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On the Convergence of an Iterative Method for Solving Linear Complementarity Problem with WGPSBD Matrix

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Abstract In this paper we propose an iterative and descent type interior point method to compute solution of linear complementarity problem LCP(q, A) given that A is real square matrix and q is a real vector. The linear complementarity problem includes many of the optimization problems and applications. In this context we consider the class of weak generalized positive subdefinite matrices (WGPSBD) which is a generalization of the class of generalized positive subdefinite (GPSBD) matrices. Though Lemke's algorithm is frequently used to solve small and medium size LCP(q, A), Lemke's algorithm does not compute solution of all problems. It is known that Lemke's algorithm is not a polynomial time bound algorithm. We show that the proposed algorithm converges to the solution of LCP(q, A) where A belongs to WGPSBD class. A numerical example is illustrated to show the performance of the proposed algorithm.

MSC: 90C51; 90C90

Keywords: Interior point algorithm; weak generalized positive subdefinite matrices (WGPSBD); generalized positive subdefinite matrices (GPSBD); linear complementarity problem

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1. Introduction

In this paper we introduce an iterative descent approach of an interior point method to compute the solution of linear complementarity problem LCP(q, A) where A is real square matrix and q is a real vector. The class of weak generalized positive subdefinite matrices (WGPSBD) is a generalization of the class of genralized positive subdefinite (GPSBD) matrices [1]. We study this algorithm for solving LCP(q, A) under $A \in$ WGPSBD class. Lemke's algorithm is well known to solve an LCP(q, A). However, Lemke's algorithm does not consider all problems. Interior point method is another approach to solve linear complementarity problem. Fathi [2] showed that if A is positive semidefinite matrix

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then LCP(q, A) is solvable in polynomial time. For details of the interior point method, see [3], [4] and references cited therein. Here we consider an interior point method to compute solution of LCP(q, A) where A belongs to WGPSBD class. The proposed algorithm is useful to compute solution of a large linear complementarity problem. In recent years, research in complementarity problems has received attention to develop efficient algorithms for solving the linear complementarity problem. The linear complementarity problem has wide application in the field of optimization. The computational method for solving linear complementarity problem can broadly be divided into two approches, pivotal method which includes Lemke's algorithm, Criss-cross method [5] and iterative method which includes ellipsoid method [6], path-following method [7], projective method [8], differentiable optimization based descent methods [9].

In Section 2, some results are presented that are used in the next sections. In Section 3, we propose an algorithm based on interior point method to solve linear complementarity problem. We establish some new results related with the proposed interior point method. We prove that the proposed algorithm converges to the solutions of the problem. Finally, we consider a numerical example to illustrate the performance of the proposed algorithm.

2. Preliminaries

R^n_{++} donotes the positive orthant in R^n. For any matrix $A \in R^{n\times n}$, A^T denotes its transpose. x_i denotes the i^{th} coordinate of the vector x. Also x^T denotes the transpose of x. $\|x\|$ denotes the norm of the vector x.

Now we start with the definition of linear complementarity problem. Suppose that a square matrix A of order n and an n dimensional vector q, we have to find n dimensional vectors u and v satisfying

$$v - Au = q, \quad u \geq 0, v \geq 0 \tag{2.1}$$
$$u^T v = 0. \tag{2.2}$$

(2.1) indicates the feasibility of the problem and (2.1), (2.2) jointly indicate the solution of the problem.

Martos [10] proposed positive subdefinite (PSBD) matrices to address pseudo-convex functions. The nonsymmetric PSBD matrices was studied to connect generalized monotonicity and the linear complementarity problem. Later Crouzeix and Komlósi [1] enlarged PSBD class by introducing the class of GPSBD matrices. This class was studied in the context of the processability of linear complementarity problem by Lemke's algorithm. We recall that A is called *PSBD matrix* if for all $u \in R^n$, $u^T Au < 0$ implies $A^T u$ is unisigned. A matrix $A \in R^{n\times n}$ is called GPSBD [1, 11] if $\exists\ e_i \geq 0$ and $f_i \geq 0$ with $e_i + f_i = 1,\ i = 1, 2, \ldots, n$ such that

$$\forall u \in R^n, u^T Au < 0 \Rightarrow \begin{cases} \text{either} & -e_i u_i + f_i(A^T u)_i \geq 0 \text{ for all } i, \\ \text{or} & -e_i u_i + f_i(A^T u)_i \leq 0 \text{ for all } i. \end{cases} \tag{2.3}$$

A matrix $A \in R^{n\times n}$ is called WGPSBD [12] if $\exists\ e_i \geq 0$ and $f_i \geq 0$ with $e_i + f_i = 1,\ i = 1, 2, \ldots, n$ such that

$$\forall u \in R^n, u^T Au < 0 \Rightarrow \begin{cases} \text{either} & -e_i u_i + f_i(A^T u)_i \geq 0 \text{ for at least } (n-1) \text{ coordinates}, \\ \text{or} & -e_i u_i + f_i(A^T u)_i \leq 0 \text{ for at least } (n-1) \text{ coordinates}. \end{cases}$$

Example 2.1. Let us consider the matrix,

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 8 & 0 & 1 & -1 \\ 4 & 0 & -1 & 1 \end{pmatrix}$$

Then for $u = [1, 1, -1, -1]^T$, $u^T Au < 0$ and

$$(-E + FA^T)u = \begin{pmatrix} 11e_1 - 12 \\ -e_2 \\ 1 \\ 1 \end{pmatrix}.$$

So for $e_2 = 0$, the matrix A satisfies the definition of WGPSBD. Hence $A \in$ WGPSBD.

Theorem 2.2 ([13]). *Suppose $u > 0$ such that $v = q + Au > 0$, $\kappa > n$ and $\psi : R^n_{++} \times R^n_{++} \to R$ such that $\psi(u, v) = \kappa\, log(u^T v) - \sum_{i=1}^n\; log\; (u_i v_i)$. Then*

$$\psi(u, v) \geq (\kappa - n) log(u^T v).$$

Proof. Note that

$$\begin{aligned} \psi(u, v) \quad &= \kappa \log(u^T v) - \textstyle\sum_{i=1}^n\; \log\; (u_i v_i) \\ &\geq \kappa \log(u^T v) - \; \log\; \left(\tfrac{1}{n} \textstyle\sum_{i=1}^n (u_i v_i)\right)^n \\ &= (\kappa - n)\log(u^T v) + n\; \log\; n \\ &\geq (\kappa - n)\log(u^T v). \end{aligned}$$

■

Theorem 2.3 ([13]). *Suppose $u > 0$ such that $v = q + Au > 0$, $\kappa > n$ and $\psi : R^n_{++} \times R^n_{++} \to R$ such that $\psi(u, v) = \kappa\, log(u^T v) - \sum_{i=1}^n\; log\; (u_i v_i)$. Then*

$$(\nabla_u \psi(u, v))_i (\nabla_v \psi(u, v))_i = u_i v_i \Big(\frac{\kappa}{u^T v} - \frac{1}{u_i v_i}\Big)^2 \;\forall\, i.$$

Proof. Note that

$$\begin{aligned} \big(\nabla_u \psi(u, v)\big)_i \quad &= \tfrac{\kappa}{u^T v} v_i - \tfrac{1}{u_i v_i} v_i \\ &= v_i \big[\tfrac{\kappa}{u^T v} - \tfrac{1}{u_i v_i}\big]. \end{aligned}$$

Similarly we show

$$\big(\nabla_v \psi(u, v)\big)_i \quad = u_i \big[\tfrac{\kappa}{u^T v} - \tfrac{1}{u_i v_i}\big].$$

Hence $(\nabla_u \psi(u, v))_i (\nabla_v \psi(u, v))_i = u_i v_i \Big(\frac{\kappa}{u^T v} - \frac{1}{u_i v_i}\Big)^2 \;\forall\, i.$ ■

Theorem 2.4 ([13]). *Suppose $u > 0$ such that $v = q + Au > 0$, $\kappa > n$ and $\psi : R^n_{++} \times R^n_{++} \to R$ such that $\psi(u, v) = \kappa\, log(u^T v) - \sum_{i=1}^n\; log\; (u_i v_i)$. Then*

$$(\nabla_u \psi(u, v))^T \nabla_v \psi(u, v) > 0.$$

3. Main Results

Let $u > 0$, $v = q + Au > 0$, $\kappa > n$ and $\psi : R^n_{++} \times R^n_{++} \to R$ such that $\psi(u, v) = \kappa \log(u^T v) - \sum_{i=1}^n \log (u_i v_i) \geq 0$. Todd et al. [14] considered this function in the context of linear programming. We propose an interior point algorithm in line with Pang [13] for finding solution of LCP (q, A) given that A is a WGPSBD $\cap C_0$ matrix. We prove the following results which are required for the proposed algorithm. Let us consider the following index sets,

$$\begin{aligned}
I_1 &= \{i : (\nabla_v \psi(u, v))_i > 0\} \\
I_2 &= \{i : (\nabla_v \psi(u, v))_i < 0\} \\
\breve{I}_1 &= \{i : -e_i(\nabla_v \psi(u, v))_i + f_i(A^T(\nabla_v \psi(u, v)))_i > 0\} \\
\breve{I}_2 &= \{i : -e_i(\nabla_v \psi(u, v))_i + f_i(A^T(\nabla_v \psi(u, v)))_i < 0\} \\
\breve{I}_3 &= \{i : -e_i(\nabla_v \psi(u, v))_i + f_i(A^T(\nabla_v \psi(u, v)))_i = 0\}.
\end{aligned}$$

Theorem 3.1. *Suppose $A \in WGPSBD \cap C_0$ with either $I_1 \cap (\breve{I}_1 \cup \breve{I}_3) \neq \emptyset$ or $I_2 \cap (\breve{I}_2 \cup \breve{I}_3) \neq \emptyset$. Then for $u, v > 0$, $\nabla_u \psi(u, v) + A^T \nabla_v \psi(u, v) \neq 0$.*

Proof. Suppose $\nabla_u \psi(u, v) + A^T \nabla_v \psi(u, v) = 0$. It follows that

$$(\nabla_v \psi(u, v))_i (A^T \nabla_v \psi(u, v))_i = -(\nabla_u \psi(u, v))_i (\nabla_v \psi(u, v))_i < 0 \; \forall \, i.$$

We consider following three cases (**Case 1, Case 2, Case 3**).

Case 1: $I_2 = \emptyset$. Since $A \in C_0$, we have

$$(\nabla_v \psi(u, v))^T A(\nabla_v \psi(u, v)) = (\nabla_v \psi(u, v))^T A^T (\nabla_v \psi(u, v)) \geq 0.$$

Hence $\max_i [(\nabla_v \psi(u, v))_i (A^T(\nabla_v \psi(u, v)))_i] \geq 0, \forall i$.

Case 2: $I_1 = \emptyset$. Again as $A \in C_0$, we have

$$(-(\nabla_v \psi(u, v)))^T A^T (-(\nabla_v \psi(u, v))) = (\nabla_v \psi(u, v))^T A^T (\nabla_v \psi(u, v)) \geq 0.$$

Hence $\max_i [(\nabla_v \psi(u, v))_i (A^T(\nabla_v \psi(u, v)))_i] \geq 0, \forall i$.

Case 3: Here $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. Suppose

$$\max_{(\nabla_v \psi(u,v))_i \neq \emptyset} [(\nabla_v \psi(u, v))_i (A^T(\nabla_v \psi(u, v)))_i] < 0.$$

Now as $I_1 \cap (\breve{I}_1 \cup \breve{I}_3) \neq \emptyset$ or $I_2 \cap (\breve{I}_2 \cup \breve{I}_3) \neq \emptyset$, by the definition of WGPSBD

$$-e_i(\nabla_v \psi(u, v))_i + f_i(A^T(\nabla_v \psi(u, v)))_i \geq 0, \text{ for at least } (n-1) \text{ coordinates}$$

or

$$-e_i(\nabla_v \psi(u, v))_i + f_i(A^T(\nabla_v \psi(u, v)))_i \leq 0, \text{ for at least } (n-1) \text{ coordinates.}$$

Suppose $-e_i(\nabla_v \psi(u, v))_i + f_i(A^T(\nabla_v \psi(u, v)))_i \geq 0$, for at least $(n-1)$ coordinates. Then for all $i \in I_1 \cap (\breve{I}_1 \cup \breve{I}_3)$,

$$-e_i(\nabla_v \psi(u, v))_i^2 + f_i(\nabla_v \psi(u, v))_i (A^T(\nabla_v \psi(u, v)))_i \geq 0.$$

But as $\max_{(\nabla_v\psi(u,v))_i\neq\emptyset}(\nabla_v\psi(u,v))_i(A^T(\nabla_v\psi(u,v)))_i<0$, this implies

$$-e_i(\nabla_v\psi(u,v))_i^2+f_i(\nabla_v\psi(u,v))_i(A^T(\nabla_v\psi(u,v)))_i<0.$$

Hence we arrive at a contradiction. So

$$\max_{(\nabla_v\psi(u,v))_i\neq\emptyset}(\nabla_v\psi(u,v))_i(A^T(\nabla_v\psi(u,v)))_i\geq 0.$$

Again if $-e_i(\nabla_v\psi(u,v))_i+f_i(A^T(\nabla_v\psi(u,v)))_i\leq 0$, for at least $(n-1)$ coordinates. Then for all $i\in I_2\cap(\breve{I}_2\cup\breve{I}_3)$,

$$-e_i(\nabla_v\psi(u,v))_i^2+f_i(\nabla_v\psi(u,v))_i(A^T(\nabla_v\psi(u,v)))_i\geq 0.$$

But as $\max_{(\nabla_v\psi(u,v))_i\neq\emptyset}[(\nabla_v\psi(u,v))_i(A^T(\nabla_v\psi(u,v)))_i]<0$, this implies

$$-e_i(\nabla_v\psi(u,v))_i^2+f_i(\nabla_v\psi(u,v))_i(A^T(\nabla_v\psi(u,v)))_i<0.$$

Hence this leads to a contradiction. So

$$\max_{(\nabla_v\psi(u,v))_i\neq\emptyset}(\nabla_v\psi(u,v))_i(A^T(\nabla_v\psi(u,v)))_i\geq 0.$$

Therefore, $[(\nabla_v\psi(u,v))_i(A^T(\nabla_v\psi(u,v)))_i]=0,\ \forall\, i$. This contradicts Theorem 2.4. Hence

$$\nabla_u\psi(u,v)+A^T\nabla_v\psi(u,v)\neq 0,$$

for $u,v>0$. ∎

Theorem 3.2. *Suppose u and v are two positive n-vectors and $U=diag(u)$ and $V=diag(v)$ and $B=(U)^{-2}+A^T(V)^{-2}A$ where $A\in R^{n\times n}$. Then B is symmetric positive definite matrix.*

Proof. Note that

$$\begin{aligned}B^T=[(U)^{-2}+A^T(V)^{-2}A]^T &= (U)^{-2}+[A^T(V)^{-2}A]^T\\ &= (U)^{-2}+A^T(V)^{-2}A\\ &= (U)^{-2}+A^T(V)^{-2}A=B.\end{aligned}$$

Hence B is symmetric. Again

$$\begin{aligned}x^TA^T(V)^{-2}Ax &= (Ax)^T(V)^{-2}Ax\\ &= (y)^T(V)^{-2}y.\end{aligned}$$

Since $(y)^T(V)^{-2}y\geq 0,\ \forall\, y\in R^n$, $A^T(V)^{-2}A$ is positive semidefinite. Hence B is positive definite. ∎

We describe an interior point algorithm for solving LCP(q,A) where $A\in$ WGPSBD $\cap\, C_0$ with either $I_1\cap(\breve{I}_1\cup\breve{I}_3)\neq\emptyset$ or $I_2\cap(\breve{I}_2\cup\breve{I}_3)\neq\emptyset$.

Algorithm

Step 1: Let $\beta,\gamma\in(0,1)$ and $\sigma\in(0,\frac{1}{2})$ following armizo type line search step and u^0 be a stictly feasible point of LCP(q,A) and $v^0=q+Au^0>0$.

$$\nabla_u\psi_k=\nabla_u\psi(u^k,v^k),\ \nabla_v\psi_k=\nabla_v\psi(u^k,v^k)$$

and

$$U^k=diag(u^k),\ V^k=diag(v^k).$$

Step 2: Now to find the search direction, consider the following problem

$$\text{minimize } (\nabla_u \psi_k)^T d_u + (\nabla_v \psi_k)^T d_v$$
$$\text{subject to } d_v = A d_u,\ \|(U^k)^{-1} d_u\|^2 + \|(V^k)^{-1} d_v\|^2 \le \beta^2.$$

We apply scaled gradient reduction method to obtain search direction (d_u, d_v).

Step 3: Find m_k to be the smallest $m \ge 0$ integer such that

$$\psi(u^k + \gamma^m d_u^k, v^k + \gamma^m d_v^k) - \psi(u^k, v^k) \le \sigma \gamma^m [(\nabla_u \psi_k)^T d_u^k + (\nabla_v \psi_k)^T d_v^k].$$

Step 4: Set

$$(u^{k+1}, v^{k+1}) = (u^k, v^k) + \gamma^{m_k} (d_u^k, d_v^k).$$

Step 5: If (u^{k+1}, v^{k+1}) satisfies the termination criterion i.e. $(u^{k+1})^T v^{k+1} \le \epsilon$, where $\epsilon > 0$ is a very small quantity, stop else $k = k + 1$.

Now to show (d_u^k, d_v^k) as descent direction for the merit function, we prove the following lemma.

Lemma 3.3. *Suppose $A \in WGPSBD \cap C_0$ with either $I_1 \cap (\breve{I}_1 \cup \breve{I}_3) \neq \emptyset$ or $I_2 \cap (\breve{I}_2 \cup \breve{I}_3) \neq \emptyset$, $u > 0$, $v = q + Au > 0$, $\kappa > n$ and $\psi : R^n_{++} \times R^n_{++} \to R$ such that*

$$\psi(u, v) = \kappa\, log(u^T v) - \sum_{i=1}^{n} log\ (u_i v_i).$$

If there is a pair of vectors (d_u^k, d_v^k) such that $(\nabla_u \psi_k)^T d_u^k + (\nabla_v \psi_k)^T d_v^k < 0$, then there exists $\gamma \in (0, 1)$ such that $\psi(u^k + \gamma^m d_u^k, v^k + \gamma^m d_v^k) - \psi(u^k, v^k) < 0$ where m is a nonnegative integer and (d_u^k, d_v^k) is said to be the descent direction.

Proof. We have $d_u^k = -\frac{(A^k)^{-1} r^k}{\tau_k}$, $d_v^k = A d_u^k$ from the algorithm. According to Theorem 3.1,

$$r^k = \nabla_u \psi_k + A^T \nabla_v \psi_k \neq 0$$

and $A^k = (U^k)^{-2} + A^T (V^k)^{-2} A$ is positive definite by Theorem 3.2. So $\tau_k = \frac{\sqrt{(r^k)^T (A^k)^{-1} r^k}}{\beta}$ is positive. Now we show that $(\nabla_u \psi_k)^T d_u^k + (\nabla_v \psi_k)^T d_v^k < 0$. We derive

$$\begin{aligned}(\nabla_u \psi_k)^T d_u^k + (\nabla_v \psi_k)^T d_v^k &= \left[\nabla_u \psi_k + A^T \nabla_v \psi_k\right]^T d_u^k \\ &= -\tfrac{1}{\tau_k} \left(\sqrt{(r^k)^t (A^k)^{-1} r^k}\right)^2 \\ &= -\tau_k \beta^2 < 0.\end{aligned}$$

Now we consider

$$\psi(u^k + \gamma^m d_u^k, v^k + \gamma^m d_v^k) - \psi(u^k, v^k) \le \sigma \gamma^m [(\nabla_u \psi_k)^T d_u^k + (\nabla_v \psi_k)^T d_v^k].$$

Since $0 < \beta, \gamma, \sigma < 1$, we say

$$\psi(u^k + \gamma^m d_u^k, v^k + \gamma^m d_v^k) - \psi(u^k, v^k) < 0. \qquad \blacksquare$$

We prove the following theorem to show that the proposed algorithm converges to the solution under some defined condition.

Theorem 3.4. *If $A \in WGPSBD \cap C_0$ with either $I_1 \cap (\breve{I}_1 \cup \breve{I}_3) \neq \emptyset$ or $I_2 \cap (\breve{I}_2 \cup \breve{I}_3) \neq \emptyset$ and LCP(q, A) has a strictly feasible solution, then every accumulation point of $\{u^k\}$ is the solution of LCP(q, A) i.e. algorithm converges to the solution.*

Proof. Let us consider the subsequences $\{u^k : k \in \omega\}$. Suppose $\tilde{u}$ is the limit of the subsequence and $\tilde{v} = q + A\tilde{u}$. Again we know $\psi(\tilde{u}, \tilde{v}) < \infty$. So either $\tilde{u}^T\tilde{v} = 0$ or $(\tilde{u}, \tilde{v}) > 0$. For $\tilde{u}^T\tilde{v} = 0$, algorithm converges to the solution. So let us consider that $(\tilde{u}, \tilde{v}) > 0$. Also suppose $\tilde{r}$ and $\tilde{A}$ are the limits of the subsequences $\{r^k : k \in \omega\}$ and $\{A^k : k \in \omega\}$ respectively. Consider τ^k converges to $\tilde{\tau} = \dfrac{\sqrt{\tilde{r}^T\tilde{A}^{-1}\tilde{r}}}{\beta}(> 0)$, where $\tilde{A}$ remains positive definite. $(\tilde{d_u}, \tilde{d_v})$ be the limits of the sequence of direction (d_u^k, d_v^k). So from the algorithm we get

$$\tilde{d_u} = -\frac{\tilde{A}^{-1}\tilde{r}}{\tilde{\tau}}, \qquad \tilde{d_v} = A\tilde{d_u}.$$

Now as $\{\psi(u^{k+1}, v^{k+1}) - \psi(u^k, v^k)\}$ converges to zero and since $\lim_{k\to\infty} m_k = \infty$, $\{(u^{k+1}, v^{k+1}) : k \in \omega\}$ and $\{(u^k + \gamma^{m_k-1}d_u^k, v^k + \gamma^{m_k-1}d_v^k) : k \in \omega\}$ converges to $(\tilde{u}, \tilde{v})$. As m_k is the smallest non-negative integers, we have,

$$\frac{\psi(u^k + \gamma^{m_k-1}d_u^k, v^k + \gamma^{m_k-1}d_v^k) - \psi(u^k, v^k)}{\gamma^{m_k-1}} > -\sigma\beta^2\tau_k.$$

Again on the other hand from the algorithm,

$$\frac{\psi(u^{k+1}, v^{k+1}) - \psi(u^k, v^k)}{\gamma^{m_k}} \leq -\sigma\beta^2\tau_k.$$

Now taking limit $k \to \infty$, From the last two inequalities, we can write,

$$\nabla_u\psi(\tilde{u}, \tilde{v})^T\tilde{d_u} + \nabla_v\psi(\tilde{u}, \tilde{v})^T\tilde{d_v} = -\sigma\tilde{\tau}\beta^2.$$

Again from Lemma 3.3 we know,

$$(\nabla_u\psi_k)^T d_u^k + (\nabla_v\psi_k)^T d_v^k = -\tau_k\beta^2.$$

Hence by taking limit $k \to \infty$, we get

$$\nabla_u\psi(\tilde{u}, \tilde{v})^T\tilde{d_u} + \nabla_v\psi(\tilde{u}, \tilde{v})^T\tilde{d_v} = -\tilde{\tau}\beta^2.$$

This is a contradiction. So our proposed algorithm converges to the solution. ■

4. Numerical Illustration

A numerical example is considered to demonstrate the effectiveness and efficiency of the proposed algorithm. We consider the example of LCP(q, A), where

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 8 & 0 & 1 & -1 \\ 4 & 0 & -1 & 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -10 \\ 5 \end{pmatrix}.$$

It is easy to show that A is not a PSD matrix. However, A satisfies the definitions of WGPSBD $\cap$ C_0-matrix as shown in Example 2.1. We apply proposed algorithm to find solution of the given problem. According to Theorem 3.4 algorithm converges to solution with $u^0, v^0 > 0$. To start with we initialize $\beta = 0.5$, $\gamma = 0.5$, $\sigma = 0.2$, $\kappa = 5$ and $\epsilon = 0.00001$. We set $u^0 = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 2 \end{pmatrix}$ and obtain $v^0 = \begin{pmatrix} 1 \\ 1 \\ 22 \\ 21 \end{pmatrix}$. We define

$$\text{diff}[\psi(u^k, v^k)] = \psi(u^k + \gamma^m d_u^k, v^k + \gamma^m d_v^k) - \psi(u^k, v^k).$$

Iteration (k)	u^k	v^k	$\psi(u^k, v^k)$	d_u^k	d_v^k	$diff[\psi(u^k, v^k)]$
1	$\begin{pmatrix} 3.78 \\ 1.97 \\ 1.92 \\ 2.03 \end{pmatrix}$	$\begin{pmatrix} 0.819 \\ 1.128 \\ 20.166 \\ 20.249 \end{pmatrix}$	13.0076	$\begin{pmatrix} -0.4307 \\ -0.0678 \\ -0.1645 \\ 0.0566 \end{pmatrix}$	$\begin{pmatrix} -0.363 \\ 0.255 \\ -3.667 \\ -1.502 \end{pmatrix}$	-0.0881
2	$\begin{pmatrix} 3.49 \\ 1.79 \\ 2.01 \\ 1.93 \end{pmatrix}$	$\begin{pmatrix} 0.707 \\ 1.237 \\ 18.035 \\ 18.898 \end{pmatrix}$	12.9195	$\begin{pmatrix} -0.58 \\ -0.357 \\ -0.188 \\ -0.192 \end{pmatrix}$	$\begin{pmatrix} -0.223 \\ 0.22 \\ -4.264 \\ -2.702 \end{pmatrix}$	-0.0565
3	$\begin{pmatrix} 3.18 \\ 1.5 \\ 2.09 \\ 1.85 \end{pmatrix}$	$\begin{pmatrix} 0.678 \\ 1.256 \\ 15.662 \\ 17.468 \end{pmatrix}$	12.8631	$\begin{pmatrix} -0.634 \\ -0.575 \\ 0.152 \\ -0.173 \end{pmatrix}$	$\begin{pmatrix} -0.0582 \\ 0.0372 \\ -4.7444 \\ -2.8592 \end{pmatrix}$	-0.0549
4	$\begin{pmatrix} 2.91 \\ 1.28 \\ 2.19 \\ 1.71 \end{pmatrix}$	$\begin{pmatrix} 0.626 \\ 1.265 \\ 13.752 \\ 16.155 \end{pmatrix}$	12.8081	$\begin{pmatrix} -0.537 \\ -0.434 \\ 0.197 \\ -0.282 \end{pmatrix}$	$\begin{pmatrix} -0.1037 \\ 0.0186 \\ -3.82 \\ -2.627 \end{pmatrix}$	-0.0523
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
21	$\begin{pmatrix} 1.485 \\ 0.278 \\ 2.485 \\ 0.619 \end{pmatrix}$	$\begin{pmatrix} 0.208 \\ 0.896 \\ 3.749 \\ 9.076 \end{pmatrix}$	12.3258	$\begin{pmatrix} -0.0697 \\ -0.0465 \\ 0.3637 \\ -0.3224 \end{pmatrix}$	$\begin{pmatrix} -0.0232 \\ 0.0645 \\ 0.1283 \\ -0.965 \end{pmatrix}$	-0.0153
22	$\begin{pmatrix} 1.45 \\ 0.258 \\ 2.304 \\ 0.694 \end{pmatrix}$	$\begin{pmatrix} 0.192 \\ 0.807 \\ 3.208 \\ 9.189 \end{pmatrix}$	12.3105	$\begin{pmatrix} -0.0713 \\ -0.0394 \\ -0.3607 \\ 0.151 \end{pmatrix}$	$\begin{pmatrix} -0.0319 \\ -0.1777 \\ -1.0821 \\ 0.2266 \end{pmatrix}$	-0.0278
23	$\begin{pmatrix} 1.419 \\ 0.238 \\ 2.471 \\ 0.549 \end{pmatrix}$	$\begin{pmatrix} 0.181 \\ 0.839 \\ 3.275 \\ 8.755 \end{pmatrix}$	12.2827	$\begin{pmatrix} -0.0611 \\ -0.0401 \\ 0.3336 \\ -0.2895 \end{pmatrix}$	$\begin{pmatrix} -0.021 \\ 0.0652 \\ 0.134 \\ -0.8676 \end{pmatrix}$	-0.0142
24	$\begin{pmatrix} 1.39 \\ 0.222 \\ 2.301 \\ 0.616 \end{pmatrix}$	$\begin{pmatrix} 0.168 \\ 0.75 \\ 2.802 \\ 8.873 \end{pmatrix}$	12.2685	$\begin{pmatrix} -0.0593 \\ -0.0321 \\ -0.3391 \\ 0.1339 \end{pmatrix}$	$\begin{pmatrix} -0.0272 \\ -0.178 \\ -0.9472 \\ 0.2359 \end{pmatrix}$	-0.0269
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
83	$\begin{pmatrix} 1.0177 \\ 0.0097 \\ 2.0521 \\ 0.0269 \end{pmatrix}$	$\begin{pmatrix} 0.0080 \\ 0.0710 \\ 0.1670 \\ 7.0457 \end{pmatrix}$	11.7529	$\begin{pmatrix} -0.0023 \\ -0.0012 \\ -0.0295 \\ 0.0031 \end{pmatrix}$	$\begin{pmatrix} -0.0011 \\ -0.0254 \\ -0.0510 \\ -0.0234 \end{pmatrix}$	-0.0010
84	$\begin{pmatrix} 1.0173 \\ 0.0095 \\ 2.0539 \\ 0.0254 \end{pmatrix}$	$\begin{pmatrix} 0.0078 \\ 0.0715 \\ 0.1671 \\ 7.0408 \end{pmatrix}$	11.7518	$\begin{pmatrix} -0.0032 \\ -0.0018 \\ 0.0145 \\ -0.0115 \end{pmatrix}$	$\begin{pmatrix} -0.0014 \\ 0.0044 \\ -0.0006 \\ -0.0387 \end{pmatrix}$	-0.0009
85	$\begin{pmatrix} 1.0171 \\ 0.0093 \\ 2.0503 \\ 0.0258 \end{pmatrix}$	$\begin{pmatrix} 0.0077 \\ 0.0685 \\ 0.1609 \\ 7.0437 \end{pmatrix}$	11.7508	$\begin{pmatrix} -0.0022 \\ -0.0011 \\ -0.0286 \\ -0.0033 \end{pmatrix}$	$\begin{pmatrix} -0.0010 \\ -0.0243 \\ -0.0491 \\ 0.0233 \end{pmatrix}$	-0.0009
86	$\begin{pmatrix} 1.0166 \\ 0.0091 \\ 2.0521 \\ 0.0245 \end{pmatrix}$	$\begin{pmatrix} 0.0075 \\ 0.0691 \\ 0.1612 \\ 7.0391 \end{pmatrix}$	11.7499	$\begin{pmatrix} -0.0029 \\ -0.0016 \\ 0.0144 \\ -0.0111 \end{pmatrix}$	$\begin{pmatrix} -0.0013 \\ 0.0046 \\ 0.0018 \\ -0.0374 \end{pmatrix}$	-0.0009

TABLE 1. Summary of computation for the proposed algorithm

Table 1 summarizes the computations for the first 4 iterations, 21st-24th iterations, 83rd-86th iterations. It is clear that the sequence $\{u^k\}$ and $\{v^k\}$ produced by the proposed algorithm converges to the solution of the given LCP(q, A) i.e.

$$u^* = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \text{ and } v^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 7 \end{pmatrix},$$

and it is a degenerate solution.

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