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Integrals Involving Product of Srivastava's Polynomials and Multiindex Bessel Function

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Abstract The present paper deals with four new generalized integral formulae involving product of Srivastava's polynomials and generalized multiindex Bessel function and are represented in terms of the Fox-Wright function. Various particular cases and consequences of our main results involving the Hermite polynomials are also pointed out.

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1. Introduction and Preliminary

The Bessel function is associated with a wide range of problems concerning the most important areas of mathematical physics and various technical problems are linked into application of Bessel functions. Bessel function theory is often used when solving, for example, problems of hydrodynamics, acoustics, radio physics, atomic and nuclear physics, information theory. These functions are also an effective tool for problem solving in areas of wave mechanics and elasticity theory. In recent years, various useful integral formulae and applications associated with the Bessel (or generalized) functions have been studied by several authors [1–7].

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Recently, Choi and Agarwal [8] introduced the generalized multiindex Bessel function in the form

$$J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\upsilon n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+1)}\frac{(-z)^n}{n!}\ (m\in\mathbb{N}), \tag{1.1}$$

where $\alpha_j,\,\beta_j,\,\gamma\in\mathbb{C}\,(j=1,\cdots,m),\upsilon>0,\,\Re(\gamma)>0,\,\Re(\beta_j)>-1,\,\sum_{j=1}^{m}\Re(\alpha_j)>\max\{0;\Re(\upsilon)-1\}$.

Clearly, for $\upsilon=0,m=1,\alpha_1=1,\beta_1=\nu$ and replacing z by $z^2/4$ in (1.1), we obtain

$$J_{\nu,0}^{1,\gamma}\left[\frac{z^2}{4}\right]=\left(\frac{2}{z}\right)^{\nu}J_\nu\,[z], \tag{1.2}$$

where $J_\nu\,[z]$ is a Bessel function of the first kind defined for complex $z\in\mathbb{C}$, $(z\neq 0)$ and $\nu\in\mathbb{C}$, $(\Re(\nu)>-1)$ ([9–11]):

$$J_\nu\,[z]=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k+1)}\frac{(z/2)^{\nu+2k}}{k!}. \tag{1.3}$$

The Srivastava's polynomials is defined by Srivastava [12, p. 1, Eq.(1)] as:

$$S_n^{\ell}\,[x\,]=\sum_{k=0}^{[n/\ell]}\frac{(-n)_{\ell k}}{k!}A_{n,k}\,x^k,\quad n=0,1,2,\ldots, \tag{1.4}$$

where ℓ is an arbitrary positive integer and the coefficients $A_{n,k}(n,k\geq 0)$ are arbitrary constants, real or complex. The polynomial family $S_n^{\ell}[x]$ gives a number of known polynomials as its special cases on suitably specializing the coefficient $A_{n,k}$.

For our present investigation, we required the following Oberhettinger's integral formula [13]:

$$\int_0^{\infty}x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}dx=2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{1.5}$$

provided $0<\Re(\mu)<\Re(\lambda)$ and Lavoie-Trottier integral formula [14]:

$$\int_0^{1}x^{\alpha-1}\,(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.6}$$

provided $\Re(\alpha)>0$, $\Re(\beta)>0$.

Here, we establish four generalized integral formulas involving product of Srivastava's polynomials and multiindex Bessel function, which are expressed in terms of the generalized (Wright) hypergeometric functions. Particular cases and consequences of our main results involving the Hermite polynomials are also considered.

2. Main Results

For our purpose, we first recall the definition of Fox-Wright function ${}_p\Psi_q(z)$ (see, for details, [15–18]), for $z\in\mathbb{C}$ complex, $a_i,b_j\in\mathbb{C}$ and $\alpha_i,\beta_j\in\Re$, where $(\alpha_i,\beta_j\neq 0;$ $i=1,\,2,\,\ldots,\,p; j=1,\,2,\,\ldots,q)$, is defined as below:

$${}_p\Psi_q\left[\begin{array}{c}(a_i,\,\alpha_i)_{\,1,p}\\(b_j,\beta_j)_{\,1,q}\end{array}\Big|z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)\,z^k}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)\,k\,!}, \tag{2.1}$$

for all values of the argument z under the condition:

$$\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1. \tag{2.2}$$

It is noted that the generalized (Wright) hypergeometric function ${}_p\Psi_q$ in (2.1) whose asymptotic expansion was investigated by Fox [19] and Wright is an interesting further generalization of the generalized hypergeometric series as follow:

$${}_p\Psi_q\left[\begin{array}{c}(a_1,\ 1)\ ,\ \ldots,\ (a_p,\ 1);\\(b_1,\ 1)\ ,\ \ldots,\ (b_q,\ 1);\end{array}\ z\right]=\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{array}{c}\alpha_1,\ \ldots,\ \alpha_p;\\\beta_1,\ \ldots,\ \beta_q;\end{array}\ z\right], \tag{2.3}$$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see [20], Section 1.5)

$${}_pF_q\left[\begin{array}{c}\alpha_1,\ \ldots,\ \alpha_p;\\\beta_1,\ \ldots,\ \beta_q;\end{array}\ z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n z^n}{(\beta_1)_n\cdots(\beta_q)_n n!}={}_pF_q(\alpha_1,\ \ldots,\ \alpha_p;\beta_1,\ \ldots,\ \beta_q;z). \tag{2.4}$$

Theorem 2.1. *Let $a\in\mathbb{N}$; λ, μ, α_j, β_j, $\gamma\in\mathbb{C}$ such that n, $k\ \geq 0$, $\Re(\beta_j)>-1$ and $0<\Re(\mu)<\Re(\lambda+k)$, then there holds the following result for $x>0$:*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}S_n^\ell\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)$$
$$\times J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=2^{1-\mu}a^{\mu-\lambda}\frac{\Gamma(2\mu)}{\Gamma(\gamma)}\sum_{k=0}^{[n/\ell]}\frac{(-n)_{\ell k}}{k!}A_{n,k}\,(y/a)^k$$
$${}_3\Psi_{m+2}\left[\begin{array}{c}(\lambda+k+1,\ 1)\,,(\lambda-\mu+k,\ 1)\,,(\gamma,\upsilon)\ ;\\(\lambda+\mu+k+1,1)\,,(\lambda+k,1)\,,(\beta_j+1,\alpha_j)_1^m\ ;\end{array}\ \frac{-y}{a}\right]. \tag{2.5}$$

Proof. By making use of definition (1.1) and (1.4) in the left-hand-side of integral (2.5) and then interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}S_n^\ell\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)$$
$$\times J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx,$$
$$=\sum_{k=0}^{[n/\ell]}\frac{(-n)_{\ell k}}{k!}A_{n,k}\,y^k\sum_{p=0}^{\infty}\frac{(\gamma)_{\upsilon p}}{\prod_{j=1}^{m}\Gamma(\alpha_j p+\beta_j+1)}\frac{(-y)^p}{p!}$$
$$\times\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-k-p}dx. \tag{2.6}$$

Now, on applying the integral formula (1.5) to the integral in (2.6), we obtain the following expression:

$$= 2^{1-\mu}a^{\mu-\lambda}\frac{\Gamma(2\mu)}{\Gamma(\gamma)}\sum_{k=0}^{[n/\ell]}\frac{(-n)_{\ell k}}{k!}A_{n,k}\,(y/a)^k$$

$$\times\sum_{p=0}^{\infty}\frac{\Gamma(\gamma+\upsilon p)\Gamma(\lambda-\mu+k+p)\,\Gamma(\lambda+k+1+p)}{\prod_{j=1}^{m}\Gamma(\alpha_j p+\beta_j+1)\,\Gamma(\lambda+\mu+k+1+p)\,\Gamma(\lambda+k+p)p!}\left(\frac{-y}{a}\right)^p.$$

In accordance with the definition of (2.1), we obtain the result (2.5). This completes the proof of the theorem. ∎

Theorem 2.2. *Let $a\in\mathbb{N}$; λ, μ, α_j, β_j, $\gamma\in\mathbb{C}$ such that $n,\,k\ \geq 0$, $\Re(\beta_j)>-1$ and $0<\Re(\mu)<\Re(\lambda+k)$, then there holds the following result for $x>0$:*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}S_n^\ell\left(\frac{x\,y}{x+a+\sqrt{x^2+2ax}}\right)$$
$$\times J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}\left(\frac{x\,y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=\frac{2^{1-\mu}a^{\mu-\lambda}}{\Gamma(\gamma)}\sum_{k=0}^{[n/\ell]}\frac{(-n)_{\ell k}\Gamma(\lambda-\mu)}{k!}A_{n,k}\,(y/2)^k$$
$$ {}_3\Psi_{m+2}\left[\begin{array}{c}(2\mu+2k,2)\,,(\lambda+k+1,1)\,,(\gamma,\upsilon)\,;\\(\lambda+k,1)\,,(\lambda+\mu+2k+1,2),(\beta_j+1,\alpha_j)_1^m\,;\end{array}\ \frac{-y}{2}\right].\tag{2.7}$$

Proof. By similar manner as in the proof of Theorem 2.1, we can prove the integral formula (2.7). ∎

Theorem 2.3. *Let $a\in\mathbb{N}$; λ, μ, α_j, β_j, $\gamma\in\mathbb{C}$ such that n, $k\ \geq 0$, $\Re(\beta_j)>-1$, $\Re(\sigma)>0$ and $\Re(\rho+k)>0$, then there holds the following result for $x>0$:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}$$
$$\times S_n^\ell\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho)}{\Gamma(\gamma)}\sum_{k=0}^{[n/\ell]}\frac{(-n)_{\ell k}}{k!}A_{n,k}\,y^k\,{}_2\Psi_{m+1}\left[\begin{array}{c}(\sigma+k,1)\,,(\gamma,\upsilon)\,;\\(\rho+\sigma+k,1)\,,(\beta_j+1,\alpha_j)_1^m\,;\end{array}\ -y\right].\tag{2.8}$$

Proof. By making use of definition (1.1) and (1.4) in the left-hand-side of integral (2.8) and then interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}$$
$$\times S_n^\ell\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$= \sum_{k=0}^{[n/\ell]} \frac{(-n)_{\ell k}}{k\,!} A_{n,k}\, y^k \sum_{p=0}^{\infty} \frac{(\gamma)_{\upsilon\, p}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j p + \beta_j + 1\right)} \frac{(-y)^p}{p!}$$

$$\times \int_0^1 x^{\rho-1} \left(1-x\right)^{2(\sigma+p+k)-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma+p+k-1} dx. \tag{2.9}$$

On applying the integral formula (1.6) in (2.9), we obtain the following expression:

$$= \sum_{k=0}^{[n/\ell]} \frac{(-n)_{\ell k}}{k!} A_{n,k} y^k \sum_{p=0}^{\infty} \frac{(\gamma)_{\upsilon p}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j p + \beta_j + 1\right)} \frac{(-y)^p}{p!} \left(\frac{2}{3}\right)^{2\rho} \frac{\Gamma\left(\rho\right)\Gamma\left(\sigma+p+k\right)}{\Gamma\left(\rho+\sigma+p+k\right)},$$

$$= \left(\frac{2}{3}\right)^{2\rho} \frac{\Gamma(\rho)}{\Gamma(\gamma)} \sum_{k=0}^{[n/\ell]} \frac{(-n)_{\ell k}}{k!} A_{n,k}\, y^k$$

$$\times \sum_{p=0}^{\infty} \frac{\Gamma(\gamma+\upsilon p)\Gamma\left(\sigma+p+k\right)}{\prod_{j=1}^{m} \Gamma\left(\alpha_j p + \beta_j + 1\right)\Gamma\left(\rho+\sigma+p+k\right)} \frac{(-y)^p}{p!},$$

which in accordance with the definition (2.1), yield to the desired result (2.8). This completes the proof of the theorem. ■

Theorem 2.4. *Let $a \in \mathbb{N}$; λ, μ, α_j, β_j, $\gamma \in \mathbb{C}$ such that n, $k \geq 0$, $\Re(\beta_j) > -1$, $\Re(\sigma) > 0$ and $\Re\left(\rho + k\right) > 0$, then for $x > 0$ the following result holds*

$$\int_0^1 x^{\rho-1} \left(1-x\right)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1}$$

$$\times S_n^{\ell} \left(yx\left(1-\frac{x}{3}\right)^2\right) J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma} \left(yx\left(1-\frac{x}{3}\right)^2\right) dx$$

$$= \left(\frac{2}{3}\right)^{2(\rho)} \frac{\Gamma(\sigma)}{\Gamma(\gamma)} \sum_{k=0}^{[n/\ell]} \frac{(-n)_{\ell k}}{k!} A_{n,k} \left(\frac{4y}{9}\right)^k$$

$${}_2\Psi_{m+1} \left[\begin{matrix} \left(\rho+k,1\right), \left(\gamma,\upsilon\right); \\ \left(\rho+\sigma+k,1\right), \left(\beta_j+1,\alpha_j\right)_1^m; \end{matrix} \; \frac{-4y}{9} \right]. \tag{2.10}$$

One can easily prove the integral formula (2.10), following the similar procedure as in proof of Theorem 2.3, so we omit its detailed proof.

3. Special Cases

In this section, we derive some new integral formulae as special cases of our main results derived in the preceding section.
If we set $n = 0$, then we observe that the Srivastava's polynomial $S_n^{\ell}[x]$ reduce to unity *i.e.* $S_0^{\ell}[x] \to 1$. Hence, we obtain the following results:

Corollary 3.1. *Let the condition of Theorem 2.1 be satisfied and $n = 0$, then Theorem 2.1 reduces in following form*

$$\int_0^{\infty} x^{\mu-1} \left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma} \left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2^{1-\mu} a^{\mu-\lambda} \frac{\Gamma(2\mu)}{\Gamma(\gamma)} {}_3\Psi_{m+2} \left[\begin{array}{c} (\lambda+1,1), (\lambda-\mu,1), (\gamma,\upsilon); \\ (\lambda+\mu+1,1), (\lambda,1), (\beta_j+1,\alpha_j)_1^m; \end{array} \frac{-y}{a} \right]. \tag{3.1}$$

Corollary 3.2. *For $n=0$, Theorem 2.2 reduces in the following result*

$$\int_0^\infty x^{\mu-1} \left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma} \left(\frac{x\,y}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= \frac{2^{1-\mu} a^{\mu-\lambda} \Gamma(\lambda-\mu)}{\Gamma(\gamma)} {}_3\Psi_{m+2} \left[\begin{array}{c} (2\mu+2k,2), (\lambda+1,1), (\gamma,\upsilon); \\ (\lambda,1), (\lambda+\mu+1,2), (\beta_j+1,\alpha_j)_1^m; \end{array} \frac{-y}{2} \right]. \tag{3.2}$$

Corollary 3.3. *Let the condition of Theorem 2.3 be satisfied, then for $n=0$, Theorem 2.3 reduces as under*

$$\int_0^1 x^{\rho-1} (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma} \left(y\left(1-\frac{x}{4}\right)(1-x)^2\right) dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \frac{\Gamma(\rho)}{\Gamma(\gamma)} {}_2\Psi_{m+1} \left[\begin{array}{c} (\sigma,1), (\gamma,\upsilon); \\ (\rho+\sigma,1), (\beta_j+1,\alpha_j)_1^m; \end{array} -y \right]. \tag{3.3}$$

Corollary 3.4. *Under the valid condition and $n=0$, Theorem 2.4 reduces in the following form*

$$\int_0^1 x^{\rho-1} (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma} \left(yx\left(1-\frac{x}{3}\right)^2\right) dx$$

$$= \left(\frac{2}{3}\right)^{2(\rho)} \frac{\Gamma(\sigma)}{\Gamma(\gamma)} {}_2\Psi_{m+1} \left[\begin{array}{c} (\rho,1), (\gamma,\upsilon); \\ (\rho+\sigma,1), (\beta_j+1,\alpha_j)_1^m; \end{array} \frac{-4y}{9} \right]. \tag{3.4}$$

Further, the polynomial's family $S_n^\ell[x]$ gives a number of known polynomials as its special cases on suitably specializing the coefficients $A_{n,k}$. To illustrate this, we give one more example.

If we set $\ell = 2$ and $A_{n,k} = (-1)^k$, then the Srivastava's polynomials

$$S_n^2[x] \to x^{n/2} H_n\left(\frac{1}{2\sqrt{x}}\right), \tag{3.5}$$

where $H_n(.)$ denotes the well known Hermite polynomials and defined by

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}. \tag{3.6}$$

Now, on putting $\ell = 2,\ A_{n,k} = (-1)^k$ and taking relation (3.5) into account, Theorem 2.1 to Theorem 2.4 yields to the following results involving the Hermite polynomial and the generalized multiindex Bessel function:

Corollary 3.5. *Let $x>0$; $a\in\mathbb{N}$, $\lambda,\gamma\in\mathbb{C}$; $\alpha_j>0$ and β_j is an arbitrary parameter be such that $n,k\geq 0$ and $0<\Re(\mu)<\Re(\lambda+k+1)$, then the following formulas holds:*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-\frac{n}{2}} y^{\left(\frac{n}{2}\right)} H_n\left(\frac{1}{2\sqrt{X}}\right) J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}(X)\,dx$$

$$=2^{1-\mu}a^{\mu-\lambda}\frac{\Gamma(2\mu)}{\Gamma(\gamma)}\sum_{k=0}^{[n/2]}\frac{(-n)_{2\,k}}{k!}(-1)^k(y/a)^k$$

$${}_3\Psi_{m+2}\left[\begin{array}{cc}(\lambda+k+1,1),(\lambda-\mu+k,1),(\gamma,\upsilon); & \\ & \frac{-y}{a}\\ (\lambda+\mu+k+1,1),(\lambda+k,1),(\beta_j+1,\alpha_j)_1^m; & \end{array}\right], \tag{3.7}$$

where $X=\frac{y}{x+a+\sqrt{x^2+2ax}}$.

Corollary 3.6. *Let $x>0$; $a\in\mathbb{N},\lambda,\gamma\in\mathbb{C};\alpha_j>0$ and β_j is an arbitrary parameter be such that $n,\ k\geq 0$ and $0<\Re(\mu)<\Re(\lambda+k+1)$, then we have*

$$\int_0^\infty x^{\mu-1+n/2}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-\frac{n}{2}} y^{\left(\frac{n}{2}\right)} H_n\left(\frac{1}{2\sqrt{Y}}\right) J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}(Y)\,dx$$

$$=\frac{2^{1-\mu}a^{\mu-\lambda}}{\Gamma(\gamma)}\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}\Gamma(\lambda-\mu)}{k!}(-1)^k(y/2)^k$$

$${}_3\Psi_{m+2}\left[\begin{array}{cc}(2\mu+2k,2),(\lambda+k+1,1),(\gamma,\upsilon); & \\ & -y\\ (\lambda+k,1),(\lambda+\mu+2k+1,2),(\beta_j+1,\alpha_j)_1^m; & \end{array}\right], \tag{3.8}$$

where $Y=\frac{xy}{x+a+\sqrt{x^2+2ax}}$.

Corollary 3.7. *Let $x>0$; $a\in\mathbb{N}$, $\lambda,\gamma\in\mathbb{C};\alpha_j>0$ and β_j is an arbitrary parameter be such that $n,k\geq 0$, $\Re(\sigma)>0$ and $\Re(\rho+k)>0$, then there holds the following result:*

$$\int_0^1 x^{\rho-1}(1-x)^{2(\sigma+\frac{n}{2})-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma+\frac{n}{2}-1} y^{\left(\frac{n}{2}\right)}$$

$$\times H_n\left(\frac{1}{2\sqrt{Z}}\right) J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}(Z)\,dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho)}{\Gamma(\gamma)}\sum_{k=0}^{[n/m]}\frac{(-1)_{2k}}{k!}(-1)^k y^k\,{}_2\Psi_{m+1}\left[\begin{array}{cc}(\sigma+k,1),(\gamma,\upsilon); & \\ (\rho+\sigma+k,1),(\beta_j+1,\alpha_j)_1^m; & -y\end{array}\right], \tag{3.9}$$

where $Z=y\left(1-\frac{x}{4}\right)(1-x)^2$.

Corollary 3.8. *Let $x>0$; $a\in\mathbb{N}$, $\lambda,\gamma\in\mathbb{C};\alpha_j>0$ and β_j is an arbitrary parameter be such that $n,k\geq 0$ and $\Re(\sigma)>0$ and $\Re(\rho+k)>0$, then*

$$\int_0^1 x^{\rho+\frac{n}{2}-1}(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2(\rho+\frac{n}{2})-1}\left(1-\frac{x}{4}\right)^{\sigma-1} y^{\left(\frac{n}{2}\right)}$$

$$\times H_n\left(\frac{1}{2\sqrt{W}}\right) J_{(\beta_j)_m,\upsilon}^{(\alpha_j)_m,\gamma}(W)\,dx$$

$$=\left(\frac{2}{3}\right)^{2(\rho)}\frac{\Gamma(\sigma)}{\Gamma(\gamma)}\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}}{k!}(-1)^k\left(\frac{4y}{9}\right)^k {}_2\Psi_{m+1}\left[\begin{matrix}(\rho+k,1),(\gamma,\upsilon); \\ (\rho+\sigma+k,1),(\beta_j+1,\alpha_j)_1^m;\end{matrix}\ \frac{-4y}{9}\right], \tag{3.10}$$

where $W = yx\,\left(1-\frac{x}{3}\right)^2$.

4. Concluding Remark

We conclude this paper by emphasizing that the various type of Bessel functions are particular cases of generalized multiindex Bessel function defined by (1.1). Further, on giving suitable special values to the coefficient $A_{n,k}$, the general class of polynomials give many known classical orthogonal polynomials as its particular cases, which includes Hermite, Leguerre, Jacobi, the Konhauser polynomials and so on. Therefore, we observe that our main results can lead to yield numerous other interesting integrals involving various Bessel functions and orthogonal polynomials by suitable specialization of arbitrary parameters in the main theorems.

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