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Generalized (T, Γ, Υ)–Derivations on c–Subtraction Algebras

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Abstract In a complicated subtraction (c-subtraction) algebra X, the notion of a new kind of derivation is introduced on X called (T, Γ, Υ)-derivation such that F is monotone, as a generalization of derivation of c-subtraction and characterized some of its relevant properties. Some equivalent conditions provided on a c-subtraction algebra by the notion of isotone (T, Γ, Υ)-derivation on X. By using the concept of isotone derivation, we study some interesting properties of it by the notion of (T, Γ, Υ)-derivation when F is monotone. Finally, generalized (T, F)-derivation on c-subtraction algebra is defined determined by a T-derivation d_T and discussed.

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1. Introduction

Schein [1] considered systems of the form $(\Phi, o, \backslash)$, where Φ is a set of functions closed under the composition "o" of functions (and hence (Φ, o) is a functions of semigroup), and set theoretic subtraction "$\backslash$" (and hence $(\Phi, \backslash)$ is a subtraction algebra in the sense of [2]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. Zelinka [3] discussed a problem proposed by Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called atomic subtraction algebras. Jun et al. [4–6] introduced the notion of ideals, prime ideals and irreducible ideals in subtraction algebras and investigated their characterizations. Again, Jun et al. [7] introduced the notion of complicated subtraction algebras and investigated some of its properties. Ceven and Öztürk [8] introduced the notion of additional concepts on subtraction algebras, so called subalgebra, bounded subtraction algebra, union of subtraction algebras and investigated some of its related properties. Jana et al. [9–17] have done detailed investigation

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in BCK/BCI-algebras,BG/G-algebras and c-subtraction algebras. Derivations is a very interesting research area in the theory of algebraic structure in mathematics. It has been applied to the classical Galois theory and the theory of invariants. An extensive and analytic theory has been developed for derivations in algebraic structure such as KUS-algebras [18], C^*-algebras, commutative Branch algebras and Galois theory of linear differential equation and also plays important role in functional analysis, algebraic geometry, algebras and linear differential equations. Many researchers generalized this idea in different algebraic structure such as near-rings, rings, Banach algebras, Subtraction algebras and lattices [19–23]. Öztürk and Ceven [24], Yon and Kim [25] studied the notion of derivation on subtraction algebras, Kim [26] and Lee et al. [27] introduced the notion of f-derivation on subtraction algebras and Lee and Kim [28] studied multiplier on subtraction algebra. Jun et al. [29, 30] studied N-ideals, Order systems, ideals and right fixed maps on subtraction algebras, Kim and Kim [31] introduced relationship between BCK-algebras and subtraction algebras. Handam [32] introduced notion of Smarandache weak subtraction algebra. Yilmaz and Öztrük [33], and Zhan and Liu [34] studied f-derivation on lattices and BCI-algebras respectively. Motivated by the above works and best of our knowledge there is no work available on generalized (T, Γ, Υ)-derivation D_T determined by associated two-sided (T, Γ)-derivation d on c-subtraction algebras. For this reason we are motivated to developed a new theoretical studies of (T, Γ, Υ)-derivation determined by associated two-sided (T, Γ)-derivation d on c-subtraction algebras.

In this paper, the notion of (T, Γ, Υ)-derivations of c-subtraction algebra is introduced which is a generalization of derivation in c-subtraction and studied some properties of it. Some equivalent condition are provided on c-subtraction algebras based on the notion of isotone (T, Γ, Υ)-derivation.

2. Subtraction Algebra

In this section, we recall some elementary aspects that are necessary to present the paper. By a subtraction algebra we mean an algebra $(X, -)$ with a single binary operation "$-$", then for any $x, y, z \in X$ that satisfies the following identities:

(S1) $(S1)$ $x - (y - x) = x$;
(S2) $(S2)$ $x - (x - y) = y - (y - x)$;
(S3) $(S3)$ $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parenthesis in expressions of the form $(x - y) - z$. The order relation determines by the subtraction as $X : a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element which independent of the choice of $a \in X$. The ordered set $(X, \leq)$ is a semi-Boolean algebra in the sense of [2], i.e. it is a meet semilattice with zero (0) in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order.
Here $a \wedge b = a - (a - b)$ and $a - b$ is the complement of $b \in [0, a]$.

In a subtraction algebra for all $x, y, z \in X$, the following are true [5, 6]

(P1) $(x - y) - y = x - y$;
(P2) $x - 0 = x$ and $0 - x = 0$;
(P3) $(x - y) - x = 0$;
(P4) $x - (x - y) \leq y$;
(P5) $(x - y) - (y - x) = x - y$
(P6) $x - (x - (x - y)) = x - y$;
(P7) $(x - y) - (z - y) \leq x - z$;

(P8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$;
(P9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$;
(P10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$;
(P11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$;
(P12) $(x - y) - z = (x - z) - (y - z)$.

In a subtraction algebra X, a non-empty subset S of X is called a *subalgebras* of X if for all $x, y \in X$ implies $x - y \in S$. A mapping $\Gamma : X \to Y$ (where, X and Y are subtraction algebras) is called a *homomorphism* of X if $\Gamma(x - y) = \Gamma(x) - \Gamma(y)$ for all $x, y \in X$. A homomorphism Γ from a subtraction algebra X to itself is called an *endomorphism* of X. If $x \leq y$ implies $\Gamma(x) \leq \Gamma(y)$, then Γ is called an *isotone* mapping.

Definition 2.1. [5] A nonempty subset I of a subtraction algebra X is called an ideal of X if it satisfies

(a) $0 \in I$;
(b) for all $x \in X$, $y \in I$ and $x - y \in I$ implies $x \in I$.

For an ideal I of a subtraction algebra X, we have that $x \leq y$ and $y \in I$ imply that $x \in I$ for any $x, y \in X$ [6].

Lemma 2.2. [27] *In a c-subtraction algebra X, the followings are hold:*

(a) $x \wedge y = y \wedge x$ *for any* $x, y \in X$*;*
(b) $x - y \leq x$ *for any* $x, y \in X$*.*

Definition 2.3. [7] Let X be a c-subtraction algebra. For any $a, b \in X$, then set $\mathcal{G}(a, b) = \{x | x - a \leq b\}$. Then X is said to be complicated subtraction algebra (c-subtraction algebra) if for any $a, b \in X$, the set $\mathcal{G}(a, b)$ has the greatest element.

Note that $0, a, b \in \mathcal{G}(a, b)$. The greatest element of $\mathcal{G}(a, b)$ is denoted by $a + b$.

Proposition 2.4. [7] *In a c-subtraction algebras the followings are hold for all* $x, y, z \in X$*,*

(a) $x \leq x + y$ *and* $y \leq x + y$*;*
(b) $x + 0 = 0 = 0 + x$*;*
(c) $x + y = y + x$*;*
(d) $x \leq y \Rightarrow x + z \leq y + z$*;*
(e) $x \leq y \Rightarrow x + y = y$*;*
(f) $x + y$ *is the least upper bound of* x *and* y*.*

Definition 2.5. [24] Let X be a c-subtraction algebra and d is a self-map on X. Then d is called a derivation of X if it satisfies the identity

$$d(x \wedge y) = (d(x) \wedge y) + (x \wedge d(y))$$

for all $x, y \in X$.

Definition 2.6. [27] Let X be c-subtraction algebra and $\Gamma : X \to X$ be a function. A function $D : X \to X$ is called a two-sided Γ-derivation on X if for all $x, y \in X$ satisfying the following identity

$$D(x \wedge y) = (Dx \wedge \Gamma(y)) + (\Gamma(x) \wedge Dy).$$

Theorem 2.7. [27] *Let X be a c-subtraction algebra and d be a two-sided Γ-derivation on X. Then for all $x, y \in X$, the following are hold:*

(a) $Dx \leq \Gamma(x)$*;*

(b) $Dx \wedge Dy \leq D(x \wedge y) \leq Dx + Dy$*;*
(c) $D(x + y) \leq \Gamma(x) + \Gamma(y)$*;*
(d) if X *has a least element* 0*, then* $\Gamma(0) = 0$ *implies* $D0 = 0$

3. Generalized (T, Γ, Υ)-Derivation on c-Subtraction Algebras

The following definitions introduce the notion of generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on c-subtraction algebra.

Definition 3.1. Let X be a c-subtraction algebra. Then for any $T \in X$, we define a self-map $d_T : X \to X$ by $d_T(x) = T \wedge x$ for all $x \in X$.

Definition 3.2. Let X be a c-subtraction algebra. Then for any $T \in X$, a self-map $d_T : X \to X$ is called T-derivation of X if it satisfy the condition $d_T(x \wedge y) = (d_T(x) \wedge y) + (x \wedge d_T(y))$ for all $x, y \in X$.

Example 3.3. [7] Let $X = \{0, a, b, c\}$ be a c-subtraction algebra given in Hasse diagram of figure 1 with the following Caley table:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

In a complicated (c-subtraction) subtraction algebra X. $0+0 = 0$, $0+a = a$, $0+b = b$, $0 + c = c$, $a + a = a$, $a + b = c$, $a + c = c$, $b + b = b$, $b + c = c$, and $c + c = c$.

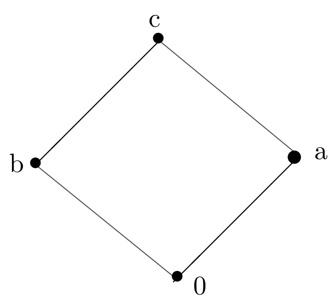

Figure 1. Hesse diagram of Example 3.3

For $T \in X$, define a self-map $d_T : X \to X$ of a subtraction algebra X. Define the mapping d_T as follows:
for $T = 0$, $d_T(x) = 0$ for all $x \in X$
for $T = a$, $d_T(0) = 0$, $d_T(a) = a$, $d_T(b) = 0$, $d_T(c) = a$
for $T = b$, $d_T(0) = 0$, $d_T(a) = 0$, $d_T(b) = b$, $d_T(c) = b$
for $T = c$, $d_T(0) = 0$, $d_T(a) = a$, $d_T(b) = b$, $d_T(c) = c$.
Hence, for each $T \in X$, d_T is a T-derivation of X.

Definition 3.4. Let X be a c-subtraction algebra. Then for any $T \in X$, a self-map $D_T : X \to X$ is defined as for any $T \in X$, $D_T(x) = T \wedge x$ for all $x \in X$. Then function

$D_T : X \to X$ is called an (T, Γ)-derivation of X if there exists a function $\Gamma : X \to X$ such that

$$D_T(x \wedge y) = (D_T(x) \wedge \Gamma(y)) + (\Gamma(x) \wedge D_T(y))$$

for all $x, y \in X$.

It is notified in the Definition 3.4 that if Γ is an identity function then $\mathcal{D}_T$ is a T-derivation on X. Therefore, according to Definition 3.4, D_T is a (T, F)-derivation on X if Γ must satisfied the identity of the Definition 3.4.

Example 3.5. For $T \in X$, define a self-map $D_T : X \to X$ of a c-subtraction algebra X in Example 3.3. Define the mapping D_T as follows,
for $T = 0$, $D_T(x) = 0$ for all $x \in X$
for $T = a$, $D_T(0) = 0$, $D_T(a) = a$, $D_T(b) = 0$, $D_T(c) = a$
for $T = b$, $D_T(0) = 0$, $D_T(a) = 0$, $D_T(b) = b$, $D_T(c) = b$
for $T = c$, $D_T(0) = 0$, $D_T(a) = a$, $D_T(b) = b$, $D_T(c) = c$.
Hence, for each $T \in X$, D_T is a (T, F)-derivation of X. If we defined the function F by $\Gamma(0) = 0$, $\Gamma(a) = a$, $\Gamma(b) = 0$, $\Gamma(c) = c$, then it is easy to cheque that for each $T \in X$, $\mathcal{D}_T$ is a (T, Γ)-derivation of X.
Where as, if we defined the function Γ by $\Gamma(0) = b$, $\Gamma(b) = 0$, $\Gamma(a) = c$ and $\Gamma(c) = a$, then it is verified that for each $T \in X$, D_T is not a (T, Γ)-derivation of X.

Now, we defined the generalized (T, Γ, Υ)-derivation on a subtraction algebras X in the following definition.

Definition 3.6. Let X be a subtraction algebras. A function $\Gamma : X \to X$ is called increasing if $x \leq y$ imply $\Gamma(x) \leq \Gamma(y)$ for all $x, y \in X$.

Definition 3.7. Let X be a c-subtraction algebra. Let $\Gamma : X \to X$ and $\Upsilon : X \to X$ be two mapping and $d : X \to X$ be a two-sided (T, Γ)-derivation on X. A function $D_T : X \to X$ is called a generalized (T, Γ, Υ)-derivation on X if for all $x, y \in X$ satisfying the following identity

$$D_T(x \wedge y) = (D_T(x) \wedge \Gamma(y)) + (\Upsilon(y) \wedge d_T(y)).$$

If $\Gamma = 1$ and $\Upsilon = 1$, which imply the identity mapping on X, then generalized $(1, 1)$-derivation D_T became a generalized T-derivation on c-subtraction algebra of X. If $\Gamma = \Upsilon$ and $D_T = d_T$, then D_T will be a two-sided (T, Γ)-derivation on c-subtraction algebra of X. In this paper, we consider generalized (T, Γ, Υ)- derivation whose associated derivation is a two-sided (T, Γ)-derivation.

Proposition 3.8. *Let X be a c-subtraction algebra and D_T be a generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X such that $\Gamma(x) \leq \Upsilon(x)$. Then for all $x, y \in X$, the following conditions are holds:*

(1) $d_T(x) \leq D_T(x) \leq \Gamma(x)$;
(2) $D_T(x) \wedge D_T(y) \leq D_T(x \wedge y) \leq D_T(x) + D_T(y)$;
(3) if I is an ideal of X with $\Gamma(I) \subseteq X$, then $D_T(I) \subseteq I$;
(4) if X has a least element 0, then $\Gamma(0) = 0$ implies that $D_T(0) = 0$;
(5) if X has a greatest element 1 and Γ is an increasing function,

$$D_T(x) = (D_T(1) \wedge \Gamma(x)) + d_T(x).$$

Proof. (1) For all $x \in X$, then

$$\begin{aligned} d_T(x) \wedge D_T(x) &= d_T(x) \wedge D_T(x \wedge x) \\ &= d_T(x) \wedge ((D_T(x) \wedge \Gamma(x)) + (\Upsilon(x) \wedge d_T(x)). \end{aligned}$$

Since $\Gamma(x) \leq \Upsilon(x)$, so

$$d_T(x) \wedge D_T(x) = (\Gamma(x) + d_T(x)) \wedge d_T(x) = d_T(x)$$

which implies $d_T(x) = d_T(x) - (d_T(x) - D_T(x)) \leq D_T(x)$ by (P4). Again,

$$\begin{aligned} D_T(x) + \Gamma(x) &= (D_T(x \wedge x) + \Gamma(x) \\ &= (D_T(x) \wedge \Gamma(x) + \Upsilon(x) \wedge d_T(x)) + \Gamma(x) \\ &= (D_T(x) \wedge \Gamma(x) + d_T(x)) + \Gamma(x) \\ &= (D_T(x) \wedge \Gamma(x)) + (d_T(x) + \Gamma(x)) \\ &= \Gamma(x). \end{aligned}$$

Hence, $D_T(x) + \Gamma(x) = \Gamma(x)$ which implies $D_T(x) \leq \Gamma(x)$. Therefore,

$$d_T(x) \leq D_T(x) \leq \Gamma(x).$$

(2) Let X be a c-subtraction algebra. Since $d_T(y) \leq \Gamma(y)$, then we have

$$D_T(x) - \Gamma(y) \leq D_T(x) - d_T(y)$$

and

$$\begin{aligned} D_T(x) - (D_T(x) - D_T(y)) &\leq D_T(x) - (D_T(x) - d_T(y)) \\ &\leq D_T(x) - (D_T(x) - \Gamma(y)), \end{aligned}$$

hence

$$D_T(x) \wedge D_T(y) \leq D_T(x) \wedge \Gamma(y).$$

Similarly, let $D_T(x) \leq \Upsilon(x)$, then

$$d_T(y) - \Upsilon(x) \leq d_T(y) - D_T(x).$$

Therefore, $d_T(y) - (d_T(y) - D_T(x)) \leq d_T(y) - (d_T(y) - \Upsilon(x))$, that is,

$$d_T(y) \wedge D_T(x) \leq d_T(y) \wedge \Upsilon(x)$$

which implies $D_T(y) \wedge D_T(x) \leq d_T(y) \wedge \Upsilon(x)$. Hence

$$\begin{aligned} D_T(x) \wedge D_T(y) &\leq D_T(x) \wedge \Gamma(y) + d_T(y) \wedge \Upsilon(x) \\ &= D_T(x \wedge y) \end{aligned}$$

Also, since $D_T(x) \wedge \Gamma(y) \leq D_T(x)$ and $d_T(y) \wedge \Upsilon(x) \leq d_T(y)$ by (P_4), and hence

$$\begin{aligned} D_T(x \wedge y) &= D_T(x) \wedge \Gamma(y) + d_T(y) \wedge \Upsilon(x) \\ &\leq D_T(x) + d_T(y) \wedge \Upsilon(x) \\ &\leq D_T(x) + d_T(y) \\ &\leq D_T(x) + D_T(y) \end{aligned}$$

(3) Let X be a c-subtraction algebra and $D_T(x)$ be a generalized (T, Γ, Υ)-derivation on X. Let $\Gamma(x) \in I$ for $x \in I$. Since $D_T(x) \leq \Gamma(x)$, then $D_T(x) - \Gamma(x) = 0 \in I$. Then by definition of ideal of X, $D_T(x) \in I$ for all $x \in I$. Thus, $D_T(I) \subseteq I$.

(4) If 0 is the least element of X and $\Gamma(0) = 0$. Since d is a two-sided (T, Γ)-derivation on X, then $d_T(0) = 0$ and so,

$$\begin{aligned} D_T(0) &= D_T(0 \wedge 0) \\ &= (D_T(0) \wedge \Gamma(0)) + (\Upsilon(0) \wedge d_T(0)) \\ &= D_T(0) \wedge 0 + \Upsilon(0) \wedge 0 \\ &= 0 \end{aligned}$$

Thus, $D_T(0) = 0$.

(5) For every $x \in X$, we have $d_T(x) \leq \Gamma(x) \leq \Gamma(1) \leq \Upsilon(1)$, then

$$\begin{aligned} D_T(x) &= D_T(1 \wedge x) \\ &= (D_T(1) \wedge \Gamma(x)) + (\Upsilon(1) \wedge d_T(x)) \\ &= (D_T(1) \wedge \Gamma(x)) + d_T(x). \end{aligned}$$

■

Proposition 3.9. *Let X be a c-subtraction algebra and D_T be a generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X. Then the following conditions hold:*

(1) if $x \leq y$ then $D_T(x + y) = D_T(y)$ for all $x, y \in X$;
(2) $D_T(\mathcal{G}(a, b)) \subseteq \mathcal{G}(a, b)$;
(3) $\mathcal{G}(D_T(a), D_T(b)) \subseteq \mathcal{G}(a, b)$.

Proof. (1) We have $x + y = y$, so we get

$$D_T(x + y) = D_T(y).$$

(2) For all $x \in \mathcal{G}(a, b)$, we have $x - a \leq b$. From Proposition 3.8 (1), we have

$$D_T(x) \leq \Gamma(x),$$

and so $D_T(x) - \Gamma(a) \leq x - a \leq b$ by (p9). Therefore, $D_T(x) \in \mathcal{G}(a, b)$.

(3) For all $x \in \mathcal{G}(D_T(a), D_T(b))$, we have $x - D_T(a) \leq D_T(b) \leq \Gamma(b)$. Then,

$$x - b \leq D_T(a) \leq \Gamma(a),$$

so $x - a \leq b$. Hence $x \in \mathcal{G}(a, b)$. ■

Corollary 3.10. *Let X be a c-subtraction algebra and D_T be a generalized (Γ, Υ)-derivation determined by a Γ-derivation d on X. Then the following properties hold:*

(1) $D_T(a + b) \leq a + b$;
(2) $D_T(a) + D_T(b) \leq \Gamma(a) + \Gamma(b) \leq a + b$.

Proof. (1) It is trivial from Proposition 3.9(2).

(2) Since the greatest element of $\mathcal{G}(D_T(a), D_T(b))$ is $D_T(a) + D_T(b)$ and the greatest of $\mathcal{G}(a, b)$ is $a + b$, then we get $D_T(a) + D_T(b) \leq \Gamma(a) + \Gamma(b) \leq a + b$, by using the Proposition 3.9(2). ■

Proposition 3.11. *Let X be a c-subtraction algebra and D_T be a generalized (T, Γ, Υ)-derivation on X with greatest element 1 and d be a (T, Γ)-derivation on X such that Γ is an increasing function satisfying the condition $\Gamma(x) \leq \Upsilon(x)$. Then following conditions are hold:*

(1) if $\Gamma(x) \leq D_T(1)$, then $D_T(x) = \Gamma(x)$;
(2) if $\Gamma(x) \geq D_T(1)$, then $D_T(1) \leq D_T(x)$;
(3) if $y \leq x$ and $D_T(x) = \Gamma(x)$, then $D_T(y) = \Gamma(y)$.

Proof. Let X be a c-subtraction algebra and D_T be a generalized (T, Γ, Υ)-derivation on X with greatest element 1. We have

$$\begin{aligned} D_T(x) &= D_T(1 \wedge x) \\ &= D_T(1) \wedge \Gamma(x) + \Upsilon(1) \wedge d_T(x) \\ &= D_T(1) \wedge \Gamma(x) + d_T(x). \end{aligned}$$

(1) If $\Gamma(x) \leq D_T(1)$, then $\Gamma(x) - D_T(x) = 0$, so we have

$$\Gamma(x) - (\Gamma(x) - D_T(1)) = \Gamma(x) \wedge D_T(1).$$

Hence, by Proposition 3.8(1) and (5), $D_T(x) = \Gamma(x)$.
(2) If $\Gamma(x) \geq D_T(1)$, then $D_T(1) - \Gamma(x) = 0$ and hence,

$$D_T(1) - (D_T(1) - \Gamma(x)) = D_T(1) \wedge \Gamma(x) \leq D_T(x).$$

(3) If $y \leq x$, then $y = x \wedge y)$. It follows that

$$\begin{aligned} D_T(y) &= D_T(x \wedge y) \\ &= (D_T(x) \wedge \Gamma(y)) + (\Upsilon(x) \wedge d_T(y)) \\ &= \Gamma(y) + d_T(y) = \Gamma(y). \end{aligned}$$

■

Proposition 3.12. *Let X be a c-subtraction algebra and D_T be a (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X. Let Γ be a increasing function such that $\Gamma(x) \leq \Upsilon(x)$ for all $x \in X$ and $y \leq x$. Then satisfying the following condition:*

$$D_T(x) = (D_T(x + y) \wedge \Gamma(x) + d_T(x).$$

Proof. Let X be a c-subtraction algebra and D_T be a (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X. Then for all $x, y \in X$ and by using the Proposition 2.4(e), we have

$$\begin{aligned} D_T(x) &= D_T((x + y) \wedge x) \\ &= (D_T(x + y) \wedge \Gamma(x)) + (\Upsilon(x + y) \wedge d_T(x)) \\ &= D_T(x + y) \wedge \Gamma(x) + d_T(x). \end{aligned}$$

■

Theorem 3.13. *Let X be a c-subtraction algebra and D_T is a generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X. If Γ is an increasing endomorphism such that $\Gamma(x) \leq \Upsilon(x)$ and $D_T(\Gamma(x)) = \Gamma(D_T(x))$. Then*

$$D_T^2(x) = \Gamma(D_T(x))$$

for all $x, y \in X$.

Proof. Let X be a c-subtraction algebra and $x \in X$. Then

$$\begin{aligned} D_T^2(x) &= D_T(\Gamma(x) \wedge D_T(x)) \\ &= (D_T(\Gamma(x)) \wedge \Gamma(D_T(x))) + (\Upsilon(\Gamma(x)) \wedge d_T(D_T(x))) \\ &= \Gamma(D_T(x)) + (\Upsilon(\Gamma(x)) \wedge d_T(D_T(x))). \end{aligned}$$

Since $d_T(D_T(x)) \le \Gamma(D_T(x)) \le \Gamma(\Gamma(x)) \le \Upsilon(D_T(x))$, it follows that

$$D_T^2(x) = \Gamma(D_T(x)) + d_T(D_T(x)) = \Gamma(D_T(x)).$$

∎

Definition 3.14. Let X be a c-subtraction algebra and D_T be a generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X. If $x \le y$ implies $D_T(x) \le D_T(y)$, then D_T is called an isotone generalized (T, Γ, Υ)-derivation.

Proposition 3.15. *Let X be a c-subtraction algebra and D_T be a generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X. Then for all $x, y \in X$, the followings hold:*

(1) if $D_T(x \wedge y) = D_T(x) \wedge D_T(y)$, then D_T be an isotone generalized (T, Γ, Υ)-derivation;

(2) if $D_T(x+y) = D_T(x)+D_T(y)$, then D_T is also an isotone generalized (T, Γ, Υ)-derivation.

Proof. Let X be a c-subtraction algebra and D_T be a generalized (T, Γ, Υ)-derivation on X. Next, we show both two conditions.

(1) Let $x \le y$. Then by (p4),

$$D_T(x) = D_T(x \wedge y) = D_T(x) \wedge D_T(y) \le D_T(y).$$

(2) Let $x \le y$. Then since $x + y = y$ from Proposition 2.4 (e),

$$D_T(y) = D_T(x + y) = D_T(x) + D_T(y).$$

Hence, we get $D_T(x) \le D_T(y)$. ∎

Definition 3.16. [8] Let X be a subtraction algebra. Then X is called bounded subtraction algebra if there is an element 1 of X satisfying the condition $x \le 1$ for all $x \in X$.

Proposition 3.17. *Let X be a bounded c-subtraction algebra with greatest element* 1 *and D_T be a generalized (T, Γ, Υ)-derivation determined by associated two sided (T, Γ)-derivation d on X. Then $D_T(x) = \Gamma(x)$ if and only if $D_T(1) = \Gamma(1)$.*

Proof. Let $D_T(x) = \Gamma(x)$, then it is obvious that $D_T(1) = \Gamma(1)$. Conversely, let $D_T(1) = \Gamma(1)$. Then from Proposition 3.11(3) and $x \le 1$ gives,

$$\begin{aligned} D_T(x) &= D_T(1 \wedge x) \\ &= (D_T(1) \wedge \Gamma(x)) + (\Upsilon(1) \wedge d_T(x)) \\ &= (D_T(1) \wedge \Gamma(x)) + d_T(x) \\ &= (\Gamma(1) \wedge \Gamma(x)) + d_T(x) \\ &= \Gamma(x) + d_T(x) \\ &= \Gamma(x). \end{aligned}$$

∎

Theorem 3.18. *Let X be a c-subtraction algebra with greatest element 1. Let D_T be a generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X. If $\Gamma : X \to X$ be a endomorphism such that $\Gamma(x) \leq \Upsilon(x)$ for all $x \in X$. Then followings are equivalent:*

(1) D_T is isotone generalized (T, Γ, Υ)-derivation;
(2) $D_T(x) = \Gamma(x) \wedge D_T(1)$;
(3) $D_T(x \wedge y) = D_T(x) \wedge D_T(y)$
(4) $D_T(x) + D_T(y) \leq D_T(x + y)$.

Proof. Let D_T be a generalized (T, Γ, Υ)-derivation on c-subtraction algebra X.

(1) $\Rightarrow$ (2). Since D_T is generalized isotone derivation. Therefore, $D_T(x) \leq D_T(1)$, then $D_T(x) \leq \Gamma(x)$. Also Proposition 3.8(5) gives

$$D_T(x) = (D_T(1) \wedge \Gamma(x)) + d_T(x)$$

which indicate $\Gamma(x) \wedge D_T(1) \leq D_T(x)$. Thus,

$$D_T(x) = \Gamma(x) \wedge D_T(1).$$

(2) $\Rightarrow$ (3). Assume that (2) holds, then

$$\begin{aligned} D_T(x) \wedge D_T(y) &= (\Gamma(x) \wedge D_T 1) \wedge (\Gamma(y) \wedge D_T 1) \\ &= (\Gamma(x) \wedge \Gamma(y)) \wedge D_T(1) \\ &= D_T(x \wedge y). \end{aligned}$$

(3) $\Rightarrow$ (1). Assume that (3) holds. Let $x \leq y$, then $x \wedge y = x$, and hence

$$D_T(x) = D_T(x \wedge y) = D_T(x) \wedge D_T(y)$$

which implies $D_T(x) \leq D_T(y)$.

(1) $\Rightarrow$ (4). Assume that (1) holds. We have $D_T(x) \leq D_T(x + y)$ and

$$D_T(y) \leq D_T(x + y),$$

so we get $D_T(x) + D_T(y) \leq D_T(x + y)$.

(4) $\Rightarrow$ (1). Assume that (4) holds. Let $x \leq y$, then

$$D_T(x) + D_T(y) \leq D_T(x + y) = D_T(y)$$

which imply that $D_T(x) \leq D_T(y)$. ∎

Theorem 3.19. *Let D_T be a (T, Γ, Υ)-derivation on a c-subtraction algebra X. Then the followings hold:*

(a) D_T is the identity (T, Γ, Υ)-derivation;
(b) $D_T(x + y) = (D_T(x) + \Gamma(y)) \wedge (\Gamma(x) + D_T(y))$;
(c) D_T is one-to-one (T, Γ, Υ)-derivation;
(d) D_T is onto (T, Γ, Υ)-derivation.

Proof. Let D_T be a (T, Γ, Υ)-derivation on c-subtraction algebra associated two sided (T, Γ)-derivation d on X. Then

$(a) \Rightarrow (b)$ is clear.
$(a) \Rightarrow (c)$ is straight forward.
$(a) \Rightarrow (d)$ is also trivial.
$(b) \Rightarrow (a)$. Let $y = x$, by (2) we get

$$D_T(x + x) = (D_T(x) + \Gamma(x)) \wedge (\Upsilon(x) + d_T(x)).$$

Since $d_T(x) \leq D_T(x) \leq \Gamma(x)$, we have

$$D_T(x) = \Gamma(x) \wedge \Upsilon(x) = \Gamma(x).$$

$(c) \Rightarrow (a)$. Let D_T be a one-to-one (T, Γ, Υ)-derivation. If there exists $x \in X$ such that $D_T(x) \neq \Gamma(x)$, then $D_T(x) \leq \Gamma(x)$. Let it be $D_T(x) = \Gamma(x_1)$. Then $x_1 \leq x$ so, we get

$$\begin{aligned} D_T(x_1) &= D_T(x \wedge x_1) \\ &= (D_T(x) \wedge \Gamma(x_1)) + (\Upsilon(x) \wedge d_T(x_1)) \\ &= D_T(x) + d_T(x_1) = D_T(x). \end{aligned}$$

Since $x_1 \neq x$ which contradicts that D_T is one-to-one (T, Γ, Υ)-derivation.

$(d) \Rightarrow (a)$. We assume that D_T is onto T-derivation, i.e., $D_T(X) = \Gamma(X)$. Then for every $x \in X$ there exists $y \in X$ such that $\Gamma(x) = D_T(y)$. Hence, we get

$$D_T(x) = D_T(D_T(y)) = D_T^2(y) = D_T(y) = \Gamma(x),$$

which imply that D_T is an identity (T, Γ, Υ)-derivation. ■

Remark 3.20. Let X be c-subtraction algebra with least element 0 and D_T be a generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation d on X. Let $\Gamma : X \to X$ be isomorphism on X and $\Upsilon : X \to X$ be a function. Then Γ is a one to one and onto generalized (T, Γ, Υ)-derivation determined by a (T, Γ)-derivation by $0 : X \to X$ such that $0(x) = 0$ for all $x \in X$.

4. Conclusions and Future Work

In this paper, we have considered the notion of generalized (T, Γ, Υ)-derivation on complicated (c-subtraction) subtraction algebra determined by associated two-sided (T, Γ)-derivation d on c-subtraction algebra and investigated some useful properties of it. In our opinion, these result can be similarly extended to the other algebraic structure such as B-algebras, BG-algebras, BF-algebras, MV-algebras, d-algebras, Q-algebras, BL-algebras, Lie algebras and so forth. The study of generalized (T, Γ, Υ)-derivations on different algebraic structures may have a lot of applications in different branches of theoretical physics, engineering, information theory, cryptanalysis and computer science etc. It is our hope that this work would serve as a foundation for further study in the theory of derivations of subtraction algebras.

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