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Existence and Uniqueness of Solution for Two-Dimensional Fuzzy Volterra-Fredholm Integral Equation

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Abstract In this work, we prove the existence and uniqueness of the solution for the two-dimensional fuzzy Volterra-Fredholm integral equation under some suitable conditions. Moreover, some properties of the solution are established.

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1. Introduction

The theory of fuzzy integral and differential equations is important in studying and solving a large ratio of the problems in many topics in applied mathematics. Many authors have studied them for several years. The study of fuzzy integral equations begins with the investigations of Kaleva [1], and Seikkala [2]. These studies continued by Friedman et al. [3], Lakshmikantham [4], Wu et al. [5], Song et al. [6], Diamond [7], Bede et al. [8], Salahshour et al. [9], Allahviranloo et al. [10], Khastan [11], Park et al. [12], Nieto et al. [13] .... Recently, the study of fuzzy integral equations in two-dimensional space was initiated. For example, Mirzaee et al. [14] studied the method of approximate the solution of linear two-dimensional fuzzy Fredholm integral equations of the second kind. In [15], Bica et al. constructed the fuzzy trapezoidal cubature rule providing its remainder estimate for the case of Lipschitzian functions and apply an iterative numerical method to approximate the solution of nonlinear fuzzy Fredholm integral equations in two-dimensional space. Similarly, in [15], authors proved the convergence of the proposed iterative numerical method for two-dimensional fuzzy Volterra linear integral equations by providing the error estimate. Sadatrasoul et al. [16] introduced an optimal quadrature formula for classes of two-dimensional fuzzy-number-valued functions of Lipschitz type.

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They also proved the convergence of successive approximations used to approximate the solution of the two-dimensional Hammerstein fuzzy integral equation. Ezzati et al. [17] discussed the existence of a solution for the fuzzy Volterra-Fredholm integral equations of mixed type via Banach's contraction principle.

In this work, we shall prove the existence and uniqueness of the solution for the two-dimensional fuzzy Volterra-Fredholm integral equation.

The remainder of the paper is organized as follows. Section 2 presents some necessary preliminaries of fuzzy analysis that will be used throughout this article. In Section 3, we concern with the existence and uniqueness of the solution for the two-dimensional fuzzy Volterra-Fredholm integral equation via Banach's contraction principle. Moreover, some properties of the solution are established. In section 4 contains an illustrative example for the validity of the obtained results.

2. Preliminaries

Denote by $\mathscr{K}_c(\mathbb{R}^d)$ the collection of all nonempty convex and compact subsets of $\mathbb{R}^d$. The *Hausdorff metric* d_H in $\mathscr{K}_c(\mathbb{R}^d)$ is defined as follows

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{\mathbb{R}^d}, \sup_{b\in B}\inf_{a\in A}\|a-b\|_{\mathbb{R}^d}\},$$

where $A, B \in \mathscr{K}_c(\mathbb{R}^d)$. It is easy to see that the space $(\mathscr{K}_c(\mathbb{R}^d), d_H)$ is a complete metric space.

Denote $E^d = \{u : \mathbb{R}^d \to [0,1]$ such that $x(z)$ satisfies (i)-(iv) stated below$\}$

(i) x is normal, i.e there exists an $z_0 \in \mathbb{R}^d$ such that $x(z_0) = 1$;
(ii) x is fuzzy convex, that is, for $0 \le \lambda \le 1$, $x(\lambda z_1 + (1-\lambda)z_2) \ge \min\{x(z_1), x(z_2)\}$, for any $z_1, z_2 \in \mathbb{R}^d$;
(iii) x is upper semicontinuous;
(iv) $[x]^0 = cl\{z \in \mathbb{R}^d : x(z) > 0\}$ is compact.

Let $\alpha \in (0,1]$, denote by $[x]^\alpha = \{z \in \mathbb{R}^d \,|\, x(z) \ge \alpha\}$. We call this set an *$\alpha$-cut of the fuzzy set* x and defined by $\hat{0} \in E^d$ as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \ne 0$.

The *supremum on* E^d is defined by

$$D(x_1, x_2) = \sup_{\alpha\in[0,1]} d_H([x_1]^\alpha, [x_2]^\alpha)$$

for $x_1, x_2 \in E^d$ and (E^d, D) is a complete metric space.

For $x, y, z \in E^d$ and $\lambda \in \mathbb{R}_+$, we have

$$D(x+z, y+z) = D(x,y), D(\lambda x, \lambda y) = \lambda D(x,y), D(x,y) \le D(x,z) + D(y,z).$$

A mapping $f : [a,b] \to E^d$ is called *integrably bounded* if there exists an integrable function h such that $||x|| \le h(t)$ for all $x \in [f(t)]^0$.

Definition 2.1 ([18])**.** Let $f : [a,b] \to E^d$. The *integral of f over* $[a,b]$, denoted by $\int_a^b f(t)dt$, is defined by

$$\begin{aligned}\left[\int_a^b f(t)dt\right]^r &= \int_a^b [f(t)]^\alpha dt \\ &= \left\{\int_a^b \tilde{f}(t)dt \big| \tilde{f} : J \to \mathbb{R}^d \text{ is a measurable selection for } [f(\cdot)]^\alpha\right\}.\end{aligned}$$

for all $\alpha \in [0,1]$. A strongly measurable and integrably bounded mapping $f : [a,b] \to E^d$ is said to be *integrable* on $[a,b]$ if $\int_a^b f(t)dt \in E^d$.

Next, we have the following some properties of the integral (see details in Lakshmikantham et al. [18]). (1) Let $f, g : [a,b] \to E^d$ be integrable and $\lambda > 0$. Then

i) $\int_a^b (f(s) + g(s))ds = \int_a^b f(s)ds + \int_a^b g(s)ds$.
ii) $\int_a^b \lambda f(s)ds = \lambda \int_a^b f(s)ds$.
iii) $D(f,g)$ is integrable and $D\big(\int_a^b f(s)ds, \int_a^b g(s)ds\big) \leq \int_a^b D(f(s), g(s))ds$.

(2) If $f : [a,b] \to E^d$ is continuous then it is integrable.

(3) If $f : [a,b] \to E^d$ is integrable and $c \in [a,b]$ then $\int_a^b f(s)ds = \int_a^c f(s)ds + \int_c^b f(s)ds$.

Let $J = [a,b] \times [c,d] \subset \mathbb{R} \times \mathbb{R}$ and $C(J, E^d)$ is denoted a space of all continuous functions $f : J \to E^d$ with the supremum metric D^* defined by

$$D^*(f,g) = \sup_{(t,s)\in J} D(f(t,s), g(t,s)).$$

It is easy to see that D^* is a metric in E^d. In fact, (E^d, D^*) is a complete metric space.

Definition 2.2 ([19]). A mapping $f : J \to E^d$ is called *continuous at* $(t_0, s_0) \in J$ if $f_\alpha(t,s) = [f(t,s)]^\alpha$ is continuous at (t_0, s_0) with respect to d_H for all $\alpha \in [0,1]$.

Definition 2.3 ([19]). A mapping $f : J \times E^d \to E^d$ is called *continuous function at* $(t_0, s_0, x_0) \in J \times E^d$ if for any fixed $\alpha \in [0,1]$ and arbitrary $\epsilon > 0$, there exists $\delta(\epsilon, \alpha) > 0$ such that

$$d_H([f(t,s,x)]^\alpha, [f(t_0,s_0,x_0)]^\alpha) < \epsilon$$

whenever $\max\{|t-t_0|, |s-s_0|\} \leq \delta(\epsilon,\alpha)$ and $d_H([x]^r, [x_0]^r) < \delta(\epsilon,\alpha)$, for $(t,s,x) \in J \times E^d$.

Definition 2.4 ([19]). Let $f : J \to E^d$. The *integral of* f *over* J, denote by $\int_a^b \int_c^d f(t,s)dsdt$, is defined levelwise by the expression

$$\left[\int_a^b \int_c^d f(t,s)dsdt\right]^r = \int_a^b \int_c^d [f(t,s)]^\alpha dsdt$$
$$= \left\{ \int_a^b \int_c^d \tilde{f}(t,s)dsdt \middle| \tilde{f} : J \to \mathbb{R}^d \text{ is a measurable selection for } [f(\cdot,\cdot)]^\alpha \right\}.$$

for all $\alpha \in [0,1]$.

3. Main Results

Let $\Omega = [a,b]$ be the given subsets of $\mathbb{R}$. Denote by $C(A,B)$ the class of continuous functions from the set A to the set B and $\Im := [0,T] \times \Omega$. Consider the fuzzy Volterra-Fredholm integral equation in two variables as follows:

$$u(t,s) = h(t,s) + \int_0^t \int_\Omega F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau, \tag{3.1}$$

for all $(t,s) \in \Im$, where

$$(\mathscr{T}u)(t,s) = \int_0^t \int_\Omega K(t,s,\xi,\vartheta,u(\xi,\vartheta))d\xi d\vartheta, \tag{3.2}$$

and $u,h \in C(\Im, E^d)$, $F \in C(\Im \times \Im \times E^d \times E^d, E^d)$, $K \in C(\Im \times \Im \times E^d, E^d)$.

Let Φ be the space of continuous function from $\Im$ into (E^d, D) with $H_1(u,\hat{0}) \le \delta$, that is,

$$\Phi = \{u \,|\, u : \Im \to E^d \text{ is continuous and } \quad H_1(u,\hat{0}) \le \delta\},$$

where

$$H_1(u,v) = \sup_{(t,s)\in\Im} \{D(u(t,s),v(t,s))e^{-\lambda(t+s)}\}, \lambda > 0.$$

It is not difficult to check that $(C(\Im, E^d), H_1)$ is also a complete metric space.

Now, we give a prove of the existence and uniqueness of solution of the problem (3.1) by using the Banach fixed point theorem.

Theorem 3.1. *Let $F \in C(\Im \times \Im \times E^d \times E^d, E^d)$ and $K \in C(\Im \times \Im \times E^d, E^d)$ be satisfy the following conditions:*

*(**c1**) there exists $L \in C(\Im \times \Im, \mathbb{R}_+)$ such that*

$$D\big(F(t,s,\sigma,\tau,u_1,v_1), F(t,s,\sigma,\tau,u_2,v_2)\big) \le L(t,s,\sigma,\tau)\{D(u_1,u_2) + D(v_1,v_2)\},$$

for all $(t,s,\sigma,\tau,u_1,v_1), (t,s,\sigma,\tau,u_1,v_1) \in \Im \times \Im \times E^d \times E^d$;

*(**c2**) there exists $M \in C(\Im \times \Im, \mathbb{R}_+)$ such that*

$$D\big(K(t,s,\sigma,\tau,u), K(t,s,\sigma,\tau,v)\big) \le M(t,s,\sigma,\tau)D(u,v),$$

for all $(t,s,\sigma,\tau,u), (t,s,\sigma,\tau,v) \in \Im \times \Im \times E^d$;

*(**c3**) there exists $\beta_1 > 0$ such that*

$$D(h(t,s),\hat{0}) + \int_0^t \int_\Omega D\big(F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau \le \beta_1 e^{\lambda(t+s)},$$

for all $(t,s,\sigma,\tau) \in \Im \times \Im$;

*(**c4**) there exists $\beta_2 \in (0,1]$ such that*

$$\int_0^t \int_\Omega L(t,s,\sigma,\tau)\Big(e^{\lambda(\sigma+\tau)} + \int_0^\sigma \int_\Omega M(\sigma,\tau,\overline{\sigma},\overline{\tau})e^{\lambda(\overline{\sigma}+\overline{\tau})}d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau \le \beta_2 e^{\lambda(t+s)},$$

for all $(t,s,\sigma,\tau), (t,s,\overline{\sigma},\overline{\tau}) \in \Im \times \Im$

Then the problem (3.1) has a unique solution on Φ, provided that $\beta_1 + \beta_2\delta < 1$.

Proof. Let us the operator $\mathscr{Q} : \Phi \to \Phi$ defined by

$$(\mathscr{Q}u)(t,s) = h(t,s) + \int_0^t \int_\Omega F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau. \tag{3.3}$$

To prove the theorem, we shall divide the proof into three steps.

Step 1. *The operator $\mathscr{Q}$ is continuous.* Indeed, for $(t,s,\sigma,\tau) \in \Im \times \Im, u \in E^d$ and $h,k>0$, we have

$$
\begin{aligned}
&D\big((\mathscr{Q}u)(t+h,s+k),(\mathscr{Q}v)(t,s)\big)\\
&\quad = D\Bigg(h(t+h,s+k)+\int_0^{t+h}\int_\Omega F\big(t+h,s+k,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau,\\
&\qquad\qquad h(t,s)+\int_0^t\int_\Omega F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau\Bigg)\\
&\quad \le D(h(t+h,s+k),h(t,s))\\
&\qquad + D\Bigg(\int_0^{t+h}\int_\Omega F\big(t+h,s+k,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau,\\
&\qquad\qquad \int_0^t\int_\Omega F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau\Bigg)\\
&\quad \le D(h(t+h,s+k),h(t,s))\\
&\qquad + \int_0^{t+h}\int_\Omega D\big(F\big(t+h,s+k,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau\\
&\qquad + \int_0^{t}\int_\Omega D\big(F\big(t+h,s+k,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big),\\
&\qquad\qquad F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)\big)d\sigma d\tau.
\end{aligned}
$$

Since h is continuous function on $\Im$ and the functions F,K are integrable on $\Im\times\Im$, so the right side of the inequality above tends to 0 as $h\to 0$, $k\to 0$. Similar inequalities are obtained for $D\big((\mathscr{Q}u)(t-h,s-k),(\mathscr{Q}v)(t,s)\big)$. Therefore, $\mathscr{Q}$ is a operator continuous.

Step 2. *The operator $\mathscr{Q}$ is maps bounded sets into bounded sets on* Φ. From (3.3) and using the conditions, we get

$$
\begin{aligned}
&D\big((\mathscr{Q}u)(t,s),\hat{0}\big) = D\Bigg(h(t,s)+\int_0^t\int_\Omega F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau,\hat{0}\Bigg)\\
&\quad \le D(h(t,s),\hat{0})+\int_0^t\int_\Omega D\big(F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau\\
&\quad \le D(h(t,s),\hat{0})+\int_0^t\int_\Omega D\big(F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau\\
&\qquad + \int_0^t\int_\Omega L(t,s,\sigma,\tau)\Bigg(D(u(\sigma,\tau),\hat{0})+D\Bigg(\int_0^\sigma\int_\Omega K(\sigma,\tau,\overline{\sigma},\overline{\tau},u(\overline{\sigma},\overline{\tau}))d\overline{\sigma}d\overline{\tau},\\
&\qquad \int_0^\sigma\int_\Omega K(\sigma,\tau,\overline{\sigma},\overline{\tau},\hat{0})d\overline{\sigma}d\overline{\tau}\Bigg)\Bigg)d\sigma d\tau\\
&\quad \le D(h(t,s),\hat{0})+\int_0^t\int_\Omega D\big(F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau\\
&\qquad + \int_0^t\int_\Omega L(t,s,\sigma,\tau)\Big(e^{\lambda(\sigma+\tau)}+\int_0^\sigma\int_\Omega M(\sigma,\tau,\overline{\sigma},\overline{\tau})e^{\lambda(\overline{\sigma}+\overline{\tau})}d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau\\
&\quad \le \big(\beta_1+\beta_2 H_1(u,\hat{0})\big)e^{\lambda(t+s)}.
\end{aligned}
$$

It implies that for all $u \in \Phi$,

$$D\big((\mathscr{Q}u)(t,s),\hat{0}\big)e^{-\lambda(t+s)} \le \beta_1 + \beta_2\delta.$$

Therefore,

$$H_1(\mathscr{Q}u,\hat{0}) \le \beta_1 + \beta_2\delta,$$

for all $u \in \Phi$. Hence, $\mathscr{Q}$ is maps bounded sets into itself.

Step 3. *The operator $\mathscr{Q}$ is a contraction on Φ.* Indeed, let $u, v \in \Phi$ and using the conditions, we have

$$\begin{aligned}
D\big((\mathscr{Q}u)(t,s),(\mathscr{Q}v)(t,s)\big) &= D\Big(h(t,s) + \int_0^t\int_\Omega F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau,\\
&\qquad h(t,s) + \int_0^t\int_\Omega F\big(t,s,\sigma,\tau,v(\sigma,\tau),(\mathscr{T}v)(\sigma,\tau)\big)d\sigma d\tau\Big)\\
&\le \int_0^t\int_\Omega D\big(F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big),F\big(t,s,\sigma,\tau,v(\sigma,\tau),(\mathscr{T}v)(\sigma,\tau)\big)\big)d\sigma d\tau\\
&\le \int_0^t\int_\Omega L(t,s,\sigma,\tau)\Big(D(u(\sigma,\tau),v(\sigma,\tau)) + D\Big(\int_0^\sigma\int_\Omega K(\sigma,\tau,\overline{\sigma},\overline{\tau},u(\overline{\sigma},\overline{\tau}))d\overline{\sigma}d\overline{\tau},\\
&\qquad \int_0^\sigma\int_\Omega K(\sigma,\tau,\overline{\sigma},\overline{\tau},v(\overline{\sigma},\overline{\tau}))d\overline{\sigma}d\overline{\tau}\Big)\Big)d\sigma d\tau\\
&\le H_1(u,v)\int_0^t\int_\Omega L(t,s,\sigma,\tau)\Big(e^{\lambda(\sigma+\tau)} + \int_0^\sigma\int_\Omega M(\sigma,\tau,\overline{\sigma},\overline{\tau})e^{\lambda(\overline{\sigma}+\overline{\tau})}d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau\\
&\le \beta_2 H_1(u,v)e^{\lambda(t+s)}. \qquad (3.4)
\end{aligned}$$

From (3.4), we obtain

$$H_1(\mathscr{Q}u,\mathscr{Q}v) \le \beta_2 H_1(u,v),$$

for all $u, v \in \Phi$ and thus $\mathscr{Q}$ is a contraction operator.

Therefore, by Banach fixed point theorem, we deduce that $\mathscr{Q}$ has a fixed point which is a solution to the problem (3.1). This proof is complete. ■

In the sequel, we shall apply the inequality established in Theorem 1 (see [20]) to study some basic estimaties of the solutions for the problem (3.1).

Theorem 3.2. *Assume that all the conditions of Theorem 3.1 hold and let $L, M : \Im\times\Im \to \mathbb{R}_+$ be a continuous functions, given by $L(t,s,\sigma,\tau) = f(t,s)g(\sigma,\tau)$, $M(t,s,\sigma,\tau) = f(t,s)k(\sigma,\tau)$, where $f, g, k \in C(\Im,\mathbb{R}_+)$. If $u(t,s)$ is a solution of the problem (3.1), then we have*

$$\begin{aligned}
D(u(t,s),\hat{0}) \le C\Big\{1 + f(t,s)\int_0^t\int_\Omega \big(g(\sigma,\tau)+k(\sigma,\tau)\big)\\
\times\exp\Big(\int_\sigma^t\int_\Omega f(\overline{\sigma},\overline{\tau})\big(g(\overline{\sigma},\overline{\tau})+k(\overline{\sigma},\overline{\tau})\big)d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau\Big\},
\end{aligned}$$

where

$$C = \sup_{(t,s)\in\Im}\Big\{D(h(t,s),\hat{0}) + \int_0^t\int_\Omega D\big(F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau\Big\}.$$

Proof. Since $u(t,s)$ is a solution of the problem (3.1) and from (3.1), we have

$$
\begin{aligned}
D(u(t,s),\hat{0}) &= D\Big(h(t,s) + \int_0^t \int_\Omega F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau, \hat{0}\Big) \\
&\le D(h(t,s),\hat{0}) + \int_0^t \int_\Omega D\big(F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau \\
&\quad + \int_0^t \int_\Omega D\big(F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big),F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big)\big)d\sigma d\tau \\
&\le D(h(t,s),\hat{0}) + \int_0^t \int_\Omega D\big(F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau \\
&\quad + \int_0^t \int_\Omega L(t,s,\sigma,\tau)\Big(D(u(\sigma,\tau),\hat{0}) + D\Big(\int_0^\sigma \int_\Omega K(\sigma,\tau,\overline{\sigma},\overline{\tau},u(\overline{\sigma},\overline{\tau}))d\overline{\sigma}d\overline{\tau}, \\
&\quad \int_0^\sigma \int_\Omega K(\sigma,\tau,\overline{\sigma},\overline{\tau},\hat{0})d\overline{\sigma}d\overline{\tau}\Big)\Big)d\sigma d\tau \\
&\le C + f(t,s)\int_0^t \int_\Omega g(\sigma,\tau)\Big(D(u(\sigma,\tau),\hat{0}) \\
&\quad + \int_0^\sigma \int_\Omega k(\overline{\sigma},\overline{\tau})D(u(\overline{\sigma},\overline{\tau}),\hat{0})d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau,
\end{aligned} \tag{3.5}
$$

where

$$
C = \sup_{(t,s)\in\Im}\Big\{D(h(t,s),\hat{0}) + \int_0^t \int_\Omega D\big(F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau\Big\}.
$$

By Theorem 1 in [20] and from (3.5), we see that

$$
\begin{aligned}
D(u(t,s),\hat{0}) \le C\Big\{&1 + f(t,s)\int_0^t \int_\Omega \big(g(\sigma,\tau)+k(\sigma,\tau)\big) \\
&\times \exp\Big(\int_\sigma^t \int_\Omega f(\overline{\sigma},\overline{\tau})\big(g(\overline{\sigma},\overline{\tau})+k(\overline{\sigma},\overline{\tau})\big)d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau\Big\}.
\end{aligned}
$$

This proof is complete. ∎

We want to emphasize that the solution to the problem (3.1) depends continuously on the initial condition and the right-hand side of the equation. Let us consider the problem (3.1) and the corresponding problem

$$
v(t,s) = \widetilde{h}(t,s) + \int_0^t \int_\Omega \widetilde{F}\big(t,s,\sigma,\tau,v(\sigma,\tau),(\widetilde{\mathscr{T}}v)(\sigma,\tau)\big)d\sigma d\tau, \tag{3.6}
$$

for all $(t,s)\in\Im$, where

$$
(\widetilde{\mathscr{T}}v)(t,s) = \int_0^t \int_\Omega \widetilde{K}(t,s,\xi,\vartheta,v(\xi,\vartheta))d\xi d\vartheta, \tag{3.7}
$$

and $v,\widetilde{h}\in C(\Im,E^d)$, $\widetilde{F}\in C(\Im\times\Im\times E^d\times E^d,E^d)$, $\widetilde{K}\in C(\Im\times\Im\times E^d,E^d)$.

Theorem 3.3. *Assume that all the conditions of Theorem 3.1 hold and let* $L, M : \Im \times \Im \to \mathbb{R}_+$ *be a continuous functions, given by* $L(t,s,\sigma,\tau) = f(t,s)g(\sigma,\tau)$, $M(t,s,\sigma,\tau) = f(t,s)k(\sigma,\tau)$, *where* $f, g, k \in C(\Im, \mathbb{R}_+)$. *Let* $u(t,s)$ *and* $v(t,s)$ *be solution of the problems* (3.1) *and* (3.6), *respectively. Suppose that there exist* $\epsilon_1, \epsilon_2 > 0$ *such that*

$$D(h(t,s), \widetilde{h}(t,s)) \le \epsilon_1,$$

for $(t,s) \in \Im$, *and*

$$D\big(F\big(t,s,\sigma,\tau,A,B\big), \widetilde{F}\big(t,s,\sigma,\tau,A,B\big) \le \epsilon_2,$$

for $\big(t,s,\sigma,\tau,A,B\big) \in \Im \times \Im \times E^d \times E^d$. *Then for the solution* $u(t,s)$ *of the problem* (3.1) *the following estimation is true*

$$\begin{aligned} D(u(t,s), v(t,s)) \le \big(\epsilon_1 + \epsilon_2(b-a)T\big)\Big\{1 + f(t,s)\int_0^t \int_\Omega \big(g(\sigma,\tau) + k(\sigma,\tau)\big) \\ \times \exp\Big(\int_\sigma^t \int_\Omega f(\overline{\sigma},\overline{\tau})\big(g(\overline{\sigma},\overline{\tau}) + k(\overline{\sigma},\overline{\tau})\big)d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau\Big\}, \end{aligned}$$

for all $(t,s) \in \Im$.

Proof. Notice that for $(t,s) \in \Im$ we have

$$\begin{aligned} D(u(t,s), v(t,s)) &= D\Big(h(t,s) + \int_0^t \int_\Omega F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big)d\sigma d\tau, \\ &\quad \widetilde{h}(t,s) + \int_0^t \int_\Omega \widetilde{F}\big(t,s,\sigma,\tau,v(\sigma,\tau),(\widetilde{\mathscr{T}}v)(\sigma,\tau)\big)d\sigma d\tau\Big) \\ &\le D(h(t,s), \widetilde{h}(t,s)) \\ &\quad + \int_0^t \int_\Omega D\big(F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big), F\big(t,s,\sigma,\tau,v(\sigma,\tau),(\mathscr{T}v)(\sigma,\tau)\big)\big)d\sigma d\tau \\ &\quad + \int_0^t \int_\Omega D\big(F\big(t,s,\sigma,\tau,v(\sigma,\tau),(\mathscr{T}v)(\sigma,\tau)\big), \widetilde{F}\big(t,s,\sigma,\tau,v(\sigma,\tau),(\widetilde{\mathscr{T}}v)(\sigma,\tau)\big)\big)d\sigma d\tau \\ &\le \int_0^t \int_\Omega L(t,s,\sigma,\tau)\Big(D(u(\sigma,\tau), v(\sigma,\tau)) + D\Big(\int_0^\sigma \int_\Omega K(\sigma,\tau,\overline{\sigma},\overline{\tau},u(\overline{\sigma},\overline{\tau}))d\overline{\sigma}d\overline{\tau}, \\ &\quad \int_0^\sigma \int_\Omega K(\sigma,\tau,\overline{\sigma},\overline{\tau},v(\overline{\sigma},\overline{\tau}))d\overline{\sigma}d\overline{\tau}\Big)\Big)d\sigma d\tau + \epsilon_1 + \epsilon_2(b-a)T \\ &\le \epsilon_1 + \epsilon_2(b-a)T + f(t,s)\int_0^t \int_\Omega g(\sigma,\tau)\Big(D(u(\sigma,\tau), v(\sigma,\tau)) \\ &\quad + \int_0^\sigma \int_\Omega k(\overline{\sigma},\overline{\tau})D(u(\overline{\sigma},\overline{\tau}, v(\overline{\sigma},\overline{\tau}))d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau, \end{aligned} \tag{3.8}$$

By Theorem 1 in [20] and from (3.8), we infer that

$$\begin{aligned} D(u(t,s), v(t,s)) \le \big(\epsilon_1 + \epsilon_2(b-a)T\big)\Big\{1 + f(t,s)\int_0^t \int_\Omega \big(g(\sigma,\tau) + k(\sigma,\tau)\big) \\ \times \exp\Big(\int_\sigma^t \int_\Omega f(\overline{\sigma},\overline{\tau})\big(g(\overline{\sigma},\overline{\tau}) + k(\overline{\sigma},\overline{\tau})\big)d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau\Big\}. \end{aligned}$$

This proof is complete. ∎

4. An Example

Consider the following fuzzy Volterra-Fredholm integral equation:

$$u(t,s) = h(t,s) + \int_0^t \int_0^1 \Big[f(t,s,\sigma,\tau) u(\sigma,\tau) + \int_0^\sigma \int_0^1 g(t,s,\overline{\sigma},\overline{\tau}) u(\overline{\sigma},\overline{\tau}) d\overline{\sigma} d\overline{\tau} \Big] d\sigma d\tau, \tag{4.1}$$

where

$$h(t,s) = e^{\lambda(t+s)} \chi_{\{1/(t+s)\}}, \; f(t,s,\sigma,\tau) = \frac{\lambda^2}{e^\lambda - 1} e^{\lambda(t+s)},$$

and

$$g(t,s,\sigma,\tau) = \frac{\lambda^2}{t(e^\lambda - 1)} e^{\lambda(t+s)}, \lambda > 0.$$

Here,

$$K(t,s,\sigma,\tau,u(\sigma,\tau)) = \int_0^t \int_0^1 g(t,s,\sigma,\tau) u(\sigma,\tau) d\sigma d\tau$$

and

$$F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big) = g(t,s,\sigma,\tau)u(\sigma,\tau) + (\mathscr{T}u)(\sigma,\tau).$$

We can see that

$$\begin{aligned} &D\big(K(t,s,\sigma,\tau,u), K(t,s,\sigma,\tau,v)\big) \\ &\quad \le \int_0^t \int_0^1 \big|g(t,s,\sigma,\tau)\big| D(u(\sigma,\tau),v(\sigma,\tau)) d\sigma d\tau \\ &\quad = \frac{\lambda^2}{e^\lambda - 1} \left(\int_0^t \int_0^1 \frac{e^{\lambda(t+s)}}{t} d\sigma d\tau \right) D^*(u,v) = \frac{\lambda^2}{e^\lambda - 1} e^{\lambda(t+s)} D^*(u,v) \end{aligned}$$

and

$$\begin{aligned} &D\Big(F\big(t,s,\sigma,\tau,u(\sigma,\tau),(\mathscr{T}u)(\sigma,\tau)\big), F\big(t,s,\sigma,\tau,v(\sigma,\tau),(\mathscr{T}v)(\sigma,\tau)\big)\Big) \\ &\quad = D\bigg(f(t,s,\sigma,\tau)u(\sigma,\tau) + \int_0^\sigma \int_0^1 g(t,s,\overline{\sigma},\overline{\tau}) u(\overline{\sigma},\overline{\tau}) d\overline{\sigma} d\overline{\tau}, \\ &\qquad\quad f(t,s,\sigma,\tau)v(\sigma,\tau) + \int_0^\sigma \int_0^1 g(t,s,\overline{\sigma},\overline{\tau}) v(\overline{\sigma},\overline{\tau}) d\overline{\sigma} d\overline{\tau} \bigg) \\ &\quad \le \big|f(t,s,\sigma,\tau)\big| D(u(\sigma,\tau),v(\sigma,\tau)) + \int_0^\sigma \int_0^1 \big|g(t,s,\overline{\sigma},\overline{\tau})\big| D\big(u(\overline{\sigma},\overline{\tau}),v(\overline{\sigma},\overline{\tau})\big) d\overline{\sigma} d\overline{\tau} \\ &\quad \le \left\{ \big|f(t,s,\sigma,\tau)\big| + \int_0^\sigma \int_0^1 \big|g(t,s,\overline{\sigma},\overline{\tau})\big| d\overline{\sigma} d\overline{\tau} \right\} D^*(u,v) \\ &\quad = \frac{2\lambda^2}{e^\lambda - 1} e^{\lambda(t+s)} D^*(u,v). \end{aligned}$$

By choosing $L(t,s,\sigma,\tau) = \dfrac{2\lambda^2}{e^\lambda - 1} e^{\lambda(t+s)}$ and $M(t,s,\sigma,\tau) = \dfrac{\lambda^2}{e^\lambda - 1} e^{\lambda(t+s)}$. It is easy to see that the functions $L(t,s,\sigma,\tau)$ and $M(t,s,\sigma,\tau)$ are continuous, and so the conditions (c1)-(c2) are satisfied.

On the order hand, we get that

$$
\begin{aligned}
&D(h(t,s),\hat{0}) + \int_0^t \int_\Omega D\big(F\big(t,s,\sigma,\tau,\hat{0},(\mathscr{T}\hat{0})(\sigma,\tau)\big),\hat{0}\big)d\sigma d\tau \\
&\quad = D\big(e^{\lambda(t+s)}\chi_{\{t+s\}},\hat{0}\big) = \frac{1}{t+s}e^{\lambda(t+s)} \le \frac{1}{1+T}e^{\lambda(t+s)},
\end{aligned}
$$

which satisfies the condition (c3) of Theorem 3.1 with $\beta_1 = 1/(1+T)$ and $\lambda > 0$.

Moreover,

$$
\begin{aligned}
&\int_0^t \int_\Omega L(t,s,\sigma,\tau)\Big(e^{\lambda(\sigma+\tau)} + \int_0^\sigma \int_\Omega M(\sigma,\tau,\overline{\sigma},\overline{\tau})e^{\lambda(\overline{\sigma}+\overline{\tau})}d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau \\
&\quad = \int_0^t \int_0^1 \Big(e^{\lambda(\sigma+\tau)} + \frac{\lambda^2}{e^\lambda - 1}e^{\lambda(\sigma+\tau)}\int_0^\sigma \int_0^1 e^{\lambda(\overline{\sigma}+\overline{\tau})}d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau \\
&\quad = \frac{2\lambda^2}{e^\lambda - 1}e^{\lambda(t+s)}\int_0^t \int_0^1 e^{\lambda(2\sigma+\tau)}d\sigma d\tau \le (e^\lambda + 1)(e^{\lambda t} - 1)e^{\lambda(t+s)}.
\end{aligned}
$$

If we chose $T = \ln\left(\dfrac{e^\lambda + 2}{e^\lambda + 1}\right)^{1/\lambda}$, $\lambda > 0$, then we have

$$
\int_0^t \int_\Omega L(t,s,\sigma,\tau)\Big(e^{\lambda(\sigma+\tau)} + \int_0^\sigma \int_\Omega M(\sigma,\tau,\overline{\sigma},\overline{\tau})e^{\lambda(\overline{\sigma}+\overline{\tau})}d\overline{\sigma}d\overline{\tau}\Big)d\sigma d\tau < e^{\lambda(t+s)}.
$$

It follows that the condition (c4) of Theorem 3.1 is satisfied for $\beta_2 = 1$, and by direct computation, we obtain that $\beta_1 + \beta_2\delta < 1$, where $\delta = 1 - 1/(1+T)$.

From the results above, we infer that the problem (4.1) satisfies the all conditions of Theorem 3.1, which guarantees the problem (4.1) has a unique solution.

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