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Fixed Point Theorems for Pseudo-Banach Contraction

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Abstract We establish the coincidence point and fixed point theorems for two new types of single-valued and multi-valued mappings in a complete metric space with a graph. These two maps are extended from the maps constructed by Khojasteh et al. It also extends some recent works on the extension of Banach contraction principle to metric spaces with a directed graph.

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1. Introduction

Throughout the rest of the paper, unless otherwise specified, (X, ρ) and $(\widetilde{X}, \rho)$ are a metric space and a complete metric space, respectively. Moreover, denote

$$CB(X) = \{A \subset X \mid A \neq \emptyset, A \text{ is closed and bounded}\}$$

and

$$Comp(X) = \{A \subset X \mid A \neq \emptyset, A \text{ is compact}\}.$$

For any element x in X and any two subsets A, B of $CB(X)$, we define functions $\mathcal{D}, \delta$ and $\mathcal{H}$ by

$$\mathcal{D}(x, A) = \inf \{\rho(x, y) \mid y \in A\},$$
$$\delta(x, A) = \sup \{\rho(x, y) \mid y \in A\},$$

and

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in B} \mathcal{D}(x, A), \sup_{x \in A} \mathcal{D}(x, B) \right\}.$$

The mapping $\mathcal{H}$ is called a Hausdorff metric induced by $\mathcal{D}$.

The classical principle of Banach contraction is the important role in studying fixed point theory, which states as follows.

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Theorem 1.1. [1] *Let* $S : (\widetilde{X}, \rho) \to (\widetilde{X}, \rho)$*. Suppose that*

$$\rho(Sx, Sy) \leq \alpha\rho(x, y),$$

for all $x, y \in \widetilde{X}$*, where* $\alpha \in [0, 1)$*, then* S *has a unique fixed point in* $\widetilde{X}$*, i.e., there exists a unique* $z \in \widetilde{X}$*.*

Such a mapping S satisfying the inequality in Theorem 1.1 is called a Banach contraction which plays a central role in studying the existence of solutions of various equations. It also has been vastly generalized in numerous trends by several authors [2, 3], and also [4, 5].

In 1969, Nadler [6] introduced the Banach contraction principle for multi-valued mappings, as the first famous theorem of multi-valued contraction, which states as follows.

Theorem 1.2. [6] *Let* $S : (\widetilde{X}, \rho) \to CB(\widetilde{X})$*. If there is* $k \in [0, 1)$ *such that* $\mathcal{H}(Sx, Sy) \leq k\rho(x, y)$*, for all* $x, y \in \widetilde{X}$*, then* S *has a fixed point.*

In 1972, Reich [7] established the following fixed point theorem as a modification of Nadler's fixed point theorem on compact sets instead of closed and bounded sets.

Theorem 1.3. [7] *Let* $S : (\widetilde{X}, \rho) \to Comp(\widetilde{X})$*. If there is a function* $\alpha : [0, \infty) \to [0, 1)$ *such that* $\limsup_{r\to t^+} \alpha(r) < 1$*, for each* $t \in (0, \infty)$ *and*

$$\mathcal{H}(Sx, Sy) \leq \alpha(\rho(x, y))\rho(x, y),$$

for all $x, y \in \widetilde{X}$*, then* S *has a fixed point.*

In 1989, Mizoguchi and Takahashi [8] also gave a generalization of Nadler's theorem but in this case they considered a multi-valued mapping S with closed and bounded value.

Theorem 1.4. [8] *Let* $S : (\widetilde{X}, \rho) \to CB(\widetilde{X})$*. If there is a function* $\alpha : [0, \infty) \to [0, 1)$ *such that* $\limsup_{r\to t^+} \alpha(r) < 1$*, for each* $t \in [0, \infty)$ *and*

$$\mathcal{H}(Sx, Sy) \leq \alpha(\rho(x, y))\rho(x, y),$$

for all $x, y \in \widetilde{X}$*, then* S *has a fixed point.*

In 2007, Berinde and Berinde [9] introduced weak contraction and its generalization for multi-valued mappings stated as the concept of following definitions.

Definition 1.5. [9] Let $S : (X, \rho) \to CB(X)$. If there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that $\mathcal{H}(Sx, Sy) \leq \theta\rho(x, y) + L\mathcal{D}(y, Sx)$, for all $x, y \in X$, then S is called a multi-valued weak contraction or a multi-valued (θ, L)-weak contraction.

Definition 1.6. [9] Let $S : (X, \rho) \to CB(X)$. If there exist a nonnegative number L and a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r\to t^+} \alpha(r) < 1$ for each $t \in [0, \infty)$ such that $\mathcal{H}(Sx, Sy) \leq \alpha(\rho(x, y))\rho(x, y) + L\mathcal{D}(y, Sx)$, for all $x, y \in X$, then S is called a generalized multi-valued (α, L)-weak contraction.

In particular, for $\widetilde{X}$, they proved that fixed points of weak contraction and generalized weak contraction for multi-valued mappings always exist. Actually, Nadler's theorem has been extended in many trends, see [10, 11], and also [12].

In 2008, Jachymski [13] newly introduced a directed graph to a metric space and considered a multi-valued mapping on it and then proved some fixed point results. Moreover, he claimed that his results derived the main theorems of many authors. We now recall some notions concerning a directed graph. In a metric space X, for any graph G, denote

$V(G) \subset X$ is the vertices of G, $E(G)$ is the set of edges of G and Δ is the set of all loops. We consider in the case that the graph G has no parallel edges.

Definition 1.7. [13] For a metric space (X, ρ), let $G = (V(G), E(G))$ be a directed graph with $V(G) = X$ and $\Delta \subset E(G)$. A self-mapping S on X is a G-contraction if $(Sx, Sy) \in E(G)$, for every $(x, y) \in E(G)$ and $x, y \in X$, and there is $\alpha \in (0, 1)$ such that $\rho(Sx, Sy) \leq \alpha\rho(x, y)$ for every $(x, y) \in E(G)$ and $x, y \in X$.

Jachymski proved in [13] that under some assumptions on X, a G-contraction S attains a fixed point if and only if there is an edge connecting x and Sx for some $x \in X$. And then Bojor [14], Chifu and Petrusel [3], and Asl et al. [15] generalized the results of Jachymski to other fashions. Recently, many mathematicians produced new types of generalized contraction multi-valued mappings related to a directed graph and proved their fixed point and coincidence point theorems, see Tiammee and Suantai [16], Alfuraidan [17], Alfuraidan and Khamsi [18], Dinevari and Frigon [19] and Nicolae et al. [20], for examples.

In 2013, Khojasteh et al. [21] introduced two new types of single-valued and multi-valued mappings and confirmed the existence of their fixed points as the following theorems.

Theorem 1.8. [21] *Let $S : (\widetilde{X}, \rho) \to (\widetilde{X}, \rho)$ be such that*

$$\rho(Sx, Sy) \leq \left(\frac{\rho(x, Sy) + \rho(y, Sx)}{\rho(x, Sx) + \rho(y, Sy) + 1} \right) \rho(x, y),$$

for all $x, y \in \widetilde{X}$. Then

(1) *S has at least one fixed point $z \in \widetilde{X}$;*
(2) *$\{S^n x\}$ approaches to a fixed point, for all $x \in \widetilde{X}$;*
(3) *if x, y are two distinct fixed points of $\widetilde{X}$, then $\rho(x, y) > \frac{1}{2}$.*

Theorem 1.9. [21] *Let $S : (\widetilde{X}, \rho) \to CB(\widetilde{X})$ be such that*

$$\mathcal{H}(Sx, Sy) \leq \left(\frac{\mathcal{D}(x, Sy) + \mathcal{D}(y, Sx)}{\delta(x, Sx) + \delta(y, Sy) + 1} \right) \rho(x, y),$$

for all $x, y \in \widetilde{X}$. Then S has a fixed point.

Motivated by all of those works mentioned above and utilizing the concept of multi-valued contraction defined by Khojasteh et al. [21], we introduce two new types of single-valued and multi-valued contractions in a metric space endowed with a directed graph which are more general than that of Khojasteh et al. [21].

2. Main Results

In this section, we now introduce new types of single-valued contraction and multi-valued contraction in a metric space endowed with a directed graph and we then prove, under some conditions, that the mappings have fixed points.

Definition 2.1. Suppose (X, ρ) is a metric space, $G = (V(G), E(G))$ is a directed graph such that $V(G) = X$ and g is a self-mapping on X. A single-valued mapping $S : X \to X$ is said to be g-Pseudo-Banach contraction if it satisfies the following conditions:

(1) $(Sx, Sy) \in E(G)$ whenever $(g(x), g(y)) \in E(G)$ for all $x, y \in X$;

(2)

$$\rho(Sx, Sy) \leq \left(\frac{\rho(g(x), Sy) + \rho(g(y), Sx)}{\rho(g(x), Sx) + \rho(g(y), Sy) + 1} \right) \rho(g(x), g(y)),$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$.

Remark 2.2. We note that the ratio

$$\frac{\rho(g(u), Sv) + \rho(g(v), Su)}{\rho(g(u), Su) + \rho(g(v), Sv) + 1}$$

in (2) of Definition 2.1 is quite interesting. In the case of $V(G) = X$, $E(G) = X \times X$, $g = i_X$, and if $\rho(u, v) < \frac{1}{2}$ for all $u, v \in X$, then

$$\begin{aligned} \rho(u, Su) + \rho(v, Sv) &\leq 2\rho(u, v) + \rho(u, Su) + \rho(v, Sv) \\ &< \rho(u, Su) + \rho(v, Sv) + 1. \end{aligned}$$

This implies that

$$\frac{\rho(u, Sv) + \rho(v, Su)}{\rho(u, Su) + \rho(v, Sv) + 1} < 1.$$

Hence a g-Pseudo-Banach contraction S is just a Banach contraction. However, if $\rho(u, v) \geq \frac{1}{2}$ for all $u, v \in X$, a g-Pseudo-Banach contraction S might not be a Banach contraction which will be illustrated in the following example.

Example 2.3. Let $X = \{0, \frac{1}{2}, 1\}$ be endowed with the Euclidean metric, and $g = i_X$. Define $S : X \to X$ by $S(0) = 0, S\left(\frac{1}{2}\right) = \frac{1}{2}$, and $S(1) = 0$.
Given $E(G) = \left\{ \left(0, \frac{1}{2}\right), (0,1), (0,0), \left(\frac{1}{2}, 0\right) \right\}$. It is clear that $(X, |\cdot|)$ is a Banach space.

First we will show that $(Su, Sv) \in E(G)$ for all $u, v \in X$ with $(g(u), g(v)) \in E(G)$. Firstly, we consider $(g(u), g(v)) = \left(0, \frac{1}{2}\right)$. Then $(Su, Sv) = \left(0, \frac{1}{2}\right) \in E(G)$. Secondly, consider $(g(u), g(v)) = (0, 1)$ or $(g(u), g(v)) = (0, 0)$. Then $(Su, Sv) = (0, 0) \in E(G)$. Thus S satisfies the condition (1) of a g-Pseudo-Banach contraction. Next we will show that for $(g(u), g(v)) \in E(G)$,

$$\rho(Su, Sv) \leq \left(\frac{\rho(g(u), Sv) + \rho(g(v), Su)}{\rho(g(u), Su) + \rho(g(v), Sv) + 1} \right) \rho(g(u), g(v)).$$

If $(g(u), g(v)) = \left(0, \frac{1}{2}\right)$ or $(g(u), g(v)) = \left(\frac{1}{2}, 0\right)$, then

$$\frac{\rho(u, Sv) + \rho(v, Su)}{\rho(u, Su) + \rho(v, Sv) + 1} = 1, \rho(g(u), g(v)) = \frac{1}{2}, \text{and } \rho(Su, Sv) = \frac{1}{2}.$$

So,

$$\rho(Su, Sv) \leq \left(\frac{\rho(g(u), Sv) + \rho(g(v), Su)}{\rho(g(u), Su) + \rho(g(v), Sv) + 1} \right) \rho(g(u), g(v)).$$

If $(g(u), g(v)) = (0, 1)$ or $(g(u), g(v)) = (0, 0)$, then $\rho(Su, Sv) = 0$ which clearly implies that

$$\rho(Su, Sv) \leq \left(\frac{\rho(g(u), Sv) + \rho(g(v), Su)}{\rho(g(u), Su) + \rho(g(v), Sv) + 1} \right) \rho(g(u), g(v)).$$

Thus, S is g-Pseudo-Banach contraction. However, S is not a Banach contraction since $\rho\left(S(0), S\left(\frac{1}{2}\right)\right) = \frac{1}{2} \nleq \alpha \cdot \rho\left(0, \frac{1}{2}\right)$ for all $\alpha \in [0,1)$

The next lemma will play crucial roles in our first main result.

Lemma 2.4. *Suppose (X, ρ) is a metric space endowed with a graph G, $g : X \to X$ is a surjection, and $S : X \to X$ is a g-Pseudo-Banach contraction. If there is $x_0 \in X$ such that $(g(x_0), Sx_0) \in E(G)$, then there exists a sequence $\{x_k\}_{k\in\mathbb{N}\cup\{0\}}$ such that $(g(x_k), g(x_{k+1})) \in E(G)$, $g(x_{n+1}) = Sx_n$, and $\{g(x_k)\}_{k\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in X.*

Proof. Assume that there is $x_0 \in X$ such that $(g(x_0), Sx_0) \in E(G)$. As g is onto, there is $x_1 \in X$ such that $g(x_1) = Sx_0$. So, $(g(x_0), g(x_1)) \in E(G)$. Since S is a g-Pseudo-Banach contraction, we have $(Sx_0, Sx_1) \in E(G)$ and

$$\begin{aligned}
\rho(Sx_0, Sx_1) &\leq \left(\frac{\rho(g(x_0), Sx_1) + \rho(g(x_1), Sx_0)}{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1) + 1}\right) \rho(g(x_0), g(x_1)) \\
&= \left(\frac{\rho(g(x_0), Sx_1)}{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1) + 1}\right) \rho(g(x_0), g(x_1)) \\
&\leq \left(\frac{\rho(g(x_0), Sx_0) + \rho(Sx_0, Sx_1)}{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1) + 1}\right) \rho(g(x_0), g(x_1)) \\
&= \left(\frac{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1)}{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1) + 1}\right) \rho(g(x_0), g(x_1)) \\
&= \beta_1 \rho(g(x_0), g(x_1)),
\end{aligned}$$

where

$$\beta_1 = \frac{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1)}{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1) + 1} < 1.$$

Next, as g is onto, there is $x_2 \in X$ such that $g(x_2) = Sx_1$. Since $g(x_1) = Sx_0$ and $g(x_2) = Sx_1$, it follows that $(g(x_1), g(x_2)) \in E(G)$. As S is a g-Pseudo-Banach contraction, we get $(Sx_1, Sx_2) \in E(G)$ and

$$\begin{aligned}
\rho(Sx_1, Sx_2) &\leq \left(\frac{\rho(g(x_1), Sx_2) + \rho(g(x_2), Sx_1)}{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2) + 1}\right) \rho(g(x_1), g(x_2)) \\
&= \left(\frac{\rho(g(x_1), Sx_2)}{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2) + 1}\right) \rho(g(x_1), g(x_2)) \\
&\leq \left(\frac{\rho(g(x_1), Sx_1) + \rho(Sx_1, Sx_2)}{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2) + 1}\right) \rho(g(x_1), g(x_2)) \\
&= \left(\frac{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2)}{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2) + 1}\right) \rho(g(x_1), g(x_2)) \\
&= \beta_2 \rho(g(x_1), g(x_2)) \\
&\leq \beta_2 \beta_1 \rho(g(x_0), g(x_1),
\end{aligned}$$

where

$$\beta_2 = \frac{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2)}{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2) + 1} < 1.$$

Continuing in this fashion, we obtain a sequence $\{x_n\}$ with the property that $(g(x_{n-1}), g(x_n)) \in E(G)$, $g(x_n) = Sx_{n-1}$ and

$$\begin{aligned}\rho(Sx_{n-1}, Sx_n) &\leq \beta_n \rho(Sx_{n-2}, Sx_{n-1})\\ &\leq \beta_n \cdots \beta_2\beta_1\rho(g(x_0), g(x_1),\end{aligned}$$

where

$$\beta_n = \frac{\rho(g(x_{n-1}), Sx_{n-1}) + \rho(g(x_n), Sx_n)}{\rho(g(x_{n-1}), Sx_{n-1}) + \rho(g(x_n), Sx_n) + 1} < 1.$$

We claim that $\{g(x_n)\}$ is a Cauchy sequence in X. Note that for $n \geq 2$, we have

$$\begin{aligned}\rho(g(x_n), Sx_n)) &= \rho(Sx_{n-1}, Sx_n)\\ &\leq \beta_n\rho(Sx_{n-2}, Sx_{n-1})\\ &= \beta_n\rho(g(x_{n-1}), Sx_{n-1})\\ &\leq \beta_n\beta_{n-1}\rho(g(x_{n-2}), Sx_{n-2})\\ &\leq \rho(g(x_{n-2}), Sx_{n-2}),\end{aligned}$$

which implies that $\beta_n \leq \beta_{n-1}$ for all $n \geq 2$. Thus,

$$\begin{aligned}\rho(g(x_n), g(x_{n+1})) &= \rho(Sx_{n-1}, Sx_n)\\ &\leq \beta_n \cdots \beta_2\beta_1\rho(g(x_0), g(x_1))\\ &\leq \beta_1^n\rho(g(x_0), g(x_1)).\end{aligned}$$

Therefore, we have

$$\sum_{n=1}^{\infty}(\rho(g(x_n), g(x_{n+1}))) \leq \rho(g(x_0), g(x_1))\sum_{n=1}^{\infty}\beta_1^n < \infty,$$

and it is obvious to assert that the sequence $\{g(x_n)\}$ is a Cauchy sequence. ■

Property A. [13] For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

Theorem 2.5. *For $(\widetilde{X}, \rho)$ endowed with a graph G, let $g : \widetilde{X} \to \widetilde{X}$ be a surjection, and $S : \widetilde{X} \to \widetilde{X}$ be a g-Pseudo-Banach contraction. If the followings hold:*

(1) *$(g(x_0), Sx_0) \in E(G)$ for some $x_0 \in \widetilde{X}$;*
(2) *$\widetilde{X}$ has the Property A,*

then there exists a coincidence point $u \in \widetilde{X}$ of g and S, i.e., there exists $u \in \widetilde{X}$ such that $g(u) = Su$. Moreover, if there exist $a, b \in \widetilde{X}$ such that $g(a) = Sa, g(b) = Sb$, and $(g(a), g(b)) \in E(G)$, then $g(a) = g(b)$ or $\rho(g(a), g(b)) \geq \frac{1}{2}$.

Proof. Since the condition (1) holds and g is surjective, there are $x_0, x_1 \in \widetilde{X}$ satisfying $(g(x_0), Sx_0) \in E(G)$ and $g(x_1) = Sx_0$. By Lemma 2.4, there is a sequence $\{x_k\}_{k\in\mathbb{N}\cup\{0\}}$ in $\widetilde{X}$ such that $(g(x_k), g(x_{k+1})) \in E(G)$, $g(x_k) = Sx_{k-1}$, and $\{g(x_k)\}$ is a Cauchy sequence. As $\widetilde{X}$ is complete, $\{g(x_k)\}$ approaches to a point w in $\widetilde{X}$. Again using the fact that g is surjective, $g(u) = w$ for some $u \in \widetilde{X}$. Hence the Property A gives a subsequence $\{g(x_{n_k})\}$

with $(g(x_{n_k}), g(u)) \in E(G)$ for any $k \in \mathbb{N}$. We claim that $g(u) = Su$. For each $g(x_{n_k+1})$, we have

$$\begin{aligned}
\rho(Su, g(x_{n_k+1})) &= \rho(Su, Sx_{n_k}) \\
&\leq \left(\frac{\rho(g(u), Sx_{n_k}) + \rho(g(x_{n_k}), Su)}{\rho(g(u), Su) + \rho(g(x_{n_k}), Sx_{n_k}) + 1} \right) \rho(g(u), g(x_{n_k})) \\
&= \left(\frac{\rho(g(u), g(x_{n_k+1})) + \rho(g(x_{n_k}), Su)}{\rho(g(u), Su) + \rho(g(x_{n_k}), g(x_{n_k+1})) + 1} \right) \rho(g(u), g(x_{n_k})).
\end{aligned}$$

Since $\{g(x_{n_k})\}$ and $\{g(x_{n_k+1})\}$ are subsequences of $\{g(x_n)\}$, they converge to $g(u)$ as $n \to \infty$, and hence $\rho(Su, g(u)) = 0$ which concludes that $g(u) = Su$. Moreover, assume that $a, b \in \widetilde{X}$ are such that $g(a) = Sa$, $g(b) = Sb$, and $(g(a), g(b)) \in E(G)$. Then claim that $g(a) = g(b)$ or $\rho(g(a), g(b)) \geq \frac{1}{2}$. Note that

$$\begin{aligned}
\rho(g(a), g(b)) &= \rho(Sa, Sb) \\
&\leq \left(\frac{\rho(g(a), Sb) + \rho(g(b), Sa)}{\rho(g(a), Sa) + \rho(g(b), Sb) + 1} \right) \rho(g(a), g(b)) \\
&= (\rho(g(a), Sb) + \rho(g(b), Sa)) \, \rho(g(a), g(b)) \\
&= 2 \left[\rho(g(a), g(b)) \right]^2 .
\end{aligned}$$

Therefore $\rho(g(a), g(b)) = 0$ or $\rho(g(a), g(b)) \geq \frac{1}{2}$, completing the proof. ∎

Example 2.6. By Example 2.3, we have that S is a g-Pseudo-Banach contraction. Let $u_0 = 0$. Then $(g(u_0), Su_0) = (0, 0) \in E(G)$. Also X satisfies the Property A. By Theorem 2.5, S has a fixed point.

Remark 2.7. By Example 2.6, we can not use Banach contraction theorem to guarantee the fixed point of S. However, we can apply our theorem to guarantee the fixed point of S.

Example 2.8. Suppose $X = \{(0,0), (0,1), (1,0), (1,1)\}$ with the usual Euclidean metric. Let

$$E(G) = \{((0,0), (1,1)), ((0,0), (0,1)), ((0,0), (0,0))\}.$$

Define $S : X \to X$ by

$$Su = \begin{cases} (0,0) & \text{if } u \in \{(0,0), (0,1)\}; \\ (0,1) & \text{if } u \in \{(1,0), (1,1)\}, \end{cases}$$

and $g : X \to X$ by

$$g(0,0) = (0,1), g(0,1) = (0,0), g(1,0) = (1,1), \text{and } g(1,1) = (1,0).$$

We first verify that S is a g-Pseudo-Banach contraction. If $(g(u), g(v)) = ((0,0), (1,1))$, then $u = (0,1)$ and $v = (1,0)$. Hence $Su = (0,0)$ and $Sv = (0,1)$ which implies that $(Su, Sv) = ((0,0), (0,1)) \in E(G)$. Also, we can see that $\rho(g(u), g(v)) = \sqrt{2}$, $\rho(Su, Sv) = 1$, and

$$\alpha = \frac{\rho((0,0), (0,1)) + \rho((1,1), (0,0))}{\rho((0,0), (0,0)) + \rho((1,1), (0,1)) + 1} = \frac{1 + \sqrt{2}}{2},$$

and so

$$\rho(Su, Sv) \leq \alpha \rho(g(u), g(v)).$$

If $(g(u), g(v)) = ((0,0),(0,1))$, then $u = (0,1)$ and $v = (0,0)$ and hence $Su = (0,0)$ and $Sv = (0,0)$ which implies that $(Su, Sv) = ((0,0),(0,0)) \in E(G)$. We also note that $\rho(g(u), g(v)) = 1$, $\rho(Su, Sv) = 0$, and so

$$\rho(Su, Sv) \leq \alpha\rho(g(u), g(v)).$$

If $(g(u), g(v)) = ((0,0),(0,0))$, then $u = (0,1)$ and $v = (0,1)$. Hence $Su = (0,0) = Sv$ which induces $(Su, Sv) = ((0,0),(0,0)) \in E(G)$. It is thus obvious that

$$\rho(Su, Sv) \leq \alpha\rho(g(u), g(v)).$$

Therefore, S is a g-Pseudo-Banach contraction. Let $u_0 = (0,1)$. It follows that $(g(u_0), Su_0) \in E(G)$. Also, X satisfies the Property A and hence Theorem 2.5 guarantees the existence of a fixed point set of S, namely $F(S) = \{(0,0)\}$.

Now, we introduce a new contraction for multi-valued mapping in a metric space endowed with a graph by combining the concept of g-graph-preserving given by Tiammee and Suantai [16] and the idea of the multi-valued contraction defined by Khojasteh et al. [21] as the following definition.

Definition 2.9. Suppose (X, ρ) is a metric space, $G = (V(G), E(G))$ is a directed graph such that $V(G) = X$ and g is a self-mapping on X. A mapping $S : X \to CB(X)$ is a g-Pseudo-Banach contraction if it satisfies the following conditions:

(1) S is a g-graph-preserving, that is, for each $x, y \in X$ with $(g(x), g(y)) \in E(G)$, $(u, v) \in E(G)$ for all $u \in Sx, v \in Sy$;

(2) for $(g(x), g(y)) \in E(G)$,

$$\mathcal{H}(Sx, Sy) \leq \left(\frac{\rho(g(x), Sy) + \rho(g(y), Sx)}{\rho(g(x), Sx) + \rho(g(y), Sy) + 1}\right)\rho(g(x), g(y)),$$

for all $x, y \in X$.

The next example will show a mapping which satisfies Definition 2.9.

Example 2.10. Let $X = [0,1]$ be endowed with the Euclidean metric, and let $g : X \to X$ be defined by $g(x) = \sqrt{x}$. Consider the directed graph defined by $V(G) = X$ and

$$E(G) = \left\{(u,u), \left(0, \frac{u}{2}\right), \left(\frac{u}{2}, 0\right) : u \in X\right\}.$$

Suppose $S : X \to CB(X)$ is defined by

$$Su = \begin{cases} \left\{0, \frac{1}{2}\right\} & \text{if } 0 \leq u \leq \frac{1}{4}; \\ \left\{0, \frac{\sqrt{u}}{2}\right\} & \text{if } \frac{1}{4} < u \leq 1. \end{cases}$$

We will show that S is a g-Pseudo-Banach constraction.

Let $(g(a), g(b)) \in E(G)$. If $(g(a), g(b)) = (u, u)$ for $u \in X$, then $(a, b) = (u^2, u^2)$. If $0 \leq u \leq \dfrac{1}{2}$, then we have $0 \leq u^2 \leq \dfrac{1}{4}$. So, $Sa = Sb = \left\{0, \frac{1}{2}\right\}$ which yields $(0,0), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right) \in E(G)$. If $\dfrac{1}{2} < u \leq 1$, then $\dfrac{1}{4} < u^2 \leq 1$. It follows that $Sa = Sb = \left\{0, \frac{\sqrt{u^2}}{2}\right\} = \left\{0, \frac{u}{2}\right\}$ and so $(0,0), \left(0, \frac{u}{2}\right), \left(\frac{u}{2}, 0\right), \left(\frac{u}{2}, \frac{u}{2}\right) \in E(G)$. Also,

$$\mathcal{H}(Sa, Sb) = 0 \leq \left(\frac{\rho(g(a), Sb) + \rho(g(b), Sa)}{\rho(g(a), Sa) + \rho(g(b), Sb) + 1}\right)\rho(g(a), g(b)).$$

If $(g(a), g(b)) = \left(0, \frac{u}{2}\right)$ for $u \in X$, then $(a, b) = (0, \frac{u^2}{4})$. Note that $b \in \left[0, \dfrac{1}{4}\right]$. It follows that $Sa = Sb = \left\{0, \frac{1}{2}\right\}$ and hence we obtain $(0,0), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right) \in E(G)$. Also,

$$\mathcal{H}(Sa, Sb) = 0 \leq \left(\frac{\rho(g(a), Sb) + \rho(g(b), Sa)}{\rho(g(a), Sa) + \rho(g(b), Sb) + 1}\right) \rho(g(a), g(b)).$$

If $(g(a), g(b)) = \left(\frac{u}{2}, 0\right)$ for $u \in X$, then $(a, b) = (\frac{u^2}{4}, 0)$. Note that $a \in \left[0, \dfrac{1}{4}\right]$. It follows that $Sa = Sb = \left\{0, \frac{1}{2}\right\}$ and hence $(0,0), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right) \in E(G)$. Also,

$$\mathcal{H}(Sa, Sb) = 0 \leq \left(\frac{\rho(g(a), Sb) + \rho(g(b), Sa)}{\rho(g(a), Sa) + \rho(g(b), Sb) + 1}\right) \rho(g(a), g(b)).$$

Therefore S is g-Pseudo-Banach contraction.

For the second part of this section, the next two lemmas will be used to prove the main theorem for multi-valued mapping.

Lemma 2.11. [6] *Suppose (X, ρ) is a metric space, $A, B \in CB(X)$ and $a \in A$, then for each $\epsilon > 0$, there is $b \in B$ such that*

$$\rho(a, b) < \mathcal{H}(A, B) + \epsilon.$$

Lemma 2.12. *Suppose (X, ρ) is a metric space with a directed graph G, a self-mapping g on X is a surjection and a mapping $S : X \to CB(X)$ is a g-Pseudo-Banach contraction. If $(g(x_0), u) \in E(G)$ for some $x_0 \in X$ and for some $u \in Sx_0$, then there exists a sequence $\{x_k\}_{k \in \mathbb{N} \cup \{0\}}$ in X with the properties :*

(1) *$\{g(x_k)\}_{k \in \mathbb{N} \cup \{0\}}$ is a Cauchy sequence;*
(2) *$(g(x_k), g(x_{k+1})) \in E(G)$ for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. Suppose there is $x_0 \in X$ such that $(g(x_0), u) \in E(G)$ for some $u \in Sx_0$. By surjectivity of g, there is $x_1 \in X$ such that $g(x_1) = u$. By assumption, we have $(g(x_0), g(x_1)) \in E(G)$. By Lemma 2.11, there is $x_2 \in X$ such that $g(x_2) \in Sx_1$ and

$$\begin{aligned}
\rho(g(x_1), g(x_2)) &\leq \mathcal{H}(Sx_0, Sx_1) + \left(\frac{1}{\sqrt{\beta_1}} - 1\right) \mathcal{H}(Sx_0, Sx_1) \\
&= \frac{1}{\sqrt{\beta_1}} \mathcal{H}(Sx_0, Sx_1) \\
&\leq \sqrt{\beta_1} \rho(g(x_0), g(x_1)),
\end{aligned}$$

where

$$\beta_1 = \frac{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1)}{\rho(g(x_0), Sx_0) + \rho(g(x_1), Sx_1) + 1} < 1.$$

Since S is a $g-$graph-preserving, it follows that $(g(x_1), g(x_2)) \in E(G)$. By Lemma 2.11 and surjectivity of g, there is $x_3 \in X$ such that $g(x_3) \in Sx_2$ and

$$\begin{aligned}
\rho(g(x_2), g(x_3)) &\leq \mathcal{H}(Sx_1, Sx_2) + \left(\frac{1}{\sqrt{\beta_2}} - 1\right) \mathcal{H}(Sx_1, Sx_2) \\
&= \frac{1}{\sqrt{\beta_2}} \mathcal{H}(Sx_1, Sx_2) \\
&\leq \sqrt{\beta_2} \rho(g(x_1), g(x_2)),
\end{aligned}$$

where

$$\beta_2 = \frac{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2)}{\rho(g(x_1), Sx_1) + \rho(g(x_2), Sx_2) + 1} < 1.$$

Continuing on this fashion, for each $n \in \mathbb{N} - \{1\}$, we get $(g(x_{n-1}), g(x_n)) \in E(G)$, and

$$\begin{aligned}\rho(g(x_n), g(x_{n+1})) &\leq \mathcal{H}(Sx_{n-1}, Sx_n) + \left(\frac{1}{\beta_n} - 1\right)\mathcal{H}(Sx_{n-1}, Sx_n) \\ &= \frac{1}{\sqrt{\beta_{n-1}}}\mathcal{H}(Sx_{n-1}, Sx_n) \\ &\leq \sqrt{\beta_{n-1}}\rho(g(x_{n-1}), g(x_n)),\end{aligned}$$

where

$$\beta_n = \frac{\rho(g(x_{n-1}), Sx_{n-1}) + \rho(g(x_n), Sx_n)}{\rho(g(x_{n-1}), Sx_{n-1}) + \rho(g(x_n), Sx_n) + 1} < 1.$$

We claim that $\{g(x_n)\}$ is a Cauchy sequence in X. Since $\beta_{n+1} \leq \beta_n$ for all n, it yields

$$\begin{aligned}\rho(g(x_n), g(x_{n+1})) &\leq \sqrt{\beta_n}\rho(g(x_{n-1}), g(x_n)) \\ &\leq \sqrt{\beta_n}\cdots\sqrt{\beta_2}\sqrt{\beta_1}\rho(g(x_0), g(x_1)) \\ &\leq \beta_1^{\frac{n}{2}}\rho(g(x_0), g(x_1)).\end{aligned}$$

Therefore, we have

$$\sum_{n=1}^{\infty}(\rho(g(x_n), g(x_{n+1}))) \leq \rho(g(x_0), g(x_1))\sum_{n=1}^{\infty}\beta_1^{\frac{n}{2}} < \infty,$$

and hence $\rho(g(x_n), g(x_{n+1}))$ approaches to zero. This guarantees $\{g(x_n)\}$ is a Cauchy sequence. ∎

Theorem 2.13. *For $(\widetilde{X}, \rho)$ with a directed graph G, let $g : \widetilde{X} \to \widetilde{X}$ be a surjection and $S : \widetilde{X} \to CB(\widetilde{X})$ be a g-Pseudo-Banach contraction. If the following conditions hold:*

(1) *$(g(x_0), u) \in E(G)$ for some $x_0 \in \widetilde{X}$ and for some $u \in Sx_0$;*
(2) *The Property A holds in $\widetilde{X}$,*

then $g(v) \in Sv$ for some $v \in \widetilde{X}$.

Proof. By (1), set $x_0 \in \widetilde{X}$ such that $(g(x_0), g(x_1)) \in E(G)$ for some $g(x_1) \in S_{x_0}$. By Lemma 2.12, we can construct a sequence there exists a sequence in $\widetilde{X}$, namely $\{x_k\}_{k\in\mathbb{N}\cup\{0\}}$, such that $g(x_k) \in Sx_{k-1}$, $(g(x_k), g(x_{k+1})) \in E(G)$, and $\{g(x_k)\}$ is a Cauchy sequence. As $\widetilde{X}$ is complete, $\{g(x_k)\}$ converges to a point w for some $w \in \widetilde{X}$. Since g is surjective and $w \in \widetilde{X}$, it surely has $v \in \widetilde{X}$ so that $g(v) = w$. By (2), there is a subsequence $\{g(x_{n_k})\}$ of $\{g(x_n)\}$ such that $(g(x_{n_k}), g(v)) \in E(G)$ for all $k \in \mathbb{N}$. We claim that $g(v) \in Sv$. For each $g(x_{n_k+1})$, we have

$$\begin{aligned}\rho(g(v), Sv) &\leq \rho(g(v), g(x_{n_k+1})) + \rho(g(x_{n_k+1}), Sv) \\ &\leq \rho(g(v), g(x_{n_k+1})) + \mathcal{H}(Sx_{n_k}, Sv) \\ &\leq \rho(g(v), g(x_{n_k+1})) \\ &\quad + \left(\frac{\rho(g(x_{n_k}), Sv) + \rho(g(v), Sx_{n_k})}{\rho(g(x_{n_k}), Sx_{n_k}) + \rho(g(v), Sv) + 1}\right)\rho(g(x_{n_k}), g(v)).\end{aligned}$$

Since $\{g(x_k)\}$ converges, so are $\{g(x_{n_k})\}$ and $\{g(x_{n_k+1})\}$ as subsequences of $\{g(x_k)\}$. It thus forces $\rho(g(v), Sv) = 0$. Using the fact that Sv is closed, we have $g(v) \in Sv$, completing the proof. ∎

The following example is illustrated the Theorem 2.13.

Example 2.14. It is already shown that the mapping S in Example 2.10 is a g-Pseudo-Banach constraction. Let $x_0 = 0$, then the condition (1) of Theorem 2.13 is satisfied. Moreover, the space X endowed with the graph G in Example 2.10 satisfies the condition (2) of Theorem 2.13. So by Theorem 2.13, S has a coincidence point $v \in X$, namely $v = 0$.

Theorem 2.13 deduces the next corollary by setting $g = i_X$.

Corollary 2.15. *For $(\widetilde{X}, \rho)$, let $G = (V(G), E(G))$ be a directed graph such that $V(G) = \widetilde{X}$. Suppose $S : \widetilde{X} \to CB(\widetilde{X})$ is a g-Pseudo-Banach contraction satisfying the following properties:*

(1) *the set $\widetilde{X}_S = \{x \in \widetilde{X} \mid (g(x), y) \in E(G)$ for some $y \in Sx\} \neq \emptyset$;*
(2) *$\widetilde{X}$ has the Property A,*

then the mapping S has a fixed point.

3. Application

As application, in the last section we prove coincidence point and fixed point results in a complete metric space endowed with a partial order.

Definition 3.1. Suppose $(X, \leq)$ is a POSET (partially ordered set). For $a, b \in X$, we write $a < b$ whenever $a \leq b$ and $a \neq b$. For subsets A, B of X with the property that $a < b$ for all $a \in A, b \in B$, we write $A \prec B$. Then $A \prec B$ if $a \leq b$ for any $a \in A,\ b \in B$.

For convenience, denote $(\widetilde{X}, \rho, \leq)$ a complete metric space with a partial order $\leq$.

Theorem 3.2. *For $(\widetilde{X}, \rho, \leq)$, let $g : \widetilde{X} \to \widetilde{X}$ be a surjective self-map, and a map $S : \widetilde{X} \to CB(\widetilde{X})$. If the followings hold:*

(1) *$Sx \prec Sy$ for any $x, y \in \widetilde{X}$ with $g(x) < g(y)$;*
(2) *$g(x_0) < u$ for some $x_0 \in \widetilde{X}$ and for some $u \in Sx_0$;*
(3) *for every sequence $\{x_k\}$ which $g(x_k) < g(x_{k+1})$ for all $k \in \mathbb{N}$ and $g(x_k)$ converges to $g(x)$ for some $x \in \widetilde{X}$, $g(x_k) < g(x)$ for all $k \in \mathbb{N}$;*
(4)
$$\mathcal{H}(Sx, Sy) \leq \left(\frac{\rho(g(x), Sy) + \rho(g(y), Sx)}{\rho(g(x), Sx) + \rho(g(y), Sy) + 1} \right) \rho(g(x), g(y)),$$
for all $x, y \in \widetilde{X}$ with $g(x) < g(y)$,

then there exists $u \in \widetilde{X}$ such that $g(u) \in Su$. If g is a injection and distance between two points in $\widetilde{X}$ less than $\frac{1}{2}$, then there is an only one $u \in \widetilde{X}$ for which $g(u) \in Su$.

Proof. Before applying the Theorem 2.13, we first define a graph $G = (V(G), E(G))$ with $V(G) = \widetilde{X}$ and $E(G) = \{(x, y) \mid x < y\}$. It is obvious that $g(x) < g(y)$ for every $x, y \in \widetilde{X}$ with $(g(x), g(y)) \in E(G)$. Hence by (1) it yields $Sx \prec Ty$ and so $u < v$ for $u \in Sx$ and

$v \in Sy$. It follows that $(u, v) \in E(G)$. By (2), there are $x_0 \in \widetilde{X}$ and $u \in Sx_0$ such that $(g(x_0), u) \in E(G)$. So it satisfies the property (1) in Theorem 2.13. Moreover, from (3) we obtain the property (2) of Theorem 2.13. By (1) and (4), we have S is a g-Pseudo-Banach contraction. Therefore, the result of this theorem is followed by Theorem 2.13. It remains to show that the fixed point is unique if g is injective and $\rho(x, y) < \frac{1}{2}$ for all $x, y \in \widetilde{X}$. Assume that g is injective and for all $x, y \in \widetilde{X}$, $\rho(x, y) < \frac{1}{2}$. Let $u, v \in \widetilde{X}$ be such that $g(u) \in Su$ and $g(v) \in Sv$. Suppose to the contrary that $u \neq v$. Since g is injective and $u \neq v$, we have without loss of generality that $g(u) < g(v)$ and this yields $\rho(g(u), g(v)) > 0$. However, since $g(u) \in Su$ and $g(v) \in Sv$, we have $\rho(g(u), Su) = \rho(g(v), Sv) = 0$. Thus

$$\begin{aligned}\rho(g(u), g(v)) &\leq \mathcal{H}(Su, Sv) \\ &\leq (\rho(g(u), Sv) + \rho(g(v), Su))\, \rho(g(u), g(v)) \\ &\leq 2\rho(g(u), g(v)) \cdot \rho(g(u), g(v)).\end{aligned}$$

It follows that $\rho(g(u), g(v)) \in (-\infty, 0] \cup [\frac{1}{2}, \infty)$. But now we $\rho(g(u), g(v)) < \frac{1}{2}$, then $\rho(g(u), g(v)) = 0$, a contradiction. Therefore, we have $u = v$. ∎

The following corollary is a consequence of Theorem 3.2 by setting $g = i_{\widetilde{X}}$.

Corollary 3.3. *For $(\widetilde{X}, \rho, \leq)$, let $S : \widetilde{X} \to CB(\widetilde{X})$. If the followings hold:*

(1) *$x < y \Rightarrow Sx \prec Sy$, for any $x, y \in \widetilde{X}$;*
(2) *there is $x_0 \in \widetilde{X}$ and $u \in Sx_0$ such that $x_0 < u$;*
(3) *for each sequence $\{x_k\}$ such that $x_k < x_{k+1}$ for all $k \in \mathbb{N}$ and $x_k \to x$ for some $x \in \widetilde{X}$, then $x_k < x$ for all $k \in \mathbb{N}$;*
(4) *for all $x, y \in \widetilde{X}$ with $x < y$,*

$$\mathcal{H}(Sx, Sy) \leq \left(\frac{\mathcal{D}(x, Sy) + \mathcal{D}(y, Sx)}{\mathcal{D}(x, Sx) + \mathcal{D}(y, Sy) + 1} \right) \rho(x, y),$$

then $u \in Su$ for some $u \in \widetilde{X}$. Moreover, if distance between any two points $x, y \in \widetilde{X}$ less than $\frac{1}{2}$, then S has a unique fixed point.

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