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Certain Classes of Analytic Functions Defined by Convolution with Varying Argument of Coefficients

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Abstract In this paper we introduce the classes $\kappa - ST(f, g, \alpha; \beta)$ and $\kappa - VCT(f, g, \alpha; \beta)$, of $\kappa-$ uniformly starlike functions of order α and type β and the $\kappa-$ uniformly convex functions of order α and type β with varying arguments of coefficient, respectively. Moreover, we give coefficient estimates, distortion theorems and extreme points for functions belonging to these classes.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

the Hadamard product (or convolution) $(f * g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.3}$$

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Definition 1.1. Let $\kappa-ST(f,g,\alpha;\beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) and function $g(z)$ of the form (1.2) and satisfy the following inequality:

$$Re\left\{\frac{z(f*g)'(z)}{(f*g)(z)}-\alpha\right\} > \kappa\left|\frac{z(f*g)'(z)}{(f*g)(z)}-\beta\right|, \quad (0 \leq \alpha<\beta\leq 1;\kappa(1-\beta)<(1-\alpha);z\in\mathbb{U}). \tag{1.4}$$

Also Let $\kappa-CT(f,g,\alpha;\beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) and function $g(z)$ of the form (1.2) and satisfy the following inequality:

$$Re\left\{1+\frac{z(f*g)''(z)}{(f*g)'(z)}-\alpha\right\} > \kappa\left|1+\frac{z(f*g)''(z)}{(f*g)'(z)}-\beta\right| \quad (0 \leq \alpha<\beta\leq 1;\kappa(1-\beta)<(1-\alpha);z\in\mathbb{U}). \tag{1.5}$$

It follows from (1.4) and (1.5) that

$$f(z)\in\kappa-CT(f,g;\alpha,\beta)\Longleftrightarrow zf^{'}(z)\in\kappa-ST(f,g;\alpha,\beta). \tag{1.6}$$

We note that

(i) $\kappa-ST(f,\frac{z}{1-z},\alpha;\beta)=\kappa-ST(\alpha,\beta)(0\leq\alpha<\beta\leq 1,\kappa(1-\beta)<(1-\alpha);z\in\mathbb{U})$ (see Sim et al. [1] and El-Ashwah et al. [2]).

(ii) $\kappa-CT(f,\frac{z}{1-z},\alpha;\beta)=\kappa-UCV(\alpha,\beta)(0\leq\alpha<\beta\leq 1,\kappa(1-\beta)<(1-\alpha);z\in\mathbb{U})$(see Sim et al. [1] and El-Ashwah et al. [2]).

We also note that, for different choices of $g(z)$ we have the following new classes:

(i) $\kappa-ST(f,z+\sum\limits_{n=2}^{\infty}\left(\frac{1+l+\lambda(n-1)}{1+l}\right)^m z^n,\alpha;\beta)=\kappa-ST(m,\alpha,\beta)$

$$\left\{\begin{array}{c} f\in\mathcal{A}: Re\left\{\frac{z(I^m(\lambda,l)f(z))'}{I^m(\lambda,l)f(z)}-\alpha\right\}>\kappa\left|\frac{z(I^m(\lambda,l)f(z))'}{I^m(\lambda,l)f(z)}-\beta\right|, \\ (0\leq\alpha<\beta\leq 1,\kappa(1-\beta)<(1-\alpha),\lambda\geq 0,l\geq 0, \\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,...\};z\in\mathbb{U}) \end{array}\right\}.$$

(ii) $\kappa-CT(f,z+\sum\limits_{n=2}^{\infty}\left(\frac{1+l+\lambda(n-1)}{1+l}\right)^m z^n,\alpha;\beta)=\kappa-UCV(m,\alpha,\beta)$

$$\left\{\begin{array}{c} f\in\mathcal{A}: Re\left\{1+\frac{z(I^m(\lambda,l)f(z))''}{(I^m(\lambda,l)f(z))'}\right\}>\kappa\left|1+\frac{z(I^m(\lambda,l)f(z))''}{(I^m(\lambda,l)f(z))'}-\beta\right| \\ (0\leq\alpha<\beta\leq 1,\kappa(1-\beta)<(1-\alpha),\lambda\geq 0,l\geq 0,m\in\mathbb{N}_0;z\in\mathbb{U}) \end{array}\right\},$$

where the operator $I^m(\lambda,l)$ was introduced and studied by Cãtas et al. [3], which generalizes other operators see ([4–7]).

(iii) $\kappa-ST(f,z+\sum\limits_{n=2}^{\infty}\Omega_n(\alpha_1)z^n;\alpha,\beta)=\kappa-TS_{q,s}(n,\alpha,\beta)$,

$$\left\{\begin{array}{l} f\in\mathcal{A}: Re\left\{\dfrac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)}-\alpha\right\} \\ >\kappa\left|\dfrac{z(H_{q,s}(\alpha_1)f(z))'}{(H_{q,s}(\alpha_1)f(z))}-\beta\right|(z\in\mathbb{U}), \end{array}\right\}$$

where

$$\Omega_n(\alpha_1)=\frac{(\alpha_1)_{n-1}...(\alpha_q)_{n-1}}{(\beta_1)_{n-1}...(\beta_s)_{n-1}}\frac{1}{(n-1)!} \tag{1.7}$$

and

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & k=0 \\ a(a+1)(a+2)...(a+k-1), & k \in \mathbb{N} \end{cases}, \tag{1.8}$$

$(0 \le \alpha < 1, \beta \ge 0, \alpha_i \in \mathbb{C}(i = 1, 2, ...q)$ and $\beta_j \in \mathbb{C}\backslash\{-1, -2, ...\}$,
$j = 1, 2, ...s, z \in \mathbb{U})$;

(iv)$\kappa - ST(f, z + \sum_{n=2}^{\infty} \Omega_n(\alpha_1)z^n; \alpha, \beta) = \kappa - TC_{q,s}(n, \alpha, \beta)$,

$$\left\{ \begin{array}{l} f \in \mathcal{A} : Re\left\{1 + \frac{z(H_{q,s}(\alpha_1)f(z))''}{(H_{q,s}(\alpha_1)f(z))'} - \alpha\right\} \\ > \kappa \left|1 + \frac{z(H_{q,s}(\alpha_1)f(z))''}{(H_{q,s}(\alpha_1)f(z))'} - \beta\right| (z \in \mathbb{U}), \end{array} \right\}$$

where $\Omega_n(\alpha_1)$ is given by (1.7) and $H_{q,s}(\alpha_1)$ was introduced and studied by Dziok-Srivastava [8].

Silverman [9] defined the class $V(\theta_n)$ of univalent functions in the form (1.1) with varying arguments of coefficient as follows:

Definition 1.2. [9] A function $f(z)$ of the form (1.1) is said to be in the class $V(\theta_n)$ if $f(z) \in \mathcal{A}$ and $\arg(a_n) = \theta_n$ for all $n \ge 2$. If furthermore there exists a real number δsuch that $\theta_n + (n-1)\delta \equiv \pi(mod 2\pi)$ for all $n \ge 2$, then $f(z)$ is said to be in the class $V(\theta_n, \delta)$. The union of $V(\theta_n, \delta)$ taken over all possible sequences $\{\theta_n\}$ and possible real numbers δ is denoted by V.

Let $\kappa - VST(f, g, \alpha; \beta)$ denote the subclass of V consisting of functions $f(z) \in \kappa - ST(f, g, \alpha; \beta)$. Also let $\kappa - VCT(f, g, \alpha; \beta)$ denote the subclass of V consisting of functions $f(z) \in \kappa - CT(f, g, \alpha; \beta)$.

In this paper we obtain coefficient bounds for functions in the classes $\kappa - VST(f, g, \alpha; \beta)$ and $\kappa - VCT(f, g, \alpha; \beta)$, further we obtain distortion bounds and the extreme points for functions in these classes.

2. Coefficient Estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \le \alpha < \beta \le 1, \kappa(1-\beta) < (1-\alpha), g(z)$ is given by (1.2) with $b_n > 0 (n \ge 2)$ and $z \in \mathbb{U}$. We shall need the following lemmas.

Theorem 2.1. *If the function $f(z)$ given by* (1.1) *satisfies the condition*

$$\sum_{n=2}^{\infty} [\kappa(n-\beta) + (n-\alpha)] |a_n| b_n \le 1 - \alpha - \kappa(1-\beta). \tag{2.1}$$

Then $f(z) \in \kappa - ST(f, g, \alpha; \beta)$.

Proof. It is sufficient to show that inequality (1.4) holds true. Using the fact that

$$Re\{w - \alpha\} > \kappa |w - \beta| \Longleftrightarrow Re\left\{(1 + \kappa e^{i\theta})w - \beta\kappa e^{i\theta}\right\} > \alpha, \tag{2.2}$$

then inequality (1.4) may be written as

$$Re\left\{(1+\kappa e^{i\theta})\frac{z(f*g)'(z)}{(f*g)(z)}-\beta\kappa e^{i\theta}\right\}>\alpha, \tag{2.3}$$

or

$$Re\left\{\frac{A(z)}{B(z)}\right\}>\alpha, \tag{2.4}$$

where $A(z)=(1+\kappa e^{i\theta})z(f*g)'(z)-\beta\kappa e^{i\theta}(f*g)(z)$ and $B(z)=(f*g)(z)$. The condition (1.4) or (2.4) is equivalent to

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\geq 0. \tag{2.5}$$

We note that

$$\begin{aligned}|A(z)+(1-\alpha)B(z)| &= \left|[(1-\beta)\kappa e^{i\theta}+2-\alpha]z-\sum_{n=2}^{\infty}[(\beta-n)\kappa e^{i\theta}+\alpha-n-1]b_na_nz^n\right|\\ &\geq [-\kappa(1-\beta)+2-\alpha]\,|z|\\ &\quad -\sum_{n=2}^{\infty}[(n-\beta)\kappa+n-\alpha+1]b_n\,|a_n|\,|z|^n. \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}|A(z)-(1+\alpha)B(z)| &= \left|[(1-\beta)\kappa e^{i\theta}-\alpha]z+\sum_{n=2}^{\infty}[(n-\beta)\kappa e^{i\theta}+n-\alpha-1]b_na_nz^n\right|\\ &\leq [\kappa(1-\beta)+\alpha]\,|z|\\ &\quad +sum_{n=2}^{\infty}[(n-\beta)\kappa+n-\alpha-1]b_n\,|a_n|\,|z|^n. \end{aligned} \tag{2.7}$$

Using (2.6) and (2.7), we obtain the following inequality:

$$\begin{aligned}&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \qquad (2.8)\\ \geq\ & 2[(1-\alpha)-\kappa(1-\beta)]\,|z|-2\sum_{n=2}^{\infty}[(n-\beta)\kappa+(n-\alpha)]b_n\,|a_n|\,|z|^n.\end{aligned}$$

The expression $|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$ is bounded below by 0 if

$$2[(1-\alpha)-\kappa(1-\beta)]\,|z|-2\sum_{n=2}^{\infty}[\kappa(n-\beta)+(n-\alpha)]b_n\,|a_n|\,|z|^n>0,$$

or

$$\sum_{n=2}^{\infty}[\kappa(n-\beta)+(n-\alpha)]b_n\,|a_n|<(1-\alpha)-\kappa(1-\beta). \tag{2.9}$$

Hence the proof of Theorem 2.1 is completed. ■

By using (1.6) and (2.1) we can obtain the following theorem.

Theorem 2.2. *If the function $f(z)$ given by* (1.1) *satisfies the condition*

$$\sum_{n=2}^{\infty}n[\kappa(n-\beta)+(n-\alpha)]b_n\,|a_n|<(1-\alpha)-\kappa(1-\beta). \tag{2.10}$$

Then $f(z)\in\kappa-CT(f,g,\alpha;\beta)$.

Theorem 2.3. *Let $f(z)$ be defined by (1.1), then $f(z) \in \kappa - VST(f, g, \alpha; \beta)$ if and only if*

$$\sum_{n=2}^{\infty}[\kappa(n-\beta)+(n-\alpha)]b_n |a_n| < (1-\alpha) - \kappa(1-\beta). \tag{2.11}$$

Proof. In view of Theorem 2.1, we need only to show that the function $f(z) \in \kappa - VST(f, g, \alpha; \beta)$ satisfies the coefficient inequality (2.11). If $f(z) \in \kappa - ST(f, g, \alpha; \beta)$, then from (1.4), we have

$$Re\left\{\frac{z(f*g)'(z)}{(f*g)(z)} - \alpha\right\} > \kappa\left|\frac{z(f*g)'(z)}{(f*g)(z)} - \beta\right|,$$

thus we have

$$Re\left\{\frac{(1-\alpha)+\sum\limits_{n=2}^{\infty}(n-\alpha)b_n a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty} b_n a_n z^{n-1}}\right\} \tag{2.12}$$

$$> \quad \kappa\left|\frac{(1-\beta)+\sum\limits_{n=2}^{\infty}(n-\beta)b_n a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty} b_n a_n z^{n-1}}\right|.$$

Since $f(z) \in V$, then $f(z)$ lies in the class $V(\theta_n, \delta)$ for some sequences $\{\theta_n\}$ and real number δ such that

$$\theta_n + (n-1)\delta \equiv \pi(mod 2\pi)\ (n \geq 2).$$

Setting $z = re^{i\delta}$ in (2.12), then we obtain

$$\frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}(n-\alpha)b_n |a_n| r^{n-1}}{1-\sum\limits_{n=2}^{\infty} b_n |a_n| r^{n-1}}$$

$$> \quad \kappa\left[\frac{(1-\beta)+\sum\limits_{n=2}^{\infty}(n-\beta)b_n |a_n| r^{n-1}}{1-\sum\limits_{n=2}^{\infty} b_n |a_n| r^{n-1}}\right].$$

Letting $r \to 1^{-}$, then we have the inequality (2.11). Hence the proof of Theorem 2.3 is completed. ∎

Corollary 2.4. *If $f(z) \in \kappa - VST(f, g, \alpha; \beta)$, then*

$$|a_n| \leq \frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(n-\beta)+(n-\alpha)]b_n}(n \geq 2). \tag{2.13}$$

The inequality holds for the function

$$f(z) = z + \frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(n-\beta)+(n-\alpha)]b_n}e^{i\theta_n}z^n(n \geq 2; z \in \mathbb{U}). \tag{2.14}$$

Using the same technique used in Theorem 2.3 we get the following theorem.

Theorem 2.5. *Let $f(z)$ be of the form* (1.1)*, then $f(z) \in \kappa - VCT(f, g, \alpha; \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n[\kappa(n-\beta) + n - \alpha]b_n |a_n| < (1-\alpha) - \kappa(1-\beta). \tag{2.15}$$

Corollary 2.6. *If $f(z) \in \kappa - VCT(f, g, \alpha; \beta)$, then*

$$|a_n| \leq \frac{(1-\alpha) - \kappa(1-\beta)}{n[\kappa(n-\beta) + (n-\alpha)]b_n} (n \geq 2). \tag{2.16}$$

The inequality holds for the function

$$f(z) = z + \frac{(1-\alpha) - \kappa(1-\beta)}{n[\kappa(n-\beta) + (n-\alpha)]b_n} e^{i\theta_n} z^n (n \geq 2; z \in \mathbb{U}). \tag{2.17}$$

3. Distortion Theorems

Theorem 3.1. *Let the function $f(z)$ defined* (1.1) *be in the class $\kappa - VST(f, g, \alpha; \beta)$. Then*

$$|z| - \frac{(1-\alpha) - \kappa(1-\beta)}{[\kappa(2-\beta) + (2-\alpha)]b_2} |z|^2 \leq |f(z)| \leq |z| + \frac{(1-\alpha) - \kappa(1-\beta)}{[\kappa(2-\beta) + (2-\alpha)]b_2} |z|^2. \tag{3.1}$$

The result is sharp.

Proof. We employ the same technique as used by Silverman [6]. In view of Theorem 2.3, since

$$\Phi(n) = [\kappa(n-\beta) + (n-\alpha)]b_n, \tag{3.2}$$

is an increasing function of $n (n \geq 2)$, we have

$$\Phi(2) \sum_{n=2}^{\infty} |a_n| \leq \sum_{k=2}^{\infty} \Phi(n) |a_n| \leq (1-\alpha) - \kappa(1-\beta), \tag{3.3}$$

that is

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(1-\alpha) - \kappa(1-\beta)}{\Phi(2)}. \tag{3.4}$$

Thus, we have

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq |z| + \frac{(1-\alpha) - \kappa(1-\beta)}{[\kappa(2-\beta) + (2-\alpha)]b_2} |z|^2. \tag{3.5}$$

Similarly, we get

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n|$$

$$\geq |z| - \frac{(1-\alpha) - \kappa(1-\beta)}{[\kappa(2-\beta) + (2-\alpha)]b_2} |z|^2. \tag{3.6}$$

This completes the proof of Theorem 3.1. Finally the result is sharp for the function:

$$f(z) = z + \frac{(1-\alpha) - \kappa(1-\beta)}{[\kappa(2-\beta) + (2-\alpha)]b_2} z^2 e^{i\theta_2} z^2, \tag{3.7}$$

at $z = \pm|z|\, e^{i\theta_2}$. ■

Corollary 3.2. *Under the hypotheses of Theorem 3.1, $f(z)$ is included in a disc with center at the origin and radius r_1 given by*

$$r_1 = 1 + \frac{(1-\alpha) - \kappa(1-\beta)}{[\kappa(2-\beta) + (2-\alpha)]b_2}. \tag{3.8}$$

Theorem 3.3. *Let the function $f(z)$ defined by* (1.1) *be in the class $\kappa - VST(f, g, \alpha; \beta)$. Then*

$$1 - \frac{2[(1-\alpha) - \kappa(1-\beta)]}{[\kappa(2-\beta) + (2-\alpha)]b_2}\,|z| \le |f'(z)| \le 1 + \frac{2[(1-\alpha) - \kappa(1-\beta)]}{[\kappa(2-\beta) + (2-\alpha)]b_2}\,|z|\,. \tag{3.9}$$

The result is sharp.

Proof. Similarly for $\Phi(n)$ defined by (3.2) it is clear that $\frac{\Phi(n)}{n}$ is an increasing function of $n(n \ge 2)$,in view of Theorem 2.3, we have

$$\frac{\Phi(2)}{2}\sum_{n=2}^{\infty} n\,|a_n| \le \sum_{n=2}^{\infty}\frac{n\Phi(n)}{n}\,|a_n| \le (1-\alpha) - \kappa(1-\beta), \tag{3.10}$$

that is

$$\sum_{n=2}^{\infty} n\,|a_n| \le \frac{2[(1-\alpha) - \kappa(1-\beta)]}{\Phi(2)}. \tag{3.11}$$

Thus, we have

$$\begin{aligned} |f'(z)| &\le 1 + |z| \sum_{n=2}^{\infty} n\,|a_n| \\ &\le 1 + \frac{2[(1-\alpha) - \kappa(1-\beta)]}{[\kappa(2-\beta) + (2-\alpha)]b_2}\,|z|\,. \end{aligned} \tag{3.12}$$

Similarly,

$$\begin{aligned} |f'(z)| &\ge 1 - |z| \sum_{n=2}^{\infty} n\,|a_n| \\ &\ge 1 - \frac{2[(1-\alpha) - \kappa(1-\beta)]}{[\kappa(2-\beta) + (2-\alpha)]b_2}\,|z|\,. \end{aligned} \tag{3.13}$$

Finally, we can see that the assertions of Theorem 3.3 are sharp for the function $f(z)$ defined by (3.7). This completes the proof of the Theorem 3.3. ■

Corollary 3.4. *Under the hypotheses of Theorem 3.3, $f'(z)$ is included in a disc with center at origin and radius r_2 given by*

$$r_2 = 1 + \frac{2[(1-\alpha) - \kappa(1-\beta)]}{[\kappa(2-\beta) + (2-\alpha)]b_2}. \tag{3.14}$$

Using the same technique used in Theorems 3.1 and 3.3 we get the following theorems.

Theorem 3.5. *Let the function $f(z)$ defined* (1.1) *be in the class $\kappa - VCT(f, g, \alpha; \beta)$. Then*

$$|z| - \frac{(1-\alpha)-\kappa(1-\beta)}{2[\kappa(2-\beta)+(2-\alpha)]b_2}|z|^2 \le |f(z)| \le |z| + \frac{(1-\alpha)-\kappa(1-\beta)}{2[\kappa(2-\beta)+(2-\alpha)]b_2}|z|^2. \tag{3.15}$$

The result is sharp for following function

$$f(z) = z + \frac{(1-\alpha)-\kappa(1-\beta)}{2[\kappa(2-\beta)+(2-\alpha)]b_2}|z|^2 e^{i\theta_2} z^2, \tag{3.16}$$

Theorem 3.6. *Let the function $f(z)$ defined by* (1.1) *be in the class $\kappa - VCT(f, g, \alpha; \beta)$. Then*

$$1 - \frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(2-\beta)+(2-\alpha)]b_2}|z| \le |f'(z)| \le 1 + \frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(2-\beta)+(2-\alpha)]b_2}|z|. \tag{3.17}$$

The result is sharp for function given by (3.16).

4. Extreme Points

Theorem 4.1. *Let the function $f(z)$ defined by* (1.1) *be in the class $\kappa - VST(f, g, \alpha; \beta)$, with $\arg(a_n) = \theta_n$, where $\theta_n + (n-1)\delta \equiv \pi (mod 2\pi) (n \ge 2)$. Let*

$$f_1(z) = z$$

and

$$f_n(z) = z + \frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(n-\beta)+(n-\alpha)]b_n} e^{i\theta_n} z^n (n \ge 2; z \in \mathbb{U}). \tag{4.1}$$

Then $f(z) \in \kappa - VST(f, g, \alpha; \beta)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \tag{4.2}$$

where $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. If $f(z)$ is given by (4.2), with $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$, then

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z).$$

$$= \left(1 - \sum_{n=2}^{\infty} \lambda_n\right) z + \sum_{n=2}^{\infty} \frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(n-\beta)+(n-\alpha)]b_n} \lambda_n e^{i\theta_n} z^n$$

$$= z + \sum_{n=2}^{\infty} \frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(n-\beta)+(n-\alpha)]b_n} \lambda_n e^{i\theta_n} z^n. \tag{4.3}$$

But according to (2.11), we see that

$$\begin{aligned}
&\sum_{n=2}^{\infty}[\kappa(n-\beta)+(n-\alpha)]b_n \cdot \left|\frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(n-\beta)+(n-\alpha)]b_n}\lambda_n e^{i\theta_n}\right| \\
=\ &\sum_{n=2}^{\infty}[\kappa(n-\beta)+(n-\alpha)]b_n \cdot \frac{(1-\alpha)-\kappa(1-\beta)}{[\kappa(n-\beta)+(n-\alpha)]b_n}\lambda_n \\
=\ &\sum_{n=2}^{\infty}(1-\alpha)-\kappa(1-\beta)\lambda_n = (1-\lambda_1)[(1-\alpha)-\kappa(1-\beta) \\
\leq\ &(1-\alpha)-\kappa(1-\beta). \qquad (4.4)
\end{aligned}$$

Then $f(z)$ satisfies (2.11), hence $f(z) \in \kappa - VST(f, g, \alpha; \beta)$.

Conversely, let the function $f(z)$ defined by (1.1) be in the class $\kappa - VST(f, g, \alpha; \beta)$, and define

$$\lambda_n = \frac{[\kappa(n-\beta)+(n-\alpha)]b_n}{(1-\alpha)-\kappa(1-\beta)}a_n, n \geq 2$$

$$\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n.$$

Form Theorem 2.3, $\sum_{n=2}^{\infty}\lambda_n \leq 1$ and so $\lambda_1 \geq 0$. Since $\lambda_n f_n(z) = \lambda_n z + a_n z^n$, then

$$\sum_{n=1}^{\infty}\lambda_n f_n(z) = z + \sum_{n=2}^{\infty}a_n z^n = f(z).$$

This completes the proof of Theorem 4.1. ■

Finally using the same technique used in Theorem 4.1 we get the following theorem.

Theorem 4.2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\kappa - VCT(f, g, \alpha; \beta)$, with $\arg(a_n) = \theta_n$, where $\theta_n + (n-1)\delta \equiv \pi(mod 2\pi)(n \geq 2)$. Define*

$$f_1(z) = z$$

and

$$f_n(z) = z + \frac{(1-\alpha)-\kappa(1-\beta)}{n[\kappa(n-\beta)+(n-\alpha)]b_n}e^{i\theta_n}z^n (n \geq 2; z \in \mathbb{U}).$$

Then $f(z) \in \kappa - VCT(f, g, \alpha; \beta)$ if and only if $f(z)$ can be expressed in the form (4.2), *where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty}\lambda_k = 1$.*

Remark 4.3. Taking $g(z) = \frac{z}{1-z}$or ($b_n = 1$,with $n \geq 2$) in our results, we obtain the results obtained by El-Ashwah et al. [2].

Remark 4.4. Specializing the function $g(z)$ in our results,we obtain new results associated to the subclasses $\kappa - ST(m, \alpha, \beta)$, $\kappa - UCV(m, \alpha, \beta)$, $\kappa - TS_{q,s}(n, \alpha, \beta)$ and $\kappa - TC_{q,s}(n, \alpha, \beta)$ defined in the introduction.

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