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On a Class of Chemically Reacting Systems Involving Nonlocal Operators

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Abstract In this article, using the sub and supersolutions method, we study the existence of a positive solution for a class of Kirchhoff type systems with singular weights.The concepts of sub- and super-solution were introduced by Nagumo [M. Nagumo, Über die differentialgleichung $y'' = f(x, y, y')$, Proceedings of the Physico-Mathematical Society of Japan 19 (1937) 861–866] in 1937 who proved, using also the shooting method, the existence of at least one solution for a class of nonlinear Sturm-Liouville problems. In fact, the premises of the sub- and super-solution method can be traced back to Picard. He applied, in the early 1880s, the method of successive approximations to argue the existence of solutions for nonlinear elliptic equations that are suitable perturbations of uniquely solvable linear problems. This is the starting point of the use of sub- and super-solutions in connection with monotone methods. Picard's techniques were applied later by Poincaré [H. Poincaré, Les fonctions fuchsiennes et l'équation $\Delta u = e^u$, J. Math. Pures Appl. 4 (1898) 137–230] in connection with problems arising in astrophysics. We refer to [V. Rădulescu, Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods, Contemporary Mathematics and Its Applications, Vol. 6, Hindawi Publishing Corporation, New York, 2008].

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1. Introduction

In this paper, we consider the existence of positive weak solutions for the following Kirchhoff systems

$$\begin{cases} -M_1\Big(\displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = \lambda a(x)\dfrac{f(v)}{u^\alpha}, & x\in\Omega,\\ -M_2\Big(\displaystyle\int_\Omega |\nabla v|^q dx\Big)\Delta_q v = \lambda b(x)\dfrac{g(u)}{v^\beta}, & x\in\Omega,\\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_r z = div(|\nabla z|^{r-2}\nabla z)$, for $(r>1)$ denotes the $r-$Laplacian operator and λ is a positive parameter and Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 1$ with smooth boundary, $\alpha,\beta\in(0,1)$. Here $a(x)$ and $b(x)$ are C^1 sign-changing functions that maybe negative near the boundary and f,g are C^1 nondecreasing functions such that $f,g:[0,\infty)\to[0,\infty)$; $f(s)>0$, $g(s)>0$ for $s>0$, $\lim_{s\to\infty} g(s)=\infty$ and

$$\lim_{s\to\infty}\frac{f(Mg(s)^{\frac{1}{q-1}})}{s^{p-1+\alpha}}=0$$

for all $M>0$. Here M_1,M_2 satisfy the following condition:

(H1) $M_i : R_0^+ \to R^+$, $i=1,2$, are two continuous and increasing functions and $0<m_i\leq M_i(t)\leq m_{i,\infty}$ for all $t\in R_0^+$, where $R_0^+ := [0,+\infty)$. System (1.1) is related to the stationary problem of a model introduced by Kirchhoff [1]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where ρ,ρ_0,h,E are all constants. This equation is an extension of the classical D'Alembert 's wave equation. A distinguishing feature of equation (1.2) is that the equations a non-local coefficient $\dfrac{P_0}{h}+\dfrac{E}{2L}\displaystyle\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx$ which depends on the average $\dfrac{1}{2L}\displaystyle\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx$; hence the equation is no longer a pointwise identity. Nonlocal problems can be used for modeling, for example, physical and biological systems for which u describes a process which depends on the average of itself, such as the population density.In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [2–12] in which the authors have used variational method and topological method to get the existence of solutions for (1.1). We study the existence of positive solution to the system with sign-changing weight functions $a(x),b(x)$. Due to these weight functions, the extensions are challenging and nontrivial. The main tool used in this study is the method of sub- and super solutions. Our result in this note improves the previous one [13] in which $M_1(t)=M_2(t)\equiv 1$. To our best knowledge, this is a new research topic for nonlocal problems, see [2, 3, 14, 15]. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} \Delta_r\phi=\lambda|\phi|^{r-2}\phi, & x\in\Omega,\\ \phi=0, & x\in\partial\Omega. \end{cases} \tag{1.3}$$

Let $\phi_{1,r}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (1.3) such that $\phi_{1,r}(x)>0$ in Ω, and $\|\phi_{1,r}\|_\infty=1$ for $r=p,q$. Let $m,\sigma,\delta>0$ be such that

$$\sigma\leq\phi_{1,r}\leq 1, \qquad x\in\Omega-\overline{\Omega}_\delta, \tag{1.4}$$

$$\lambda_{1,r}\phi_{1,r}^r - \Big(1 - \frac{sr}{r-1+s}\Big)|\nabla\phi_{1,r}|^r \le -m, \qquad x \in \overline{\Omega}_\delta, \tag{1.5}$$

for $r = p, q$, and $s = \alpha, \beta$, where $\overline{\Omega} = \{x \in \Omega \mid d(x, \partial\Omega) \le \delta\}$. (This is possible since $|\nabla\phi_{1,r}|^r \ne 0$ on $\partial\Omega$ while $\phi_{1,r} = 0$ on $\partial\Omega$ for $r = p, q$.) We also consider the unique solution $\zeta_r \in W_0^{1,r}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_r\zeta_r = 1, & x \in \Omega, \\ \zeta_r = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result, it is known that $\zeta_r > 0$ in Ω and $\dfrac{\partial\zeta_r}{\partial n} < 0$ on $\partial\Omega$. Here we assume that the weight functions $a(x)$ and $b(x)$ takes negative values in $\overline{\Omega}_\delta$, but require $a(x)$ and $b(x)$ are strictly positive in $\Omega - \overline{\Omega}_\delta$. To be precise we assume that there exist positive constants a_0, a_1, b_0 and b_1 such that $a(x) \ge -a_0$, $b(x) \ge -b_0$ on $\overline{\Omega}_\delta$ and $a(x) \ge a_1$, $b(x) \ge b_1$ on $\Omega - \overline{\Omega}_\delta$.

2. Existence of Positive Solutions

In this section, we shall establish our existence result via the method of sub - supersolution. A pair of nonnegative functions $(\psi_1, \psi_2) \in W^{1,p} \cap C(\overline{\Omega}) \times W^{1,q} \cap C(\overline{\Omega})$ and $(z_1, z_2) \in W^{1,p} \cap C(\overline{\Omega}) \times W^{1,q} \cap C(\overline{\Omega})$ are called a subsolution and supersolution of (1.1) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$\begin{aligned}
M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w dx &\le \lambda\int_\Omega a(x)\frac{f(\psi_2)}{\psi_1^\alpha} w dx,\\
M_2\Big(\int_\Omega |\nabla\psi_2|^q dx\Big)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w dx &\le \lambda\int_\Omega b(x)\frac{g(\psi_1)}{\psi_2^\beta} w dx,\\
M_1\Big(\int_\Omega |\nabla z_1|^p dx\Big)\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w dx &\ge \lambda\int_\Omega a(x)\frac{f(z_2)}{z_1^\alpha} w dx,\\
M_2\Big(\int_\Omega |\nabla z_2|^q dx\Big)\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w dx &\ge \lambda\int_\Omega b(x)\frac{g(z_1)}{z_2^\beta} w dx,
\end{aligned}$$

for all $w \in W = \{w \in C_0^\infty(\Omega) \mid w \ge 0, x \in \Omega\}$. A key role in our arguments will be played by the following auxiliary result. Its Proof is similar to those presented in [16], the reader can consult further the papers [2, 3, 14, 15].

Lemma 2.1. *Assume that $M : R^+ \to R^+$ is a continuous and increasing function satisfying*

$$M(t) \ge M_0 > 0 \text{ for all } t \in R^+.$$

If the functions $u, v \in W_0^{1,p}(\Omega)$ satisfies

$$M\left(\int_\Omega |\nabla u|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\, dx \le M\left(\int_\Omega |\nabla v|^p\, dx\right)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\, dx \tag{2.1}$$

for all $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \ge 0$, then $u \le v$ in Ω.

Proof. Our proof is based on the arguments presented in [17, 18]. Define the functional $\Phi : W_0^{1,p}(\Omega) \to R$ by the formula

$$\Phi(u) := \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p\, dx\right), \quad u \in W_0^{1,p}(\Omega).$$

It is obviously that the functional Φ is a continuously Gáteaux differentiable whose Gáteaux derivative at the point $u \in W_0^{1,p}(\Omega)$ is the functional $\Phi' \in W_0^{-1,p}(\Omega)$, given by

$$\Phi'(u)(\varphi) = M\left(\int_\Omega |\nabla u|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\, dx, \quad \varphi \in W_0^{1,p}(\Omega).$$

It is obvious that Φ' is continuous and bounded since the function M is continuous. We will show that Φ' is strictly monotone in $W_0^{1,p}(\Omega)$. Indeed, for any $u, v \in W_0^{1,p}(\Omega)$, $u \neq v$, without loss of generality, we may assume that

$$\int_\Omega |\nabla u|^p\, dx \geq \int_\Omega |\nabla v|^p\, dx.$$

(otherwise, changing the role of u and v in the following proof). Therefore, we have

$$M\left(\int_\Omega |\nabla u|^p\, dx\right) \geq M\left(\int_\Omega |\nabla v|^p\, dx\right) \tag{2.2}$$

since $M(t)$ is a monotone function. Using Cauchy's inequality, we have

$$\nabla u\cdot\nabla v \leq |\nabla u||\nabla v| \leq \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2). \tag{2.3}$$

Using (2.3) we get

$$\int_\Omega |\nabla u|^p\, dx - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\, dx \geq \frac{1}{2}\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2)\, dx \tag{2.4}$$

and

$$\int_\Omega |\nabla v|^p\, dx - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\, dx \geq \frac{1}{2}\int_\Omega |\nabla v|^{p-2}(|\nabla v|^2 - |\nabla u|^2)\, dx. \tag{2.5}$$

If $|\nabla u| \geq |\nabla v|$, using (2.2)-(2.5) we have

$$\begin{aligned}
I_1 &:= \Phi'(u)(u) - \Phi'(u)(v) - \Phi'(v)(u) + \Phi'(v)(v)\\
&= M\left(\int_\Omega |\nabla u|^p\, dx\right)\left(\int_\Omega |\nabla u|^p\, dx - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\, dx\right)\\
&\qquad - M\left(\int_\Omega |\nabla v|^p\, dx\right)\left(\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\, dx - \int_\Omega |\nabla v|^p\, dx\right)\\
&\geq \frac{1}{2}M\left(\int_\Omega |\nabla u|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2)\, dx\\
&\qquad - \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2)\, dx\\
&= \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\, dx\right)\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2)\, dx\\
&\geq \frac{M_0}{2}\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2)\, dx.
\end{aligned} \tag{2.6}$$

If $|\nabla v| \geq |\nabla u|$, changing the role of u and v in (2.2)-(2.5) we have

$$\begin{aligned} I_2 &:= \Phi'(v)(v) - \Phi'(v)(u) - \Phi'(u)(v) + \Phi'(u)(u) \\ &= M\left(\int_\Omega |\nabla v|^p\,dx\right)\left(\int_\Omega |\nabla v|^p\,dx - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\,dx\right) \\ &\qquad - M\left(\int_\Omega |\nabla u|^p\,dx\right)\left(\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx - \int_\Omega |\nabla u|^p\,dx\right) \\ &\geq \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}(|\nabla v|^2 - |\nabla u|^2)\,dx \\ &\qquad - \frac{1}{2}M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla v|^2 - |\nabla u|^2)\,dx \\ &= \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega (|\nabla v|^{p-2} - |\nabla u|^{p-2})(|\nabla v|^2 - |\nabla u|^2)\,dx \\ &\geq \frac{M_0}{2}\int_\Omega (|\nabla v|^{p-2} - |\nabla u|^{p-2})(|\nabla v|^2 - |\nabla u|^2)\,dx. \end{aligned} \tag{2.7}$$

From (2.6) and (2.7) we have

$$\Big(\Phi'(u) - \Phi'(v)\Big)(u-v) = I_1 = I_2 \geq 0, \quad \forall u, v \in W_0^{1,p}(\Omega). \tag{2.8}$$

Moreover, if $u \neq v$ and $\Big(\Phi'(u) - \Phi'(v)\Big)(u-v) = 0$, then we have

$$\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2)\,dx = 0,$$

so $|\nabla u| = |\nabla v|$ in Ω. Thus, we deduce that

$$\begin{aligned} \Big(\Phi'(u) - \Phi'(v)\Big)(u-v) &= \Phi'(u)(u-v) - \Phi'(v)(u-v) \\ &= M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}|\nabla u - \nabla v|^2\,dx \\ &= 0, \end{aligned} \tag{2.9}$$

i.e., $u-v$ is a constant. In view of $u = v = 0$ on $\partial\Omega$ we have $u \equiv v$ which is contrary with $u \neq v$. Therefore $\Big(\Phi'(u) - \Phi'(v)\Big)(u-v) > 0$ and Φ' is strictly monotone in $W_0^{1,p}(\Omega)$.

Let u, v be two functions such that (2.1) is verified. Taking $\varphi = (u-v)^+$, the positive part of $u-v$, as a test function of (2.1), we have

$$\begin{aligned} (\Phi'(u) - \Phi'(v))(\varphi) &= M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \\ &\qquad - M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \\ &\leq 0. \end{aligned} \tag{2.10}$$

Relations (2.9) and (2.10) mean that $u \leq v$. ■

From Lemma 2.1 we can establish the basic principle of the sub- and supersolutions method for nonlocal systems. Indeed, we consider the following nonlocal system

$$\begin{cases} -M_1\Big(\displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = h(x,u,v) \text{ in } \Omega, \\ -M_2\Big(\displaystyle\int_\Omega |\nabla v|^q dx\Big)\Delta_q v = k(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } x \in \partial\Omega, \end{cases} \tag{2.11}$$

where Ω is a bounded smooth domain of R^N and $h,k : \overline{\Omega} \times R \times R \to R$ satisfy the following conditions

(HK1) $h(x,s,t)$ and $k(x,s,t)$ are Carathéodory functions and they are bounded if s,t belong to bounded sets.

(KH2) There exists a function $g : R \to R$ being continuous, nondecreasing, with $g(0) = 0$, $0 \le g(s) \le C(1 + |s|^{\min\{p,q\}-1})$ for some $C > 0$, and applications $s \mapsto h(x,s,t) + g(s)$ and $t \mapsto k(x,s,t) + g(t)$ are nondecreasing, for a.e. $x \in \Omega$.

If $u,v \in L^\infty(\Omega)$, with $u(x) \le v(x)$ for a.e. $x \in \Omega$, we denote by $[u,v]$ the set $\{w \in L^\infty(\Omega) : \ u(x) \le w(x) \le v(x)$ for a.e. $x \in \Omega\}$. Using Lemma 2.1 and the method as in the proof of Theorem 2.4 of [19] (see also Section 4 of [20]), we can establish a version of the abstract lower and upper-solution method for our class of the operators as follows.

Proposition 2.2. *Let $M_1, M_2 : R_0^+ \to R^+$ be two functions satisfying the condition **(H1)**. Assume that the functions h,k satisfy the conditions (HK1) and (HK2). Assume that $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$, are respectively, a weak subsolution and a weak supersolution of system (2.11) with $\underline{u}(x) \le \overline{u}(x)$ and $\underline{v}(x) \le \overline{v}(x)$ for a.e. $x \in \Omega$. Then there exists a minimal (u_*,v_*) (and, respectively, a maximal (u^*,v^*)) weak solution for system (2.11) in the set $[\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$. In particular, every weak solution $(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$ of system (2.11) satisfies $u_*(x) \le u(x) \le u^*(x)$ and $v_*(x) \le v(x) \le v^*(x)$ for a.e. $x \in \Omega$.*

To state our results precisely we introduce the following hypotheses :
$(H_2) f,g : [0,\infty) \to [0,\infty)$ are C^1 nondecreasing functions such that
$f(s), g(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} g(s) = \infty$.

(H_3) $\displaystyle\lim_{s\to\infty} \frac{f(Mg(s)^{\frac{1}{q-1}})}{s^{p-1+\alpha}} = 0$, for all $M > 0$.

(H_4) suppose that there exist $\epsilon > 0$ such that:

$$\frac{m_{1,\infty}\epsilon^{\frac{p-1+\alpha}{p-1}}\lambda_{1,p}}{a_1 f\Big(\frac{q-1+\beta}{q}\epsilon^{\frac{1}{q-1}}\sigma^{\frac{q}{q-1+\beta}}\Big)} \le \min\left\{\frac{mm_{1,\infty}\epsilon^{\frac{\alpha+p-1}{p-1}}\left(\frac{p-1+\alpha}{p}\right)^\alpha}{a_0 f(\epsilon^{\frac{1}{q-1}})}, \frac{mm_{2,\infty}\epsilon^{\frac{\beta+q-1}{q-1}}\left(\frac{q-1+\beta}{q}\right)^\beta}{b_0 g(\epsilon^{\frac{1}{p-1}})}\right\},$$

$$\frac{m_{2,\infty}\epsilon^{\frac{q-1+\beta}{q-1}}\lambda_{1,q}}{b_1 g\Big(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\sigma^{\frac{p}{p-1+\alpha}}\Big)} \le \min\left\{\frac{mm_{1,\infty}\epsilon^{\frac{\alpha+p-1}{p-1}}\left(\frac{p-1+\alpha}{p}\right)^\alpha}{a_0 f(\epsilon^{\frac{1}{q-1}})}, \frac{mm_{2,\infty}\epsilon^{\frac{\beta+q-1}{q-1}}\left(\frac{q-1+\beta}{q}\right)^\beta}{b_0 g(\epsilon^{\frac{1}{p-1}})}\right\}.$$

We are now ready to give our existence result.

Theorem 2.3. *Assume that*
(a) $p \ge n$ or (b) $p < n$ and $\alpha < \frac{p}{n}$,
(c) $q \ge n$ or (d) $q < n$ and $\beta < \frac{q}{n}$.

Let $(H_1)-(H_4)$ *hold. Then there exists a positive weak solution of* (1.1) *for every* $\lambda \in [\lambda_*(\epsilon), \lambda^*(\epsilon)]$, *where*

$$\lambda^* = \min\left\{ \frac{mm_{1,\infty}\epsilon^{\frac{\alpha+p-1}{p-1}}\left(\frac{p-1+\alpha}{p}\right)^{\alpha}}{a_0 f\left(\epsilon^{\frac{1}{q-1}}\right)}, \frac{mm_{2,\infty}\epsilon^{\frac{\beta+q-1}{q-1}}\left(\frac{q-1+\beta}{q}\right)^{\beta}}{b_0 g\left(\epsilon^{\frac{1}{p-1}}\right)} \right\},$$

and

$$\lambda_* = \max\left\{ \frac{m_{1,\infty}\epsilon^{\frac{\alpha+p-1}{p-1}}\lambda_{1,p}}{a_1 f\left(\frac{q-1+\beta}{q}\epsilon^{\frac{1}{q-1}}\sigma^{\frac{q}{q-1+\beta}}\right)}, \frac{m_{2,\infty}\epsilon^{\frac{\beta+q-1}{q-1}}\lambda_{1,q}}{b_1 g\left(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\sigma^{\frac{p}{p-1+\alpha}}\right)} \right\}.$$

Remark 2.4. Note that (H_4) implies $\lambda_* < \lambda^*$.

Proof. We shall verify that

$$(\psi_1, \psi_2) = \left(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p}{p-1+\alpha}}, \frac{q-1+\beta}{q}\epsilon^{\frac{1}{q-1}}\phi_{1,q}^{\frac{q}{q-1+\beta}}\right),$$

is a sub-solution of (1.1). Let $w \in W$. Then a calculation shows that

$$\nabla\psi_1 = \epsilon^{\frac{1}{p-1}}\nabla\phi_{1,p}\phi_{1,p}^{\frac{1-\alpha}{p-1+\alpha}},$$

and we have

$$\begin{aligned} M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\nabla w dx \\ &\le m_{1,\infty}\epsilon\int_\Omega \phi_{1,p}^{1-\frac{\alpha p}{p-1+\alpha}}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\nabla w dx \\ &\le m_{1,\infty}\epsilon\int_\Omega |\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\Big[\nabla(\phi_{1,p}^{1-\frac{\alpha p}{p-1+\alpha}}w) - w\nabla(\phi_{1,p}^{1-\frac{\alpha p}{p-1+\alpha}})\Big]dx \\ &= m_{1,\infty}\epsilon\int_\Omega \Big[\lambda_{1,p}\phi_{1,p}^{p-\frac{\alpha p}{p-1+\alpha}} - |\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\nabla(\phi_{1,p}^{1-\frac{\alpha p}{p-1+\alpha}})\Big]w dx \\ &= m_{1,\infty}\epsilon\int_\Omega \Big[\lambda_{1,p}\phi_{1,p}^{p-\frac{\alpha p}{p-1+\alpha}} - |\nabla\phi_{1,p}|^p\Big(1-\frac{\alpha p}{p-1+\alpha}\Big)\phi_{1,p}^{\frac{-\alpha p}{p-1+\alpha}}\Big]w dx \\ &= m_{1,\infty}\epsilon\phi_{1,p}^{\frac{-\alpha p}{p-1+\alpha}}\left\{\int_\Omega \Big[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\Big(1-\frac{\alpha p}{p-1+\alpha}\Big)\Big]w dx\right\}. \end{aligned}$$

Similarly

$$\begin{aligned} M_2\Big(\int_\Omega |\nabla\psi_2|^q dx\Big)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w dx \\ &\le m_{2,\infty}\epsilon\phi_{1,q}^{\frac{-\beta q}{q-1+\beta}}\left\{\int_\Omega \Big[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\Big(1-\frac{\beta q}{q-1+\beta}\Big)\Big]w dx\right\}. \end{aligned}$$

First we consider the case when $x \in \overline{\Omega}_\delta$. we have

$$\lambda_{1,p}\phi_{1,p}^p - \Big(1-\frac{\alpha p}{p-1+\alpha}\Big)|\nabla\phi_{1,p}|^p \le -m.$$

Since $\lambda \le \lambda^*$ then

$$\lambda \le \frac{mm_{1,\infty}\epsilon^{\frac{\alpha+p-1}{p-1}}\left(\frac{p-1+\alpha}{p}\right)^{\alpha}}{a_0 f(\epsilon^{\frac{1}{q-1}})}.$$

Hence

$$\begin{aligned}
&mm_{1,\infty}\epsilon\phi_{1,p}^{\frac{-\alpha p}{p-1+\alpha}}\Big(\lambda_{1,p}\phi_{1,p}^{p}-\Big(1-\frac{\alpha p}{p-1+\alpha}\Big)|\nabla\phi_{1,p}|^{p}\Big)\\
&\qquad\le -m_{1,\infty}m\epsilon\phi_{1,p}^{\frac{-\alpha p}{p-1+\alpha}}\\
&\qquad\le -\lambda a_0\frac{f(\epsilon^{\frac{1}{q-1}})\Big(\frac{p-1+\alpha}{p}\Big)^{-\alpha}\phi_{1,p}^{-\frac{\alpha p}{p-1+\alpha}}}{\epsilon^{\frac{\alpha}{p-1}}}\\
&\qquad\le \lambda a_0\frac{f\Big(\frac{q-1+\beta}{q}\epsilon^{\frac{1}{q-1}}\phi_{1,q}^{\frac{q}{q-1+\beta}}\Big)\Big(\frac{p-1+\alpha}{p}\Big)^{-\alpha}\phi_{1,p}^{-\frac{\alpha p}{p-1+\alpha}}}{\epsilon^{\frac{\alpha}{p-1}}}\\
&\qquad\le \lambda a(x)\frac{f(\psi_2)}{\psi_1^{\alpha}}.
\end{aligned}$$

A similar argument shows that

$$m_{2,\infty}\epsilon\phi_{1,q}^{-\frac{\beta q}{q-1+\beta}}\Big(\lambda_{1,q}\phi_{1,q}^{q}-\Big(1-\frac{\beta q}{q-1+\beta}\Big)|\nabla\phi_{1,q}|^{q}\Big)\le\lambda b(x)\frac{g(\psi_1)}{\psi_2^{\beta}}.$$

On the other hand, on $\Omega-\overline{\Omega}_\delta$, we have $1\ge\phi_{1,r}\ge\sigma$ for $r=p,q$. Also $a(x)\ge a_1$, $b(x)\ge b_1$ and since $\lambda>\lambda_*$, we have

$$\lambda\ge\frac{m_{1,\infty}\epsilon^{\frac{\alpha+p-1}{p-1}}\lambda_{1,p}}{a_1 f\Big(\frac{q-1+\beta}{q}\epsilon^{\frac{1}{q-1}}\sigma^{\frac{q}{q-1+\beta}}\Big)}.$$

Hence

$$\begin{aligned}
&m_{1,\infty}\epsilon\phi_{1,p}^{-\frac{\alpha p}{p-1+\alpha}}\Big(\lambda_{1,p}\phi_{1,p}^{p}-\Big(1-\frac{\alpha p}{p-1+\alpha}\Big)|\nabla\phi_{1,p}|^{p}\Big)\\
&\qquad\le m_{1,\infty}\epsilon\lambda_{1,p}\phi_{1,p}^{p-\frac{\alpha p}{p-1+\alpha}}\\
&\qquad\le m_{1,\infty}\epsilon\lambda_{1,p}\phi_{1,p}^{-\frac{\alpha p}{p-1+\alpha}}\\
&\qquad\le \lambda a_1\frac{f\Big(\frac{q-1+\beta}{q}\epsilon^{\frac{1}{q-1}}\sigma^{\frac{q}{q-1+\beta}}\Big)\phi_{1,p}^{-\frac{\alpha p}{p-1+\alpha}}\Big(\frac{p-1+\alpha}{p}\Big)^{-\alpha}}{\epsilon^{\frac{\alpha}{p-1}}}\\
&\qquad\le \lambda a(x)\frac{f(\psi_2)}{\psi_1^{\alpha}}.
\end{aligned}$$

A similar argument shows that

$$m_{2,\infty}\epsilon\phi_{1,q}^{-\frac{\beta q}{q-1+\beta}}\Big(\lambda_{1,q}\phi_{1,q}^{q}-\Big(1-\frac{\beta q}{q-1+\beta}\Big)|\nabla\phi_{1,q}|^{q}\Big)\le\lambda b(x)\frac{g(\psi_1)}{\psi_2^{\beta}}.$$

Hence

$$M_1\Big(\int_\Omega|\nabla\psi_1|^{p}dx\Big)\int_\Omega|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w dx\le\lambda\int_\Omega a(x)\frac{f(\psi_2)}{\psi_1^{\alpha}}w dx,$$

and

$$M_2\Big(\int_\Omega|\nabla\psi_2|^{q}dx\Big)\int_\Omega|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w dx\le\lambda\int_\Omega b(x)\frac{g(\psi_1)}{\psi_2\beta}w dx$$

Thus $(\psi_1,\psi_2)=\Big(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p}{p-1+\alpha}},\frac{q-1+\beta}{q}\epsilon^{\frac{1}{q-1}}\phi_{1,q}^{\frac{q}{q-1+\beta}}\Big)$ is a positive subsolution of (1.1). Now we construct a supersolution $(z_1,z_2)\geq(\psi_1,\psi_2)$. When
(a) $p\geq n$ or (b) $p<n$ and $\alpha<\frac{p}{n}$,
(c) $q\geq n$ or (d) $q<n$ and $\beta<\frac{q}{n}$,
from [21], we know that are functions $w_1\in W_0^{1,p}(\Omega)\cap C(\overline{\Omega})$ and $w_2\in W_0^{1,q}(\Omega)\cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta_p w_1=\frac{1}{w_1^\alpha}, & x\in\Omega,\\ w_1=0, & x\in\partial\Omega,\end{cases}$$

and

$$\begin{cases} -\Delta_q w_2=\frac{1}{w_2^\beta}, & x\in\Omega,\\ w_2=0, & x\in\partial\Omega,\end{cases}$$

are satisfying $w_1\geq\theta\zeta_p$ and $w_2\geq\theta\zeta_q$ for some $\theta>0$. Now, we will prove there exists $c\gg1$ such that

$$(z_1,z_2)=\Big(cw_1,g(c\|w_1\|)^{\frac{1}{q-1}}w_2\Big),$$

is a supersolution of (1.1). A calculation shows that :

$$\int_\Omega|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w dx=c^{p-1}\int_\Omega|\nabla w_1|^{p-2}\nabla w_1\cdot\nabla w dx=c^{p-1}\int_\Omega\frac{w}{w_1^\alpha}dx,$$

by $(H3)$ we know that, for $c\gg1$,

$$\frac{m_1}{\lambda\|a(x)\|_\infty}\geq\frac{f(\|w_2\|_\infty)(g(c\|w_1\|))^{\frac{1}{q-1}})}{c^{p-1+\alpha}}.$$

Hence

$$\begin{aligned}\frac{m_1c^{p-1}}{w_1^\alpha} &\geq \lambda\|a(x)\|_\infty\frac{f(\|w_2\|_\infty)(g(c\|w_1\|_\infty))^{\frac{1}{q-1}}}{(cw_1)^\alpha}\\ &\geq \lambda a(x)\frac{f(w_2(g(c\|w_1\|_\infty))^{\frac{1}{q-1}})}{(cw_1)^\alpha}\\ &= \lambda a(x)\frac{f(z_2)}{z_1^\alpha},\end{aligned}$$

now from $(H2)$, we know that $g(s)\to\infty$ as $s\to\infty$. Thus, for $c\gg1$

$$\frac{\lambda\|b(x)\|_\infty}{g(c\|w_1\|_\infty)^{\frac{\beta}{q-1}}}\leq m_2,$$

and we have for $c \gg 1$,

$$\begin{aligned} M_2\Big(\int_\Omega |\nabla z_2|^q\Big)\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w dx &\geq m_2\int_\Omega \frac{g(c\|w_1\|_\infty)}{w_2^\beta}wdx \\ &\geq \lambda\|b(x)\|_\infty\int_\Omega \frac{g(cw_1)}{g(c\|w_1\|_\infty)^{\frac{\beta}{q-1}}w_2^\beta}wdx \\ &\geq \lambda\int_\Omega b(x)\frac{g(z_1)}{z_2^\beta}wdx, \end{aligned}$$

i.e., (z_1, z_2) is a supersolution of (1.1). Furthermore, c can be chosen large enough so that $(z_1, z_2) \geq (\psi_1, \psi_2)$, since $g(s) \to \infty$ as $s \to \infty$. Thus, by Proposition 2.2 there exist a positive solution (u, v) of (1.1) such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$. This completes the proof of Theorem 2.3. ∎

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