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Characterizations of Non-Singular Cycles, Path and Trees

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Abstract: A simple graph is said to be non-singular if its adjacency matrix is non-singular. In this paper, we find the characterization of non-singular cycles and trees. Main Theorems:
1. A cycle C_n of n points is non-singular iff n is not divided by 4.
2. A path P_n is non-singular if and only if n is even.
3. A tree T is non-singular iff T has an even number of points and contains a sesquivalent spanning subgraph.

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1 Basic Knowledge.

Let G be a simple graph whose vertex-set $V(G)$ is the set $\{v_1, v_2, ..., v_n\}$ and let $E(G)$ denote its edge-set. The adjacency matrix $A(G)$ of G is the matrix $[a_{ij}]_{n\times n}$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

We shall usually refer to the eigenvalues of $A(G)$ as the eigenvalues of G and consider G as non-singular if its adjacency matrix is non-singular, see [1].

Example 1.1 *Each complete graph K_n with $n \geq 2$ is non-singular. Indeed,*

$$a_{ij} = \begin{cases} 1 & \text{if} \quad i \neq j \\ 0 & i = j, \end{cases}$$

and hence $\det(A(K_n)) = (-1)^{n-1}(n-1)$.

In contrast, each complete bipartite graph $K_{r,s}$ with $r \geq 2$ or $s \geq 2$ is singular.

Our aim is to characterize non-singular cycles and non-singular trees. To perform this work we will refer to standard results and methods used in matrix theory, see for instance [3]. First of all, let us mention

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Theorem 1.2 [3] *Let $\lambda_0, \ldots, \lambda_{n-1}$ be eigenvalues of an $n \times n$ matrix A. Then*

$$\det(A) = \prod_{r=0}^{n-1} \lambda_r.$$

Thus, the knowledge of eigenvalues is sufficient to calculate the determinant of the matrix. It suffices to know if 0 is an eigenvalue of the adjacency matrix to decide if a graph is singular or is not. It is relatively easy to compute eigenvalues of the so-called circulant graphs, see [2]:

Definition 1.3 *An $n \times n$ matrix A is said to be circulant if $a_{ij} = a_{1k}$ provided that $k = j - i + 1$ (mod n). A circulant graph is a graph G whose vertices can be ordered in such a way that the adjacency matrix $A(G)$ is a circulant matrix.*

Theorem 1.4 [1] *Suppose $[0, a_2, ..., a_n]$ is the first row of the adjacency matrix of a circulant graph G. Then the eigenvalues of G are*

$$\lambda_r = \sum_{j=2}^{n} a_j \omega^{(j-1)r}, \ r = 0, 1, ..., n-1$$

where $\omega = e^{(2\pi i/n)}$.

To calculate the determinant of the adjacency matrix we can also use the following result due to [4].

Definition 1.5 *A sesquivalent graph is a simple graph, each component of which is regular and has the degree 1 or 2; in the other words, the components are single edges or cycles.*

Definition 1.6 *If Γ is a subgraph of a graph G such that $V(\Gamma) = V(G)$, then Γ is said to be a spanning subgraph of the graph G.*

Theorem 1.7 [4] *Let $A(G)$ be the adjacency matrix of a graph G. Then*

$$det(A(G)) = \sum_{\Gamma} (-1)^{r(\Gamma)} (2)^{s(\Gamma)}$$

where the summation is over all sesquivalent spanning subgraphs Γ of G with $c(\Gamma)$ = the number of components of the graph Γ , $r(\Gamma) = |V(\Gamma)| - c(\Gamma)$ and $s(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + c(\Gamma)$.

2 Characterization of Non-singular Cycles.

It is easy to notice that cycles are circulant graphs. Therefore we can apply Theorem to calculate their eigenvalues.

Lemma 2.1 *Let $n \geq 3$. If $A(C_n)$ is the adjacency matrix of a cycle C_n of n vertices, then its eigenvalues are the numbers $\lambda_r = 2cos(\dfrac{2\pi r}{n})$ where $r = 0, 1, ..., n-1$.*

Proof. Let C_n be a cycle of n vertices. The adjacency matrix $A(C_n)$ is a circulant matrix with the first row $[0, 1, 0, ..., 1]$. Thus, by Theorem 1.4, we have $\lambda_r = \omega^r + \omega^{r(n-1)}$ for each $r \in \{0, 1, ..., n-1\}$. But

$$\begin{aligned} \omega^r &= (cos\frac{2\pi}{n} + isin\frac{2\pi}{n})^r \\ &= cos\frac{2\pi r}{n} + isin\frac{2\pi r}{n} \\ and \quad \omega^{r(n-1)} &= (cos\frac{2\pi}{n} + isin\frac{2\pi}{n})^{r(n-1)} \\ &= (cos\frac{2\pi r(n-1)}{n} + isin\frac{2\pi r(n-1)}{n}) \\ &= (cos(2\pi r - \frac{2\pi r}{n}) + isin(2\pi r - \frac{2\pi r}{n}) \\ &= cos\frac{2\pi r}{n} - isin\frac{2\pi r}{n}. \end{aligned}$$

Then $\lambda_r = 2cos\frac{2\pi r}{n}$ for each $r \in \{0, 1, ..., n-1\}$. □

Theorem 2.2 *A cycle C_n of n vertices is non-singular if and only if n is not divided by 4.*

Proof. Let C_n be a cycle of n vertices. Assume that $det(A(C_n)) = 0$. By Lemma 2.1 and Theorem 1.2, we have $\lambda_r = 0$ for some $r \in \{0, 1, ..., n-1\}$. Then $\frac{2\pi r}{n} = \frac{\pi}{2}m$ for some odd number m. Therefore $r = \frac{mn}{4}$, i.e. 4 divides n. Conversely, assume that $n = 4k$ for some $k \in \mathbb{Z}^+$. Then choose $r = k$. Thus, $\lambda_r = 2cos\frac{2\pi k}{4k} = 2cos\frac{\pi}{2} = 0$ and therefore $\det(A(C_n)) = 0$. □

Not only we can characterize non-singular cycles, but using Theorem 1.7 we can also compute the determinant of the adjacency matrix of all cycles.

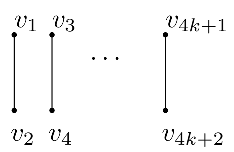
Theorem 2.3 *Let $n \geq 3$. Then*

$$\det(A(C_n)) = \begin{cases} 0 & \textit{if } n = 4k \textit{ for some } k \in \mathbb{Z}^+, \\ -4 & \textit{if } n = 4k+2 \textit{ for some } k \in \mathbb{Z}^+, \\ 2 & \textit{otherwise.} \end{cases}$$

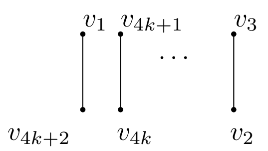
Proof. Case I : If n = 4k, then we use Theorem 2.2

Case II: Let n = 4k + 2 for some $k \in \mathbb{Z}^+$. Then there are exactly three sesquivalent spanning subgraphs of C_n

(i)



(ii)



(iii) The graph C_n itself which is the only subgraph containing a cycle.

For the graphs (i) and (ii), we have $c = 2k+1, r = 2k+1$ and $s = 0$. For the graph (iii), we have $c = 1, r = 4k+1$ and $s = 1$. Therefore, we have $\det(A(C_{4k+2}))= 2[(-1)^{2k+1}2^0] + (-1)^{4k+1}2^1 = -4$.

Case III: If $n = 4k+1$ or $4k+3$, then C_n is the only sesquivalent spanning subgraph of C_n. Thus, $c = 1, r = 4k$ (or $= 4k+2$), and $s = 1$. Therefore, we have $\det(C_{4k+1}) = (-1)^{4k}2^1 = 2 = \det(C_{4k+3})$.

□

As an immediate consequence of the above theorems and Theorem 1.2 we obtain the following formula which is difficult to prove by means of purely analytic methods:

Corollary 2.4 *Let $n \geq 1$. Then*

$$2^n \cdot \prod_{r=0}^{n-1} cos(\frac{2\pi r}{n}) = \begin{cases} 0 & \text{if } n = 4k \text{ for some } k \in \mathbb{Z}^+, \\ -4 & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{Z}^+, \\ 2 & \text{otherwise.} \end{cases}$$

3 Characterization of Non-singular Trees.

Let T be a tree. As usual, we denote by $V(T)$ the set of its vertices and by $E(T)$ the set of its edges. Since we will apply Theorem 1.7 to determine $det(A(T))$, we must identify all sesquivalent spanning subgraphs of the tree T. It turns out that it is not so common that a tree contains such a subgraph.

Lemma 3.1 *If a tree T has a sesquivalent spanning subgraph, then $|V(T)|$ must be even, that is the tree T contains an even number of vertices.*

Proof. No cycle can be a subgraph of a tree. Thus, if P is a sesquivalent spanning subgraph of a tree T, then P consists of a number of separate (single) edges. Each edge contains two (different) vertices, hence P must contain an even number of vertices. If, additionally, P is a spanning subgraph of T, then $V(T) = V(P)$ and hence $|V(T)|$ is even. □

Suppose that $x \in V(T)$ and $N(x) = \{x_1, x_2, ..., x_{n_x}\}$ is the set of all neighbors of x in T. Let us define, for each $x_i \in N(x)$, a subset T_{x_i} of the set $V(T)$:

$$T_{x_i} = \{y : y = x_i \text{ or there is a path from } y \text{ to } x_i \text{ which does not pass x } \}.$$

Now, one can prove

Lemma 3.2 *Let P be a sesquivalent spanning subgraph of the tree T. Then $\{x, x_i\} \in E(P)$ if and only if $|T_{x_i}|$ is odd.*

Proof: It is easy to notice that T_{x_i}'s are disjoint subsets of $V(T)$. Since they are connected subsets of $V(T)$, they may also be regarded as subtrees of the tree T. Let P_i denote the subgraph of P induced on the set T_{x_i}. Now, it is obvious that P_i is a sesquivalent spanning subgraph of the tree T_{x_i} if $\{x, x_i\} \notin E(P)$. If $\{x, x_i\} \in E(P)$, then the subgraph of P induced on the set $T_{x_i} \cup \{x\}$ is a sesquivalent spanning subgraph of the appropriate subgraph of T. Thus, by the above lemma, we conclude that $\{x, x_i\} \in E(P)$ if and only if $|T_{x_i}|$ is odd. □

If P is a sesquivalent spanning subgraph of the tree T, then there is only one $x_i \in N(x)$ such that $\{x, x_i\} \in E(P)$. Thus, by the above lemma, if a tree T has a sesquivalent spanning subgraph, then for every $x \in V(T)$ there is exactly one $x_i \in N(x)$ such that $|T_{x_i}|$ is odd. Let us prove that this is also a sufficient condition for a tree to have a sesquivalent spanning subgraph.

Theorem 3.3 *A tree T has a sesquivalent spanning subgraph if and only if for every $x \in V(T)$ there is exactly one $x_i \in N(x)$ such that $|T_{x_i}|$ is odd.*

Proof: One implication follows from the above lemma. To prove the reverse implication let us assume that T is a tree such that for every $x \in V(T)$ there is exactly one element in $N(x)$, denote this element by $f(x)$, such that $|T_{f(x)}|$ is odd. We define a spanning subgraph P of T taking

$$E(P) = \{\{x, f(x)\} : x \in \ V(T)\}.$$

To prove that P is a sesquivalent subgraph of T, it suffices to show that $f(f(x)) = x$ for each $x \in V(T)$.

Let $x \in V(T)$ and let $N(x) = \{x_1, x_2, ..., x_{n_x}\}$. We know that $f(x) \in N(x)$. It also means that $x \in N(f(x))$. Thus, it suffices to prove that the set

$$(\star) \qquad \{y : y = x \text{ or there is a path from } y \text{ to } x \text{ which does not pass f(x) }\}$$

has an odd number of elements. But any path that joins y with x (and does not pass $f(x)$) must contain an element of $N(x)$ (different from $f(x)$). Thus, the set $(\star)$ equals to

$$(\star\star) \qquad \bigcup\{T_{x_i} : x_i \neq f(x)\} \cup \{x\}.$$

The sets T_{x_i} are disjoint and each of them contains an even number of elements if $x_i \neq f(x)$ (we recall that among the sets T_{x_i} only $T_{f(x)}$ contains an odd number of elements). Moreover, the element x belongs to none of these sets. Therefore, we conclude that the set $(\star\star)$ (and hence $(\star)$ as well) contains an even number of elements which was to be proved. □

Since the right hand side of the above equivalence depends only on the tree T, and it is either fulfilled or is not for any given tree, a sesquivalent spanning subgraph (if it exists) is defined uniquely by Lemma 3.2. Thus, we get

Corollary 3.4 *Every tree has at most one sesquivalent spanning subgraph.*

Now, we can apply Theorem 1.7 to compute the determinant of the adjacency matrx of a tree. We get

Theorem 3.5 *Let T be a tree of n vertices. Then*

$$\det(A(T)) = \begin{cases} 1 & \textit{if } T \textit{ has a sesquivalent spanning subgraph} \\ & \textit{and } n = 4k \textit{ for some } k \in \mathbb{Z}^+, \\ -1 & \textit{if } T \textit{ has a sesquivalent spanning subgraph} \\ & \textit{and } n = 4k{+}2 \textit{ for some } k \in \mathbb{Z}^+ \cup \{\ 0\ \}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Suppose that T has a sesquivalent spanning subgraph P and $n = 4k$. Then the number of components, c of P and the number of edges both are $\frac{n}{2} = \frac{4k}{2} = 2k$. So, $r(P) = n - c = 4k - 2k = 2k$ and $s(P) = 2k - 4k + 2k = 0$. Thus, we get $\det(A(T)) = (-1)^{2k}(2)^0 = 1$.

Similarly, if T has a sesquivalent spanning subgraph P and $n = 4k+2$, then $c = 2k+1$ and hence $r(P) = 2k + 1$ and $s(P) = 0$. Thus, we get $\det(A(T)) = (-1)^{2k+1}(2)^0 = -1$. □

Path is a special case of tree, and a path of n vertices has a sesquivalent spanning subgraph if and only if n is even. Therefore,

Corollary 3.6 *Let P_n be a path of n vertices. Then*

$$\det(A(P_n)) = \begin{cases} (-1)^k & \text{if } n = 2k \text{ for some } k \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Non-singular trees and paths can be now characterized using the following equivalence (see Theorem 3.3, 3.5 and Corollary 3.6 above):

Corollary 3.7 *A tree T is non-singular if and only if T has a sesquivalent spanning subgraph.*

Corollary 3.8 *A path P_n is non-singular if and only if n is even.*

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