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Coupled Coincidence Points of Almost Generalized (ψ, ϕ)-Weakly Contractive Maps under an (F, g)-Invariant Set

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Abstract Let $(X,\ d)$ be a metric space and $F : X \times X \to X$ and $g : X \to X$ be maps. In this paper, we define almost generalized (ψ, ϕ)-weakly contractive maps and prove the existence of coupled coincidence points of such maps under an $(F,\ g)$-invariant set without using the mixed g-monotone property in a metric space setting. We also, apply our results to obtain the existence of coupled coincidence points in partially ordered metric spaces. Our results generalize the results of Choudhury, Metiya and Kundu [B.S. Choudhury, N. Metiya, A. Kundu, Coupled coincidence point theorems in ordered metric spaces, Ann. Univ. Ferrara 57 (2011) 1–16]. We also provide examples in support of our results.

MSC: 47H10; 54H25

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1. Introduction

In 1997, Alber and Guerrer-Delabriere [1] introduced the concept of weakly contractive maps in Hilbert spaces as a generalization of contraction maps and proved the existence of fixed points of weakly contractive maps in the Hilbert space setting. Rhoades [2] extended this concept to Banach spaces and proved the existence of fixed points of weakly contractive maps. In 2009, Dutta and Choudhury [3] proved a fixed point theorem by introducing a new generalization of weakly contractive maps by using the altering distance functions. For more works in this line of research we refer [4, 5] and references therein.

Let (X, d) be a metric space. A function $f : X \to R$ is said to be lower semi-continuous at a point x_0 in X if, for any sequence $\{x_n\}$ in X with $x_n \to x_0$ as $n \to \infty$, implies that $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$.

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If f is lower semicontinuous at every point of X, then we say that f is lower semicontinuous on X.

Throughout this paper, we denote $R^+= [0,\infty)$, $\mathbb{Z}^+$, the set of all natural numbers,
$\Psi = \{\psi : R^+ \to R^+ /\ \psi$ is continuous, nondecreasing and $\psi(t) = 0 \Leftrightarrow t = 0\}$ and
$\Phi = \{\phi : R^+ \to R^+ /\ \phi$ is lower semi-continuous and $\phi(t) = 0 \Leftrightarrow t = 0\}$.

In 1987, Guo and Lakshmikantham [6] introduced the notion of mixed monotone operators. In 2006, Bhaskar and Lakshmikantham[7] introduced the notion of coupled fixed points and proved some coupled fixed point theorems for a mapping with mixed monotone property in the setting of partially ordered metric spaces.

In 2009, Harjani, Sadarangani [8] extended the concept of weakly contractive maps to partially ordered sets and proved fixed point results in the setting of partially ordered sets.

Definition 1.1. [7] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Then F is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y, *i.e.*, for any x, y in X,
$x_1,\ x_2 \in X,\ \ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$ and
$y_1,\ y_2 \in X,\ \ y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$.

Definition 1.2. [7] Let X be a nonempty set and $F : X \times X \to X$. An element (x, y) in $X \times X$ is said to be a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

In 2009, Lakshmikantham and Ciric [9] extended the notion of mixed monotone property to mixed g-monotone property and proved the existence of coupled coincidence and coupled common fixed points in partially ordered metric spaces by using the concept of commuting maps in the context of coupled fixed points.

Definition 1.3. (Lakshmikantham and Ciric [9]) Let $(X, \preceq)$ be a partially ordered set. Let $F : X \times X \to X$ and $g : X \to X$ be maps. We say that F has the mixed g-monotone property if for any $x, y \in X$,
$x_1,\ x_2 \in X,\ \ gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$,
$y_1,\ y_2 \in X,\ \ gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$.

Definition 1.4. [9] Let X be a nonempty set and $F : X \times X \to X$ and $g : X \to X$ be maps. An element $(x, y) \in X \times X$ is said to be a coupled coincidence point of F and g if $gx = F(x, y)$ and $gy = F(y, x)$.

Definition 1.5. [9] Let X be a nonempty set and $F : X \times X \to X$ and $g : X \to X$ be maps. An element $(x, y) \in X \times X$ is said to be a coupled common fixed point of F and g if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.6. [9] Let X be a nonempty set and $F : X \times X \to X$ and $g : X \to X$ be maps. We say that F and g are commutative if $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$.

In 2010, Choudhury and Kundu [10] introduced the concept of compatible maps in the context of coupled fixed points and proved the existence of coupled coincidence points in partially ordered metric spaces.

Definition 1.7. [10] Let (X, d) be a metric space. The maps $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if $\lim\limits_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$ and $\lim\limits_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0$ *whenever* $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = x$ and $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = y$ for some $x, y \in X$.

In 2011, Harjani, Lopez and Sadarangani [11] extended (ψ, ϕ)-contraction to the mixed monotone operators and proved the existence of coupled fixed points in partially ordered sets.

Theorem 1.8. (Harjani, Sadarangani [11]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist functions $\psi, \phi \in \Psi$ such that*

$$\psi(d(F(x, y), F(u, v)) \leq \psi(\max\{d(x, u), d(y, v)\}) - \phi(\max\{(d(x, u), d(y, v)\}) \quad (1.1)$$

for all $x, y, u, v \in X$ with $x \succeq u$, $y \preceq v$. Also, suppose that either
(i) F is continuous (or)
(ii) X has the following properties:
(a) if $\{x_n\}$ is a nondecreasing sequence in X with $x_n \to x$ then $x_n \preceq x$ for all n,
(b) if $\{y_n\}$ is a non-increasing sequence in X with $y_n \to y$ then $y_n \succeq y$ for all n.
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

In 2011, Choudhury, Metiya and Kundu [12] extended Theorem 1.8 to a pair of compatible mappings, and proved the existence of coupled coincidence points.

Theorem 1.9. (Choudhury, Metiya and Kundu [12]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be maps such that F has the mixed g monotone property on X. Assume that there exist $\psi \in \Psi$ and $\phi : R^+ \to R^+$ satisfying ϕ is continuous, $\phi(t) = 0$ if and only if $t = 0$ such that*

$$\begin{aligned}\psi(d(F(x, y), F(u, v)) &\leq \psi(\max\{d(gx, gu), d(gy, gv)\}) \\ &\quad - \phi(\max\{(d(gx, gu), d(gy, gv)\})\end{aligned} \quad (1.2)$$

for all $x, y, u, v \in X$ with $gx \succeq gu$, $gy \preceq gv$. Suppose that $F(X \times X) \subseteq gX$, g is continuous and F and g are compatible mappings. Also, assume that

- *(i) F is continuous or*
- *(ii) X has the following properties:*
  - *(a) if $\{x_n\}$ is a nondecreasing sequence in X with $x_n \to x$ then $gx_n \preceq gx$ for all $n \geq 0$,*
  - *(b) if $\{y_n\}$ is a non-increasing sequence in X with $y_n \to y$ then $gy_n \succeq gy$ for all $n \geq 0$.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$. i.e., F and g have a coupled coincidence point in X.

On the otherhand, in 2004, Berinde [13] introduced 'almost contraction maps' as a generalization of contraction maps in complete metric spaces. In 2008, Babu, Sandhya and Kameswari [14] modified this definition by introducing ' condition (B)' and proved the existence and uniqueness of fixed points in metric spaces. For more works in this direction we refer [15–17].

Samet and Vetro [18] introduced an F-invariant set in four variables and proved the existence of coupled fixed points.

Definition 1.10. [18] Let X be a nonempty set and M be a nonempty subset of X^4. We say that M is an F-invariant subset of X^4 if the following two conditions hold:
For all $x, y, z, w \in X$,

(i) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$,
(ii) $(x, y, z, w) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

Sintunavarat, Petrusel and Kumam [19] extended the notion of an F-invariant set to a pair of mappings and proved common coupled fixed point results for w^*-compatible mappings without using mixed monotone property in the setting of cone metric spaces.

Definition 1.11. [19] Let X be a nonempty set and $F : X \times X \to X$ and $g : X \to X$ be given maps. Let M be a nonempty subset of X^4. We say that M is an (F, g)-invariant subset of X^4 if for all $x, y, z, w \in X$,

(i) $(gx, gy, gz, gw) \in M \Leftrightarrow (gw, gz, gy, gx) \in M$,
(ii) $(gx,\ gy,\ gz,\ gw) \in M \Rightarrow (F(x, y),\ F(y, x),\ F(z, w),\ F(w, z)) \in M$.

Here we observe that if g is the identity mapping, then M is an F-invariant subset of X^4.

Recently, Sintunavarat, Kumam, Cho [20] introduced new property called 'transitivity property' and proved coupled fixed point results for nonlinear contractions without using mixed monotone property.

Definition 1.12. [20] Let X be a nonempty set and M be a nonempty subset of X^4. We say that M satisfies 'transitivity property' if for all $x, y, z, w, a, b \in X$, $(x, y, z, w) \in M$ and $(z, w, a, b) \in M \Rightarrow (x, y, a, b) \in M$.

For more literature on the works of coupled fixed point results under an F-invariant set, we refer [18, 21–23].

Example 1.13. [20] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping satisfying the mixed monotone property, that is, for all $x, y \in X$, we have
$x_1,\ x_2 \in X,\ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$
and $y_1,\ y_2 \in X,\ y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq F(x, y_1)$.
Let us define a subset $M \subseteq X^4$ by $M = \{(a,\ b,\ c,\ d) \in X^4 : a \succeq c,\ b \preceq d\}$. Then M is an F-invariant subset of X^4, which satisfies the transitivity property.

Example 1.14. [19] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be maps. Let F satisfy the mixed g-monotone property, that is, for all $x, y \in X$, we have $x_1,\ x_2 \in X,\ gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$ and
$y_1,\ y_2 \in X,\ gy_1 \preceq gy_2 \Rightarrow F(x, y_2) \preceq F(x, y_1)$.

Let us define a subset $M \subseteq X^4$ by $M = \{(a,\ b,\ c,\ d) \in X^4 : a \succeq c,\ b \preceq d\}$. Then M is an (F, g)-invariant subset of X^4 which satisfies the transitivity property.

Motivated by the works of Doric, Kadelburg and Radenovic [21], Sintunavarat, Kumam, Cho [20] and Sintunavarat, Petrusel and Kumam [19], in Section 2 of this paper we define an almost generalized (ψ, ϕ)-weakly contractive maps and prove the existence of coupled coincidence points of such maps under an $(F,\ g)$-invariant set without using the mixed g-monotone property in a metric space setting. Also, we apply our results to obtain the existence of coupled coincidence points in partially ordered metric spaces in

Section 3. Our results generalize the results of Choudhury, Metiya and Kundu [12]. We also provide examples in support of our results.

2. Main Results

Definition 2.1. Let (X, d) be a metric space and $F : X \times X \to X$ and $g : X \to X$ be maps. Let M be a nonempty (F, g)-invariant subset of X^4. If there exist functions $\psi \in \Psi$, $\phi \in \Phi$ and a constant $L \geq 0$ such that

$$\begin{aligned}\psi(d(F(x,y),F(u,v))) \leq{}& \psi(\max\{d(gx,gu),d(gy,gv)\}) - \phi(\max\{d(gx,gu),d(gy,gv)\}) \\ &+ L\min\{d(gx,F(x,y)),d(gx,F(u,v)),d(gu,F(x,y)),d(gu,F(u,v))\}\end{aligned} \tag{2.1}$$

for every $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$, then we say that the maps F and g are *almost generalized* (ψ, ϕ)*-weakly contractive* maps.

The following two lemmas are useful in proving the main results in this paper.

Lemma 2.2. *Suppose that* (X, d) *is a metric space and let* $F : X \times X \to X$ *and* $g : X \to X$ *be maps. Let* $\{gx_n\}$ *and* $\{gy_n\}$ *be sequences in* X *such that*

$$\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} \to 0 \text{ as } n \to \infty. \tag{2.2}$$

If at least one of $\{gx_n\}$ *and* $\{gy_n\}$ *is not a Cauchy sequence, then there exist an* $\epsilon > 0$ *and sequences of positive integers* $\{n_k\}$ *and* $\{m_k\}$ *with* $n_k > m_k > k$ *such that*
$\max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \geq \epsilon$ *and*
$\max\{d(gx_{m_k}, gx_{n_k-1}), d(gy_{m_k}, gy_{n_k-1})\} < \epsilon$
and the following four identities hold:

$$\begin{aligned}
(i)\ & \lim_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} = \epsilon \\
(ii)\ & \lim_{k\to\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\} = \epsilon \\
(iii)\ & \lim_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} = \epsilon \\
(iv)\ & \lim_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\} = \epsilon.
\end{aligned}$$

Proof. Suppose that at least one of $\{gx_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. This implies that either
$\lim_{m,n\to\infty} d(gx_m, gx_n) \nrightarrow 0$ or $\lim_{m,n\to\infty} d(gy_m, gy_n) \nrightarrow 0$.
Hence $\max\{\lim_{m,n\to\infty} d(gx_m, gx_n), \lim_{m,n\to\infty} d(gy_m, gy_n)\} \nrightarrow 0$.
i.e., $\lim_{m,n\to\infty} \max\{d(gx_m, gx_n), d(gy_m, gy_n)\} \nrightarrow 0$. Thus there exists an $\epsilon > 0$, for which we can find two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers with $n_k > m_k > k$ such that

$$\max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \geq \epsilon \tag{2.3}$$

for all $k \in \{1, 2, 3, \cdots\}$.
We now choose n_k the smallest number exceeding m_k for which (2.3) holds.
Hence, we have $\max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\} \geq \epsilon$ and

$$\max\{d(gx_{m_k}, gx_{n_k-1}), d(gy_{m_k}, gy_{n_k-1})\} < \epsilon. \tag{2.4}$$

First we prove (i).
From the triangle inequality, we have

$$d(gx_{n_k}, gx_{m_k}) \leq d(gx_{n_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{m_k}) < d(gx_{n_k}, gx_{n_k-1}) + \epsilon \tag{2.5}$$

and also we have

$$d(gy_{n_k}, gy_{m_k}) \leq d(gy_{n_k}, gy_{n_k-1}) + d(gy_{n_k-1}, gy_{m_k}) < d(gy_{n_k}, gy_{n_k-1}) + \epsilon. \quad (2.6)$$

From (2.3), (2.5) and (2.6), we get

$$\begin{aligned} \epsilon \leq\ & \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \\ \leq\ & \max\{d(gx_{n_k}, gx_{n_k-1}), d(gy_{n_k}, gy_{n_k-1})\} + \epsilon \end{aligned} \quad (2.7)$$

On taking limit superior as $k \to \infty$ in (2.7) and from (2.2), it follows that
$\epsilon \leq \limsup\limits_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \leq \epsilon.$
Hence

$$\limsup_{k\to\infty}\ \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} = \epsilon \quad (2.8)$$

Again, on taking limit inferior as $k \to \infty$ in (2.3) and from (2.8), we have

$$\begin{aligned} \epsilon &\leq \liminf_{k\to\infty}\ \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \\ &\leq \limsup_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} = \epsilon. \end{aligned}$$

Hence

$$\liminf_{k\to\infty}\ \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} = \epsilon. \quad (2.9)$$

From (2.8) and (2.9), it follows that
$\lim\limits_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\}$ exists and
$\lim\limits_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{mk})\} = \epsilon$ so that (i) holds.

We now prove (ii).
By using the triangle inequality, we get

$$d(gx_{n_k+1}, gx_{m_k+1}) \leq d(gx_{n_k+1}, gx_{n_k}) + d(gx_{n_k}, gx_{m_k}) + d(gx_{m_k}, gx_{m_k+1}) \quad (2.10)$$

and

$$d(gy_{n_k+1}, gy_{m_k+1}) \leq d(gy_{n_k+1}, gy_{n_k}) + d(gy_{n_k}, gy_{m_k}) + d(gy_{m_k}, gy_{m_k+1}). \quad (2.11)$$

From (2.10)and (2.11), we obtain

$$\begin{aligned} &\max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\} \\ &\quad \leq\ \max\{d(gx_{n_k+1}, gx_{n_k}), d(gy_{n_k+1}, gy_{n_k})\} \\ &\quad + \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \\ &\quad + \max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\}. \end{aligned} \quad (2.12)$$

Again, by the triangle inequality, we have

$$d(gx_{n_k}, gx_{m_k}) \leq d(gx_{n_k}, gx_{n_k+1}) + d(gx_{n_k+1}, gx_{m_k+1}) + d(gx_{m_k+1}, gx_{m_k}) \quad (2.13)$$

and

$$d(gy_{n_k}, gy_{m_k}) \leq d(gy_{n_k}, gy_{n_k+1}) + d(gy_{n_k+1}, gy_{m_k+1}) + d(gy_{m_k+1}, gy_{m_k}). \quad (2.14)$$

Now, from (2.3), (2.13) and (2.14), we obtain

$$\begin{aligned} \epsilon \leq\ & \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \\ \leq\ & \max\{d(gx_{n_k}, gx_{n_k+1}), d(gy_{n_k}, gy_{n_k+1})\} \\ & + \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\} \\ & + \max\{d(gx_{m_k+1}, gx_{m_k}), d(gy_{m_k+1}, gy_{m_k})\}. \end{aligned} \quad (2.15)$$

On taking limit superior as $k \to +\infty$ in (2.12), (2.15) and from (2.2), it follows that

$$\epsilon \leq \limsup_{k\to\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\} \leq \epsilon.$$

Hence

$$\limsup_{k\to\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{mk+1})\} = \epsilon. \tag{2.16}$$

Also, from (2.15), we have,

$$\begin{aligned}\max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\}& \\ \geq \epsilon - \max\{d(gx_{n_k+1}, gx_{n_k}), d(gy_{n_k+1}, gy_{n_k})\}& \\ - \max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\}.&\end{aligned} \tag{2.17}$$

On taking limit inferior as $k \to +\infty$ in (2.17) and from (2.2), it follows that

$$\begin{aligned}\liminf_{k\to+\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\}& \\ \geq \liminf_{k\to+\infty}[\epsilon - \max\{d(gx_{n_k+1}, gx_{n_k}), d(gy_{n_k+1}, gy_{n_k})\}& \\ - \max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\}]& \\ \geq \epsilon + \liminf_{k\to+\infty}[- \max\{d(gx_{n_k+1}, gx_{n_k}), d(gy_{n_k+1}, gy_{n_k})\}]& \\ + \liminf_{k\to+\infty}[- \max\{d(gx_{m_k+1}, gx_{m_k}), d(gy_{m_k+1}, gy_{m_k})\}]& \\ = \epsilon - \limsup_{k\to+\infty} \max\{d(gx_{n_k+1}, gx_{n_k}), d(gy_{n_k+1}, gy_{n_k})\}& \\ - \limsup_{k\to+\infty} \max\{d(gx_{m_k+1}, gx_{m_k}), d(gy_{m_k+1}, gy_{m_k})\}& \\ = \epsilon.&\end{aligned} \tag{2.18}$$

On the other hand, we have

$$\begin{aligned}\liminf_{k\to+\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\}& \\ \leq \limsup_{k\to+\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\} = \epsilon \text{ (from (2.16))}.&\end{aligned} \tag{2.19}$$

Hence, from (2.18) and (2.19), we get

$$\liminf_{k\to\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\} = \epsilon. \tag{2.20}$$

Therefore, from (2.16) and (2.20), we have
$\lim_{k\to+\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\}$ exists
and $\lim_{k\to+\infty} \max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\} = \epsilon$.
Hence (ii) holds.
Now we prove (iii).
From the triangle inequality, we have

$$\begin{aligned}d(gx_{m_k}, gx_{n_k+1}) &\leq d(gx_{m_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) + d(gx_{n_k}, gx_{n_k+1}) \\ &\leq \epsilon + d(gx_{n_k-1}, gx_{n_k}) + d(gx_{n_k}, gx_{n_k+1})\end{aligned} \tag{2.21}$$

and

$$d(gy_{m_k}, gy_{n_k+1}) \leq d(gy_{m_k}, gy_{n_k-1}) + d(gy_{n_k-1}, gy_{n_k}) + d(gy_{n_k}, gy_{n_k+1}) \leq \epsilon + d(gy_{n_k-1}, gy_{n_k}) + d(gy_{n_k}, gy_{n_k+1}). \tag{2.22}$$

Hence from (2.21) and (2.22), we have

$$\max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} \leq \epsilon + \max\{d(gx_{n_k-1}, gx_{n_k}), d(gy_{n_k-1}, gy_{n_k})\} + \max\{d(gx_{n_k}, gx_{n_k+1}), d(gy_{n_k}, gy_{n_k+1})\}. \tag{2.23}$$

Also, we have

$$d(gx_{n_k}, gx_{m_k}) \leq d(gx_{n_k}, gx_{n_k+1}) + d(gx_{n_k+1}, gx_{m_k}) \tag{2.24}$$

$$d(gy_{n_k}, gy_{m_k}) \leq d(gy_{n_k}, gy_{n_k+1}) + d(gy_{n_k+1}, gy_{m_k}). \tag{2.25}$$

Now from (2.3), (2.24) and (2.25), we have

$$\begin{aligned}\epsilon \leq\ & \max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \\ \leq\ & \max\{d(gx_{n_k}, gx_{n_k+1}), d(gy_{n_k}, gy_{n_k+1})\} \\ & + \max\{d(gx_{n_k+1}, gx_{m_k}), d(gy_{n_k+1}, gy_{m_k})\}.\end{aligned} \tag{2.26}$$

On taking limit superior as $k \to \infty$ in (2.23) and (2.26), we obtain
$\epsilon \leq \limsup_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} \leq \epsilon.$
Hence

$$\limsup_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} = \epsilon. \tag{2.27}$$

Also, we have from (2.26)

$$\max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} \geq \epsilon - \max\{d(gx_{n_k}, gx_{n_k+1}), d(gy_{n_k}, gy_{n_k+1})\}. \tag{2.28}$$

On taking limit inferior as $k \to \infty$ in (2.28), we have

$$\begin{aligned}&\liminf_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} \\ &\geq \liminf_{k\to\infty} [\epsilon - \max\{d(gx_{n_k}, gx_{n_k+1}), d(gy_{n_k}, gy_{n_k+1})\}] \\ &\geq \epsilon + \liminf_{k\to\infty} [-\max\{d(gx_{n_k}, gx_{n_k+1}), d(gy_{n_k}, gy_{n_k+1})\}] \\ &= \epsilon - \limsup_{k\to\infty} \max\{d(gx_{n_k}, gx_{n_k+1}), d(gy_{n_k}, gy_{n_k+1})\} \\ &= \epsilon.\end{aligned} \tag{2.29}$$

Also, we have

$$\liminf_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} \leq \limsup_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} = \epsilon. \tag{2.30}$$

Hence, from (2.29) and (2.30), we have

$$\liminf_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} = \epsilon. \tag{2.31}$$

Hence, from (2.27) and (2.31), we have
$\lim_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\}$ exists and
$\lim_{k\to\infty} \max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\} = \epsilon$.
Thus (iii) holds.
Now we prove (iv).
By using the triangle inequality, we have

$$\begin{aligned} d(gx_{n_k}, gx_{m_k+1}) &\le d(gx_{n_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{m_k}) + d(gx_{m_k}, gx_{m_k+1}) \\ &\le \epsilon + d(gx_{n_k}, gx_{n_k-1}) + d(gx_{m_k}, gx_{m_k+1}) \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} d(gy_{n_k}, gy_{m_k+1}) &\le d(gy_{n_k}, gy_{n_k-1}) + d(gy_{n_k-1}, gy_{m_k}) + d(gy_{m_k}, gy_{m_k+1}) \\ &\le \epsilon + d(gy_{n_k-1}, gy_{n_k}) + d(gy_{m_k}, gy_{m_k+1}). \end{aligned} \tag{2.33}$$

Hence, from (2.32) and (2.33), we have
$\max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\}$

$$\begin{aligned} &\le \epsilon + \max\{d(gx_{n_k}, gx_{n_k-1}), d(gy_{n_k}, gy_{n_k-1})\} \\ &+ \max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\}. \end{aligned} \tag{2.34}$$

Also, we have

$$d(gx_{m_k}, gx_{n_k}) \le d(gx_{m_k}, gx_{m_k+1}) + d(gx_{m_k+1}, gx_{n_k}) \tag{2.35}$$

and

$$d(gy_{m_k}, gy_{n_k}) \le d(gy_{m_k}, gy_{m_k+1}) + d(gy_{m_k+1}, gy_{n_k}). \tag{2.36}$$

Hence, from (2.3), (2.35) and (2.36), we have

$$\begin{aligned} \epsilon \le\ & \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\} \\ \le\ & \max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\} \\ &+ \max\{d(gx_{m_k+1}, gx_{n_k}), d(gy_{m_k+1}, gy_{n_k})\}. \end{aligned} \tag{2.37}$$

On taking limit superior as $k \to \infty$ in (2.34) and (2.37), we obtain

$$\epsilon \le \limsup_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\} \le \epsilon.$$

Hence we have

$$\limsup_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\} = \epsilon. \tag{2.38}$$

Also, from (2.37), we have
$\max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\}$

$$\ge \epsilon - \max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\}. \tag{2.39}$$

On taking limit inferior as $k \to \infty$, in (2.39), we have

$\liminf_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\}$

$$\geq \liminf_{k\to\infty} [\epsilon - \max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\}]$$
$$\geq \epsilon + \liminf_{k\to\infty} [-\max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\}]$$
$$= \epsilon - \limsup_{k\to\infty} \max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\} = \epsilon. \tag{2.40}$$

Also, we have

$$\liminf_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\}$$
$$\leq \limsup_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\} = \epsilon. \tag{2.41}$$

Hence, from (2.40) and (2.41), we have

$$\liminf_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\} = \epsilon. \tag{2.42}$$

Therefore, from (2.38) and (2.42), we have
$\lim_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\}$ exists and
$\lim_{k\to\infty} \max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\} = \epsilon.$
This proves (iv). ■

Lemma 2.3. *Let X be a nonempty set. Let $F : X \times X \to X$ and $g : X \to X$ be maps such that $F(X \times X) \subseteq gX$. Assume that M is an (F, g)-invariant subset of X^4 which satisfies transitivity property.*
If there exist $x_0, y_0 \in X$ with $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$ then we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that for any $n, m \in \mathbb{Z}^+$ with $n > m$ we have $(gx_n, gy_n, gx_m, gy_m) \in M$.

Proof. Let $x_0, y_0 \in X$ such that

$$(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M. \tag{2.43}$$

Since $F(X \times X) \subseteq gX$ we can choose x_1, $y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$.
Again, since $F(X \times X) \subseteq gX$ we can choose x_2, $y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$.
On continuing this process, inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n) \tag{2.44}$$

for all $n \geq 0$.
Now, since M is an (F, g)-invariant set, from (2.43) and (2.44), we have $(gx_2, gy_2, gx_1, gy_1) = (F(x_1, y_1), F(y_1, x_1), F(x_0, y_0), F(y_0, x_0)) \in M$. Again, by the same argument, we can choose $x_3, y_3 \in X$ such that
$(gx_3, gy_3, gx_2, gy_2) = (F(x_2, y_2), F(y_2, x_2), F(x_1, y_1), F(y_1, x_1)) \in M$.
On continuing this process, we get by induction that

$$(gx_{n+1}, gy_{n+1}, gx_n, gy_n) \in M \tag{2.45}$$

for all $n \geq 0$.
Now, for $n > m$ where $n, m \in \mathbb{Z}^+$ we prove $(gx_n, gy_n, gx_m, gy_m) \in M$.

We let $n = m + k$ for some $k \geq 1$.
By (2.45) we have

$$(gx_{m+1}, gy_{m+1}, gx_m, gy_m) \in M. \tag{2.46}$$

Since M is an (F, g)-invariant set we have from (2.46)

$$(gx_{m+2}, gy_{m+2}, gx_{m+1}, gy_{m+1}) = (F(x_{m+1}, y_{m+1}), F(y_{m+1}, x_{m+1}), F(x_m, y_m), F(y_m, x_m)) \in M. \tag{2.47}$$

Now, from (2.46) and (2.47) and transitivity property of M, we have

$$(gx_{m+2}, gy_{m+2}, gx_m, gy_m) \in M. \tag{2.48}$$

Since M is an (F, g)-invariant set, we have from (2.48)

$$(gx_{m+3}, gy_{m+3}, gx_{m+1}, gy_{m+1}) \in M. \tag{2.49}$$

Again by the transitivity property of M and from (2.46) and (2.49), we have

$$(gx_{m+3}, gy_{m+3}, gx_m, gy_m) \in M. \tag{2.50}$$

On continuing this process inductively we get

$$(gx_{m+k}, gy_{m+k}, gx_m, gy_m) \in M \tag{2.51}$$

for every $k \geq 1$. Now, on taking $n = m + k$ in (2.51) it follows that
$(gx_n, gy_n, gx_m, gy_m) \in M$ for $n > m$. ■

In the following, we prove our main results of this section.

Theorem 2.4. *Let (X, d) be a metric space. Let $F : X \times X \to X$ and $g : X \to X$ be maps and M be a nonempty (F, g)-invariant subset of X^4. Assume that the maps F and g are almost generalized (ψ, φ)-weakly contractive maps. Further assume that*
(a) $F(X \times X) \subseteq gX$;
(b) gX is a complete subspace of X;
(c) M satisfies transitivity property;
(d) X has the following property: for any two sequences $\{x_n\}$ and $\{y_n\}$ in X with $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$, $x_n \to x$, $y_n \to y$, then $(x, y, x_n, y_n) \in M$ for all n,
(e) there exist $x_0, y_0 \in X$ with $(F(x_0, y_0), F(y_0, yx_0), gx_0, gy_0) \in M$. Then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, i.e., F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Let $x_0, y_0 \in X$ be such that $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$.
By Lemma 2.3, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for every $n = 0, 1, 2, \ldots$.
If there exists a natural number $n \geq 1$ such that $gx_{n-1} = gx_n$ and $gy_{n-1} = gy_n$, then
$gx_{n-1} = gx_n = F(x_{n-1},\ y_{n-1})$ and $gy_{n-1} = gy_n = F(y_{n-1},\ x_{n-1})$.
Thus $(x_{n-1},\ y_{n-1})$ is a coupled coincidence point of F and g.
Hence without lose of generality, we assume that $gx_{n-1} \neq gx_n$ or $gy_{n-1} \neq gy_n$ for all $n \in \{1, 2, 3, \cdots\}$.
By Lemma 2.3, we have
$(gx_n, gy_n, gx_{n-1}, gy_{n-1}) \in M$ for every $n \geq 1$. Now, we denote

$$R_n = \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}. \tag{2.52}$$

Since $(gx_n, gy_n, gx_{n-1}, gy_{n-1}) \in M$ for all $n \geq 1$, by using, (2.1) we have

$$\begin{aligned}
\psi(d(gx_{n+1},\ gx_n)) &= \psi(d(F(x_n,\ y_n), F(x_{n-1},\ y_{n-1}))) \\
&\leq \psi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}) \\
&- \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}) \\
&+ L\min\{d(gx_n,\ F(x_n,\ y_n)), d(gx_n,\ F(x_{n-1}, y_{n-1})), \\
&d(gx_{n-1}, F(x_n, y_n)), d(gx_{n-1},\ F(x_{n-1}, y_{n-1}))\} \\
&= \psi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}) \\
&- \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}) \\
&+ L\min\{d(gx_n,\ gx_{n+1}), d(gx_n,\ gx_n), d(gx_{n-1}, gx_{n+1}), d(gx_{n-1},\ gx_n)\} \\
&= \psi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}) \\
&- \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}).
\end{aligned} \tag{2.53}$$

Again, since $(gy_{n-1}, gx_{n-1}, gy_n, gx_n) \in M$ for every $n \geq 1$, we have

$$\begin{aligned}
\psi(d(gy_n,\ gy_{n+1})) &= \psi(d(F(y_{n-1},\ x_{n-1}), F(y_n,\ x_n))) \\
&\leq \psi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}) \\
&- \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}) \\
&+ L\min\{d(gy_{n-1},\ F(y_{n-1},\ x_{n-1})), d(gy_{n-1},\ F(y_n, x_n)), \\
&d(gy_n, F(y_{n-1}, x_{n-1})), d(gy_n,\ F(y_n, x_n))\} \\
&= \psi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}) \\
&- \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}) \\
&+ L\min\{d(gy_{n-1},\ gy_n), d(gy_{n-1},\ gy_{n+1}), d(gy_n, gy_n), d(gy_n,\ gy_{n+1})\} \\
&= \psi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}) \\
&- \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}).
\end{aligned} \tag{2.54}$$

Now, from (2.53) and (2.54) and using the monotone property of ψ, we have

$$\begin{aligned}
&\psi(\max\{d(gx_{n+1},\ gx_n), d(gy_{n+1},\ gy_n)\}) \\
&= \max\{\psi(d(gx_{n+1},\ gx_n)), \psi(d(gy_n,\ gy_{n+1}))\} \\
&\leq \psi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}) \\
&- \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}),
\end{aligned} \tag{2.55}$$

i.e.,

$$\begin{aligned}
\psi(R_n) &\leq \psi(R_{n-1}) - \phi(R_{n-1}) \\
&< \psi(R_{n-1})
\end{aligned} \tag{2.56}$$

and hence by the monotone property of ψ, we have $R_n \leq R_{n-1}$.
Therefore, $\{R_n\}$ is a non-increasing sequence of nonnegative real numbers.
Hence it converges to some real number r (say), $r \geq 0$.
Now we show that

$$r = 0. \tag{2.57}$$

Suppose that $r > 0$. On taking limit superior as $n \to +\infty$ in both sides of (2.56), by using continuity of ψ and lower semi-continuity of ϕ, we get $\psi(r) \leq \psi(r) - \phi(r)$, a contradiction. Thus $r = 0$. *i.e.,* $\lim_{n\to\infty} \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} = 0$.

Now, we show that the sequences $\{gx_n\}$ and $\{gy_n\}$ are Cauchy. Suppose that at least one of $\{gx_n\}$ and $\{gy_n\}$ is not Cauchy. Then there exists an $\epsilon > 0$, for which we can find two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers with $n_k > m_k > k$ such that $\max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\} \geq \epsilon$ for all $k \in \{1, 2, 3, \cdots\}$. We may also assume that $\max\{d(gx_{m_k}, gx_{n_k-1}), d(gy_{m_k}, gy_{n_k-1})\} < \epsilon$.
Now, since $n_k > m_k$ and M satisfies transitivity property, by Lemma 2.3 we have, $(gx_{n_k}, gy_{n_k}, gx_{m_k}, gy_{m_k}) \in M$.
Hence, on using (2.1) we have

$$\begin{aligned}\psi(d(gx_{n_k+1}, gx_{m_k+1})) &= \psi(d(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k}))) \\ &\leq \psi(\max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\}) \\ &- \phi(\max\{d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k})\}) \\ &+ L\min\{d(gx_{n_k}, gx_{n_k+1}), d(gx_{n_k}, gx_{m_k+1}), \\ &d(gx_{m_k}, gx_{n_k+1}), d(gx_{m_k}, gx_{m_k+1})\}.\end{aligned} \tag{2.58}$$

Again, we have

$$\begin{aligned}\psi(d(gy_{m_k+1}, gy_{n_k+1})) &= \psi(d(F(y_{m_k}, x_{m_k}), F(y_{n_k}, x_{n_k}))) \\ &\leq \psi(\max\{d(gy_{m_k}, gy_{n_k}), d(gx_{m_k}, gx_{n_k})\}) \\ &- \phi(\max\{d(gy_{m_k}, gy_{n_k}), d(gx_{m_k}, gx_{n_k})\}) \\ &+ L\min\{d(gy_{m_k}, gy_{m_k+1}), d(gy_{m_k}, gy_{n_k+1}), \\ &d(gy_{n_k}, gy_{m_k+1}), d(gy_{n_k}, gy_{n_k+1})\}.\end{aligned} \tag{2.59}$$

Now, from (2.58), (2.59) and the monotone property of ψ, we have
$\psi(\max\{d(gx_{n_k+1}, gx_{m_k+1}), d(gy_{n_k+1}, gy_{m_k+1})\})$

$$\begin{aligned}&= \max\{\psi(d(gx_{n_k+1}, gx_{m_k+1})), \psi(d(gy_{m_k+1}, gy_{n_k+1}))\} \\ &\leq \psi(\max\{d(gy_{m_k}, gy_{n_k}), d(gx_{m_k}, gx_{n_k})\}) \\ &- \phi(\max\{d(gy_{m_k}, gy_{n_k}), d(gx_{m_k}, gx_{n_k})\}) \\ &+ L\min\{\max\{d(gx_{n_k}, gx_{n_k+1}), d(gy_{n_k}, gy_{n_k+1})\}, \\ &\max\{d(gx_{n_k}, gx_{m_k+1}), d(gy_{n_k}, gy_{m_k+1})\} \\ &\max\{d(gx_{m_k}, gx_{m_k+1}), d(gy_{m_k}, gy_{m_k+1})\}, \\ &\max\{d(gx_{m_k}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k+1})\}\}.\end{aligned} \tag{2.60}$$

Now, on taking limit superior as $k \to \infty$ in both sides of (2.60), by using Lemma 2.2, the continuity of ψ and lower semi-continuity of ϕ, we get $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) + L.0$ $= \psi(\epsilon) - \phi(\epsilon)$, a contradiction.
Therefore, the sequences $\{gx_n\}$ and $\{gy_n\}$ are Cauchy in gX. Since gX is complete, there exist $x, y \in X$ such that $\lim_{n\to+\infty} gx_n = gx$ and $\lim_{n\to+\infty} gy_n = gy$.
Since $(gx_{n+1}, gy_{n+1}, gx_n, gy_n) \in M$ and $gx_n \to gx$ and $gy_n \to gy$ it follows that $(gx, gy, gx_n, gy_n) \in M$ for all n. Now, we prove that $gx = F(x, y)$ and $gy = F(y, x)$.
Since $(gx, gy, gx_n, gy_n) \in M$ for all n, we have

$$\begin{aligned}
&\psi(d(F(x,y), gx_{n+1})) \\
&\qquad = \psi(d(F(x,y), F(x_n, y_n)) \\
&\qquad \leq \psi(\max\{d(gx, gx_n), d(gy, gy_n)\}) - \phi(\max\{d(gx, gx_n), d(gy, gy_n)\}) \qquad (2.61) \\
&\qquad + L\min\{d(gx, F(x,y)), d(gx, gx_{n+1}), d(gx_n, F(x,y)), d(gx_n, gx_{n+1})\}.
\end{aligned}$$

On taking limit superior as $n \to \infty$ in both sides of (2.61), by using the continuity of ψ and lower semi-continuity of ϕ, we get $\psi(d(F(x,y), gx)) \leq 0$. Now, by the property of ψ, we have $d(F(x,y), gx) = 0$, *i.e.*, $F(x,y) = gx$.

Similarly, we have $gy = F(y,x)$. Therefore (x,y) is a coupled coincidence point of F and g. ∎

Theorem 2.5. *Let (X,d) be a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be continuous maps and M be a nonempty (F,g)-invariant subset of X^4. Assume that the maps F and g are almost generalized (ψ, ϕ)-weakly contractive maps. Further assume that*

- *(a) $F(X \times X) \subseteq gX$;*
- *(b) F and g are compatible ;*
- *(c) M satisfies transitivity property; and*
- *(d) there exist $x_0, y_0 \in X$ with $(F(x_0,y_0), F(y_0,x_0), gx_0, gy_0) \in M$.*

Then F and g have a coupled coincidence point $(x,y) \in X \times X$.

Proof. On proceeding as in the proof of Theorem 2.4, the sequences $\{gx_n\}$ and $\{gy_n\}$ defined by $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ are Cauchy in X.
Since X is complete, there exist $x, y \in X$ such that $\lim\limits_{n\to\infty} gx_n = x$ and $\lim\limits_{n\to\infty} gy_n = y$.
Since F and g are compatible maps we have
$\lim\limits_{n\to\infty} d(g(F(x_n,y_n)), F(gx_n, gy_n)) = 0$ and $\lim\limits_{n\to\infty} d(g(F(y_n,x_n)), F(gy_n, gx_n)) = 0$.
By using the triangle inequality, we have

$$d(gx, F(gx_n, gy_n)) \leq d(gx, g(F(x_n,y_n)) + d(g(F(x_n,y_n), F(gx_n, gy_n))).$$

On taking limits as $n \to +\infty$, by using the continuity of F and g, we get

$$d(gx, F(x,y)) = 0, \ i.e., \ gx = F(x,y).$$

Similarly, we have $d(gy, F(y,x)) = 0$, *i.e.*, $gy = F(y,x)$.
Therefore, $gx = F(x,y)$ and $gy = F(y,x)$.
Hence (x,y) is a coupled coincidence point of F and g. ∎

3. Corollaries and Examples

Corollary 3.1. *Let (X,d) be a metric space. Let $F : X \times X \to X$ and $g : X \to X$ be maps and M be a nonempty (F,g)-invariant subset of X^4. Assume that there exist functions $\psi \in \Psi$, $\phi \in \Phi$ such that*

$$\psi(d(F(x,y), F(u,v))) \leq \psi(\max\{d(gx, gu), (gy, gv)\}) - \phi(\max\{d(gx, gu), (gy, gv)\}) \quad (3.1)$$

for every $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$.
Further assume that

- *(a) $F(X \times X) \subseteq gX$;*
- *(b) gX is a complete subspace of X;*

(c) M satisfies transitivity property;
(d) X has the following property:
for any two sequences $\{x_n\}$ and $\{y_n\}$ in X with $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$, $x_n \to x$, $y_n \to y$, then $(x, y, x_n, y_n) \in M$ for all n,
(e) there exist $x_0, y_0 \in X$ with $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$.

Then F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Follows by choosing $L = 0$ in the inequality (2.1) and by Theorem 2.4. ∎

Corollary 3.2. *Let (X, d) be a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be maps and M be a nonempty (F, g)-invariant subset of X^4. Assume that there exist functions $\psi \in \Psi$, $\phi \in \Phi$ such that*

$$\psi(d(F(x,y), F(u,v))) \leq \psi(\max\{d(gx, gu), (gy, gv)\}) - \phi(\max\{d(gx, gu), (gy, gv)\}) \quad (3.2)$$

for every $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$.
Further assume that
(a) $F(X \times X) \subseteq gX$;
(b) g is continuous ;
(c) M satisfies transitivity property;
(d) F is continuous ;
(e) F and g are compatible;
(f) there exist $x_0, y_0 \in X$ with $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$.
Then F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Follows by choosing $L = 0$ in the inequality (2.1) and by Theorem 2.5. ∎

In the following, we deduce coupled coincidence point theorems in partially ordered metric spaces as corollaries.

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a metric space. Let $F : X \times X \to X$ and $g : X \to X$ be maps such that F has the mixed g-monotone property. Assume that there exist functions $\psi \in \Psi$, $\phi \in \Phi$ and a constant $L \geq 0$ such that*

$$\begin{aligned} \psi(d(F(x,y), F(u,v))) \leq\ & \psi(\max\{d(gx, gu), (gy, gv)\}) \\ & - \phi(\max\{d(gx, gu), (gy, gv)\}) \\ & + L \min\{d(gx, F(x,y)), d(gx, F(u,v)), \\ & d(gu, F(x,y)), d(gu, F(u,v))\} \end{aligned} \quad (3.3)$$

for every $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$.
Further assume that
(a) $F(X \times X) \subseteq gX$;
(b) gX is a complete subspace of X;
(c) X has the following properties:

(i) if $\{x_n\}$ is a nondecreasing sequence $\in X$ with $x_n \to x$, then $x_n \preceq x$ for all n;
(ii) if $\{y_n\}$ is a non-increasing sequence $\in X$ with $y_n \to y$, then $y_n \succeq y$ for all n.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$, $F(y_0, x_0) \succeq gy_0$ then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.
i.e., F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. We define a set $M = \{(a, b, c, d) \in X^4 : a \succeq c, b \preceq d\}$. From Example 1.14. we conclude that M is an (F, g)-invariant set which satisfies transitivity property.
By (3.3) we have
$$\psi(d(F(x,y),F(u,v))) \leq \psi(\max\{d(gx,gu),(gy,gv)\}) - \phi(\max\{d(gx,gu),(gy,gv)\})$$
$$+L\min\{d(gx,F(x,y)),d(gx,F(u,v)),d(gu,F(x,y)),d(gu,F(u,v))\}$$
for every $x, y, u, v \in X$ such that $(gx, gy, gu, gv) \in M$. Since there exist $x_0, y_0 \in X$, $gx_0 \preceq F(x_0, y_0)$, $gy_0 \succeq F(y_0, x_0)$, we have $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$. From assumption (c) we get the condition (d) of Theorem 2.4. Now, all the hypotheses of Theorem 2.4 are satisfied. Thus F and g have a coupled coincidence point in X. ∎

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exist a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be maps such that F has the mixed g-monotone property. Assume that the inequality (3.3) holds. Further assume that*
(a) $F(X \times X) \subseteq gX$;
(b) F and g are continuous;
(c) F and g are compatible;
If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$, $F(y_0, x_0) \succeq gy_0$ then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.
i.e., F and g have a coupled coincidence point $x, y \in X$.

Proof. We define a set $M = \{(a, b, c, d) \in X^4 : a \succeq c, b \preceq d\}$. Now, proceeding as in the proof of the Corollary 3.3, we see that all the hypotheses of Theorem 2.5 are satisfied. Hence F and g have a coupled coincidence point in X. ∎

Remark 3.5. By choosing $L = 0$ in (3.3) clearly Theorem 1.9 follows as a corollary to Corollary 3.3 and Corollary 3.4.

The following is an example in support of Theorem 2.5.

Example 3.6. Let $X = \{0, 1, 2, 4\}$ with the usual metric.
Let $A = \{(0, 0), (0, 1), (1, 0), (1, 1), (1, 2), (2, 2), (2, 4), (4, 2), (4, 4)\}$,
$B = \{(0, 2), (2, 0), (2, 1), (0, 4), (4, 0), (1, 4), (4, 1)\}$.

We define $F : X \times X \to X$ and $g : X \to X$ by
$$F(x,y) = \begin{cases} 1 & if\ (x,y) \in A \\ \\ 0 & if\ (x,y) \in B \end{cases}$$
and $g0 = 0,\ g1 = 1,\ g2 = 4,\ g4 = 2$.
We define $\psi, \phi : R^+ \to R^+$ by $\psi(t) = \frac{3}{4}t$ for $t \geq 0$ and
$$\phi(t)) = \begin{cases} \frac{1}{4}t & if\ 0 \leq t \leq 1 \\ \\ \frac{1}{3}t & if\ t > 1. \end{cases}$$
Then $\psi \in \Psi$ and $\phi \in \Phi$.
Here we observe that $F(X \times X) = \{0, 1\} \subseteq gX = \{0,\ 1,\ 2,\ 4\}$; and F and g are compatible. We write
$M = \{(0, 0, 1, 1), (1, 1, 0, 0), (1, 1, 1, 1), (1, 1, 4, 4), (4, 4, 1, 1), (2, 1, 2, 4), (4, 2, 1, 2)$
$(4, 4, 4, 4), (4, 4, 0, 0), (0, 0, 4, 4), (1, 1, 2, 2), (2, 2, 1, 1), (2, 2, 2, 2), (2, 1, 4, 2),$
$(2, 4, 1, 2), (0, 0, 2, 2), (2, 2, 0, 0), (0, 1, 1, 1), (1, 1, 1, 0), (2, 2, 4, 4), (4, 4, 2, 2),$

$(2,1,1,4),(4,1,1,2),(2,1,1,2),(0,1,1,0),(0,0,0,0),(4,4,1,0),(0,1,4,4)$
$(2,2,1,0),(0,1,2,2),(0,0,1,0),(0,1,0,0),(4,1,4,2),(2,4,1,4),(4,1,2,4)$
$(4,2,1,4),(4,1,1,4)\}$.
Clearly M is an $(F,\ g)$-invariant subset of X^4 which satisfies transitivity property.
We choose $x_0 = 1$, $y_0 = 0$, then $(F(1,0), F(0,1), g1, g0) = (1,1,1,0) \in M$.
We now verify the inequality (2.1) with $L = 1$. For $(x,y,u,v) = (2,1,2,4)$, we have

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &= \psi(1) \\
&= \tfrac{3}{4} \\
&\leq \tfrac{7}{2} \\
&= \psi(1) - \phi(1) + 1.3 \\
&= \psi(\max\{(d(gx,gu),(gy,gv))\}) - \phi(\max\{(d(gx,gu),(gy,gv))\}) \\
&\quad + L\min\{d(gx,F(x,y)), d(gx,F(u,v)), d(gu,F(x,y)), d(gu,F(u,v))\}
\end{aligned}$$

so that the inequality (2.1) holds.
Similarly, it is easy to verify the inequality for the points $(2,1,4,2)$, $(4,1,1,2)$, $(2,1,1,2)$, $(4,1,4,2)$, $(2,4,1,4)$, $(4,1,2,4)$ and $(4,2,1,4)$.
At the remaining points of M, the inequality (2.1) holds trivially.
Therefore F and g satisfy all the hypotheses of the Theorem 2.5 and $(1,1)$ is a coupled coincidence point of F and g.
Here we observe that, when $L = 0$ then the inequality (2.1) fails to hold for any $\psi \in \Psi$ and $\phi \in \Phi$, by choosing $(x,y,u,v) = (2,1,2,4)$, which indicates the importance of L in the inequality (2.1) of Theorem 2.5
Now, if we consider $X = \{0,1,2,4\}$ with the usual order $\leq$ and usual metric, the map F does not satisfy the mixed g- monotone property, for let $y_1 = 0$, $y_2 = 4$ and $x = 4$, then $0 = g0 \leq g4 = 2$ but $0 = F(4,0) \ngeq F(4,4) = 1$.
Also, when $(x,y,u,v) = (2,1,2,4)$ the inequality (1.2) fails to hold for any $\psi \in \Psi$ and $\phi \in \Phi$. Hence by Remark 3.5 it follows that Corollary 3.3 and Corollary 3.4 generalize Theorem 1.9.

Now, we give an example in support of Corollary 3.1.

Example 3.7. Let $X = [0,2)$ with the usual metric.
We define $F : X \times X \to X$ and $g : X \to X$ by

$$F(x,y) = \begin{cases} \frac{x^2+y^2}{4} & if\ (x,y) \in [0,1] \\ \frac{3}{2} & \text{otherwise} \end{cases}$$

and

$$gx = \begin{cases} x^2 & if\ x \in [0,1] \\ \frac{3}{2} & if\ x \in (1,2). \end{cases}$$

Here we note that $F(X \times X) = [0,\frac{1}{2}]\bigcup\{\frac{3}{2}\} \subseteq [0,1]\bigcup\{\frac{3}{2}\} = gX$, gX is a complete subspace of X.
We choose $M = [0,1]^4$. Clearly M is an (F,g)-invariant set which satisfies transitivity property. Also, we choose $x_0 = 0$ and $y_0 = 1$, then $(F(0,1), F(1,0), g0, g1) = (\frac{1}{4}, \frac{1}{4}, 0, 1) \in M$.
We define $\psi, \phi : R^+ \to R^+$ by $\psi(t) = \frac{1}{2}t$ for $t \geq 0$ and

$$\phi(t)) = \begin{cases} \frac{1}{6}t & if\ 0 \leq t \leq 1 \\ \frac{1}{4}t & if\ t > 1. \end{cases}$$

Then $\psi \in \Psi$ and $\phi \in \Phi$.
Now we verify the inequality (3.1). Let $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$.
i.e., $(x^2, y^2, u^2, v^2) \in M$.
In this case,
$\psi(d(F(x,y), F(u,v))) = \frac{1}{2}|\frac{x^2+y^2}{4} - \frac{u^2+v^2}{4}|$

$$\leq \frac{1}{4}(\frac{|x^2-u^2|+|y^2-v^2|}{2}) \leq \frac{1}{4}\max\{|x^2-u^2|, |y^2-v^2|\}. \tag{3.4}$$

On the other hand, we have
$\psi(\max\{d(gx, gu), d(gy, gv)\}) - \phi(\max\{d(gx, gu), d(gy, gv)\})$
$= \frac{1}{2}\max\{|x^2-u^2|, |y^2-v^2|\} - \frac{1}{6}\max\{|x^2-u^2|, |y^2-v^2|\}$

$$= \frac{1}{3}\max\{|x^2-u^2|, |y^2-v^2|\}. \tag{3.5}$$

Hence, from (3.4) and (3.5), we have

$$\psi(d(F(x,y), F(u,v))) = |\frac{x^2+y^2}{4} - \frac{u^2+v^2}{4}| \leq \frac{1}{3}\max\{|x^2-u^2|, |y^2-v^2|\}$$
$$= \psi(\max\{d(gx, gu), d(gy, gv)\}) - \phi(\max\{d(gx, gu), d(gy, gv)\}).$$

Hence, the inequality (3.1) holds for all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$. Therefore, F and g satisfy all the hypotheses of Corollary 3.1; and F and g have coupled coincidence points. In fact $(0,0)$ and every point of $(1,2) \times (1,2)$ are coupled coincidence points.

Here we observe that F does not satisfy the mixed g-monotone property under the usual order on X, for, by choosing $y_1 = 0$ and $y_2 = 1$ then
$0 = g0 \leq 1 = g1$, but at $x = 1$, $\frac{1}{4} = F(1,0) = F(x, y_1) \not\geq F(x, y_2) = F(1,1) = \frac{1}{2}$.

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