G with the vertex set $Fixf$ be connected and $x', y' \in X$ be a fixed point of f. Then there exists a path $\{x_i\}_{i=0}^N$ in G from x' to y' such that $x_1, \ldots, x_{N-1} \in Fixf$; that is, $x_0 = x'$, $x_N = y'$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, \ldots, N-1$. By (2.1) and $q(x_{i+1}, x_{i+1}) = q(x_i, x_i) = \theta$, we have

$$\begin{aligned} q(x_i, x_{i+1}) = q(fx_i, fx_{i+1}) &\preceq \alpha_1(x_i)q(x_i, x_{i+1}) + \alpha_2(x_i)q(x_i, fx_i) \\ &\quad + \alpha_3(x_i)q(x_{i+1}, fx_{i+1}) + \alpha_4(x_i)q(x_i, fx_{i+1}) \\ &= (\alpha_1 + \alpha_4)(x_i)q(x_i, x_{i+1}), \end{aligned}$$

which implies that $q(x_i, x_{i+1}) = \theta$ (by (p_3)). On the other hand, $q(x_i, x_i) = \theta$. Thus, by Lemma 1.3 (c_1), we obtain $x_i = x_{i+1}$. Consequently,

$$x' = x_0 = x_1 = \cdots = x_{N-1} = x_N = y'$$

and hence the fixed point of f is unique and the restriction of f to X_f is a Picard operator. ■

Question. Can you consider another property instead of given condition in Theorem 2.5 to obtain the uniqueness of the fixed point of f?

In Theorems 2.3 and 2.5, set $\alpha_i(x) = \alpha_i$ for $i = 1, 2, 3, 4$. Then we get the following theorem.

Theorem 2.6. *Let (X, d) be a complete cone metric space endowed with a graph G, q be a c-distance on X and $f : X \to X$ be an orbitally G-continuous mapping and preserves the edges of G. Suppose that there exist nonnegative constants α_i for $i = 1, 2, 3, 4$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1$ such that*

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) + \alpha_4 q(x, fy).$$

If $X_f \neq \emptyset$, then f has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$. Also, if the subgraph of G with the vertex set $Fixf$ is connected, then the restriction of f to X_f is a Picard operator.

Note that each w-distance in a metric space (in the sense of Kada et al. [6]) is a c-distance in the cone metric space (X, d) (in the sense of Cho et al. [7]) with $E = \mathbb{R}$ and $P = [0, \infty)$. But the converse does not hold. Thus, the c-distance is a generalization of the w-distance. Consequently, our theorems are true for mappings with respect to a w-distance in metric spaces endowed with a graph.

Remark 2.7. (i) Since we need to use neither the continuity of mapping nor the normality of cone, the method of mentioned theorems generalize, extend and unify all of research paper on fixed point theorems in cone metric spaces associated with a c-distance such as: Rahimi et al. [11, 13], Cho et al. [7], Wang and Guo [8], Kada et al. [6], Ćirić et al. [16] (and also, all references contained in them about w-distance and c-distance).

(ii) Sometimes the constant numbers which satisfy Theorem 2.6 are difficult to find. Thus, it is better to define such mappings $\alpha_i(x)$ instead of constants α_i for $i = 1, 2, 3, 4$ as another auxiliary tool of the cone metric (such as Theorem 2.3).

(iii) In 2012, Ćirić et al. [16] show that the method of Du [17] for cone contraction mappings cannot be applied for a c-distance contraction mappings. Thus, our results are new and cannot to derived from the version of w-distance in metric spaces.

Several consequences of Theorem 2.6 follow now for particular choices of the graph. For example, consider cone metric space (X, d) endowed with the complete graph G_0 whose vertex set coincides with X; that is, $V(G_0) = X$ and $E(G_0) = X \times X$. Then we get the following corollary.

Corollary 2.8. *Let (X, d) be a complete cone metric space associated endowed with the graph G_0, q be a c-distance on X and $f : X \to X$ be an orbitally G_0-continuous mapping. Suppose that there exist nonnegative constants α_i for $i = 1, 2, 3, 4$ with $\alpha_1+\alpha_2+\alpha_3+2\alpha_4 < 1$ such that*

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) + \alpha_4 q(x, fy).$$

Then f has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Suppose now that $(X, \sqsubseteq)$ is a poset. Consider on the poset X, the graph G_1 given by $V(G_1) = X$ and $E(G_1) = \big\{(x, y) \in X \times X : x \sqsubseteq y\big\}$. Since $\sqsubseteq$ is reflexive, it follows that

$E(G_1)$ contain all loops. Now, let $G = G_1$ in Theorem 2.6. Then we obtain the following result.

Corollary 2.9. *Let $(X, \sqsubseteq)$ be a poset and (X, d) be a complete cone metric space endowed with the graph G_1. Also, let q be a c-distance on X and $f : X \to X$ be a nondecreasing and orbitally G_1-continuous mapping. Suppose that there exist nonnegative constants α_i for $i = 1, 2, 3, 4$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1$ such that*

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) + \alpha_4 q(x, fy).$$

for all $x, y \in X$ with $x \sqsubseteq y$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$.

For our next consequence, consider on the poset X, the graph G_2 defined by $V(G_2) = X$ and $E(G_2) = \{(x, y) \in X \times X : x \sqsubseteq y \ \vee \ y \sqsubseteq x\}$. Then, an ordered pair $(x, y) \in X \times X$ is an edge of G_2 if and only if x and y are comparable elements of $(X, \sqsubseteq)$. If we set $G = G_2$ in Theorem 2.6, then we obtain another fixed point theorem in complete cone metric spaces associated with a c-distance q and endowed with a partial order.

Corollary 2.10. *Let $(X, \sqsubseteq)$ be a poset and (X, d) be a complete cone metric space endowed with the graph G_2. Also, let q be a c-distance on X and $f : X \to X$ be an orbitally G_2-continuous mapping which maps comparable elements of X onto comparable elements. Suppose that there exist nonnegative constants α_i for $i = 1, 2, 3, 4$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1$ such that*

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) + \alpha_4 q(x, fy).$$

for all $x, y \in X$ such that x and y are comparable. If there exists $x_0 \in X$ such that x_0 and $\sqsubseteq fx_0$ are comparable, then f has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Let $e \in int\ P$ be a fixed element. Recall that two elements $x, y \in X$ are said to be e-close if $d(x, y) \preceq e$. Define the e-graph G_3 by $V(G_3) = X$ and $E(G_3) = \{(x, y) \in X \times X : \ d(x, y) \preceq e\}$. We see that $E(G_3)$ contains all loops. Finally, if we set $G = G_3$ in Theorem 2.6, then we get the following consequence.

Corollary 2.11. *Let (X, d) be a complete cone metric space, q be a c-distance on X and $e \in intP$. Also, let $f : X \to X$ be a mapping which maps e-close elements of X onto e-close elements and f be orbitally G_3-continuous on X. Suppose that there exist nonnegative constants α_i for $i = 1, 2, 3, 4$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1$ such that*

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) + \alpha_4 q(x, fy).$$

for all $x, y \in X$ such that x and y are e-close elements. If there exists $x_0 \in X$ such that x_0 and fx_0 are e-close elements, then f has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Question. Can one obtain common fixed point results for mappings with respect to the c-distance in cone metric spaces endowed with the graph G?

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