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On Dislocated Symmetric Spaces

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Abstract In this paper, we discuss some topological properties of dislocated symmetric space (simply d-symmetric space) and establish fixed point theorem by using cyclic contraction.

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1. Introduction

Definition 1.1. A Distance space (X, d) is said to be a *symmetric space* if it satisfies,

(1) $d(x, y) \geq 0$;
(2) $d(x, y) = 0$ iff $x = y$;
(3) $d(x, y) = d(y, x)$, for all $x, y \in X$.

In this case d is called a *symmetric function* on X and (X, d) called a symmetric space. If further d satisfies the triangle inequality:

(4) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Then d is called a *metric* and (X, d) is a *metric space*.

Investigations on the replacement of the triangular inequality in a metric space by various weaker forms have been under active considerations since long time. Several mathematicians including Frechet [1], E.W.Chittenden [2], Fisher [3], W.A.Wilson [4], V.Niemytzki [5] and others made extensive investigations in this regard and contributed substantively on the topological aspects by replacing the triangular inequality with appropriate reasonable conditions.

V.Niemytzki [5] calls a *distance function coherent* if for any sequences $\{x_n\}, \{y_n\}$ and x in X.

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(C_1):lim $d(x_n, y_n) = 0 =$lim $d(x_n, x) \Rightarrow$ lim $d(y_n, x) = 0$.

V.Niemytzki [5] calls (X, d) *metrizable* if there exists a metric δ on X such that

$$\lim d(x_n, x) = 0 \Leftrightarrow \lim \delta(x_n, x) = 0.$$

Also he proved that if (X, d) is coherent, then (X, d) is metrizable. His proof is however lengthy, an elegant proof was given by A. H. Frink [6] in 1937. A. D. Pitcher and E. W. Chittenden [7] consider the following two axioms for a distance space (X, d).

(C_2): $\lim d(x_n, x) = 0, \lim d(y_n, x) = 0 \Rightarrow \lim d(x_n, y_n) = 0$

(C_3): $\lim d(x_n, y_n) = 0, \lim d(y_n, z_n) = 0 \Rightarrow \lim d(x_n, z_n) = 0$

Seong Cho [8] discussed the implications and non implications among (C_1) and (C_2) and introduced the following axiom:

(C_4): for a sequence $\{x_n\} \in X, x, y \in X$,

$$\lim_{n\to\infty} d(x_n, x) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, y) = d(x, y).$$

W. A. Wilson [4] introduced following axioms:

(C_5): for a sequence $\{x_n\} \in X, x, y \in X, \lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

W_1 : for each pair of distinct points a, b in X there corresponds a positive number $r = r(a, b)$ such that $r < \inf_{c\in X} d(a, c) + d(b, c)$.

W_2 : for each $a \in X$, for each $k > 0$, there corresponds a positive number $r = r(a, k)$ such that if b is a point of X such that $d(a, b) \geq k$ and c is any point of X then $d(a, c) + d(c, b) \geq r$.

W_3 : for each positive number k there is a positive number $r = r(k)$ such that $d(a, c) + d(c, b) \geq r$ for all c in X and all a, b in X with $d(a, b) \geq k$.

We note that (C_2), (C_3), (C_4) are respectively equivalent to the axioms IV, V, III of W. A. Wilson [4]. He proved that a semi metric space is uniformly homeomorphic with a metric space if and only if axiom V holds. It was also proved that a semi metric space satisfying axiom IV is homeomorphic with a semi metric space with axiom V.

Recently, I. R. Sarma et al. [9] introduced d-symmetric space by deleting $d(x, x) = 0$ from the axioms of a symmetric space. Moreover, he introduced axiom C as below.

Axiom C : Every convergent sequence satisfies Cauchy criterion. That is, if (x_n) is a sequence in X, $x \in X$ and $\lim d(x_n, x) = 0$; then given $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $m, n \geq N(\epsilon)$.

Furthermore, he presented implications and non-implications among such convergence axioms as,

$C_3 \Rightarrow C_1 \Rightarrow C_5,\ C_3 \Rightarrow C_2, C_4 \Rightarrow C_5, C_1 \Rightarrow C \Rightarrow C_2, W_1 \Leftrightarrow C_5, W_2 \Leftrightarrow C_1$ and $W_3 \Leftrightarrow C_3$.

Also he presented examples for following relationships:

$C_5 \not\Rightarrow C_1, C_5 \not\Rightarrow C_4, C_1 \not\Leftrightarrow C_2, C_2 \not\Leftrightarrow C_5$ and $C_1 \not\Leftrightarrow C_4$.

In 2003, Kirk et al. [10] introduced cyclic contractions in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mappings.

In this paper, we discuss some topological properties of d-symmetric space and derive fixed point theorem by using cyclic contraction.

2. Preliminaries and Topological Aspects on d-Symmetric Spaces

Definition 2.1 ([9]). A distance space (X, d_s) is said to be d-*symmetric space* if d satisfies
(i). $d_s(x,y) = 0 \Rightarrow x = y$;
(ii). $d_s(x,y) = d(y,x)$, for all x, y.

If $x \in X$ and $\epsilon > 0$, then the set $\mathcal{B}_\epsilon(x) = \{y/y \in X\}$ and $d_s(x,y) < \epsilon$ is called the *ball* with center x and radius ϵ.
Notation: $V_\epsilon^x = \mathcal{B}_\epsilon^x \cup \{x\}$.

Example 2.2. 1. Let $X = [0,1]$. Define

$$d_s(x,y) = \begin{cases} \frac{1}{x} + \frac{1}{y}, \text{ if } x \neq 0 \neq y \\ 1, \text{ if } x = y \end{cases} \tag{2.1}$$

$d_s(x,0) = d_s(0,x) = x;\ if\ x \neq 0.$
Clearly d_s is d-symmetric and does not satisfy the triangle inequality since $d_s(\frac{1}{2}, 1) > d_s(\frac{1}{2}, 0) + d(0,1)$.

2. Let $X = [-1,1]$. Define

$$d_s(x,y) = \begin{cases} (x-y)^2, \text{ if } x \neq 0 \neq y \\ 1, \text{ if } x = y \end{cases} \tag{2.2}$$

$d_s(x,0) = d_s(0,x) = |x|;\ if\ x \neq 0.$
Clearly d_s is d-symmetric and does not satisfy the triangle inequality since $d_s(-1,1) > d_s(-1,0) + d_s(0,1)$.

Remark 2.3. If triangle inequality does not hold, then limits need not be unique.

Proof. Let $X = \mathbb{R}$. Define

$$d_s(x,y) = \begin{cases} |x-y|, \text{ if } x \neq 0 \neq y \\ 1, \text{ if } x = y \end{cases} \tag{2.3}$$

$d_s(x,0) = d_s(0,x) = \frac{|x-1|}{3};\ if\ x \neq 0.$ (X, d_s) is a d-symmetric space.
Triangular inequality fails since $d_s(-1,2) \geq d_s(-1,0) + d_s(0,2)$.
Further, $\lim d_s(1 - \frac{1}{n}, 1) \to 0$ and $\lim d_s(1 - \frac{1}{n}, 0) \to 0$.
Hence, limits are not unique. ∎

Define the set $\mathcal{C}(d_s, X, x) := \{\{x_\alpha\} \subset X : \lim_{n\to\infty} d_s(x_\alpha, x) = 0\}$.

In order to obtain below propositions, fixed point theorem on a d-symmetric space (X, d_s), we need below additional axiom.

P.S-property: Let X be a non empty set and $d_s : X \times X \longrightarrow [0, +\infty)$ be a mapping. For every $x \in X$, define the set $\mathcal{C}(d_s, X, x) = \{\{x_\alpha\} \subset X : \lim_{n\to\infty} d_s(x_\alpha, x) = 0\}$. There exist a real number $\mathcal{C} > 0$ such that $d_s(x,y) \leq \mathcal{C} \lim_{n\to\infty} \sup d_s(x_\alpha, y)$ for all $(x,y) \in X \times X; \{x_\alpha\} \in \mathcal{C}(d_s, X, x)$.

A net $\{x_\alpha/\alpha \in \Delta\}$ *converges* in X if $\{x_\alpha\} \in \mathcal{C}(d_s, X, x)$ and $x \in X$ is a *limit point* of $A \subset X$ iff there is a net $\{x_\alpha/\alpha \in \Delta\}$ in A such that $\{x_\alpha\} \in \mathcal{C}(d_s, X, x)$.

We write $\mathcal{D}(A)$ for the set of all limit points of A and $\overline{A} = A \cup \mathcal{D}(A)$. We show that $A \to \overline{A}$ satisfies *Kuratowski's axioms* [11]. Further, $\overline{A} = A$ iff A is closed. Also A is open iff $X - A$ is closed. Thus when d satisfies the triangle inequality or (C_1); the two topologies become the same.

Definition 2.4. If (X, d_s) be a d-symmetric space with P.S-property. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ d_s-converges to x and $\{x_n\}$ d_s-converges to y, then $x = y$.

Proposition 2.5. *Let (X, d) be a d-symmetric space satisfying (C_1). If $A \subseteq X$ and $B \subseteq X$, then*
(i). $\mathcal{D}(A) = \phi$ if $A = \phi$;
(ii). $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ if $A \subseteq B$;
(iii). $\mathcal{D}(A \cup B) = \mathcal{D}(A) \cup \mathcal{D}(B)$;
(iv). $\mathcal{D}(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$.

Proof. (i) and (ii) are clear. That $\mathcal{D}(A) \cup \mathcal{D}(B) \subseteq \mathcal{D}(A \cup B)$ follows from (ii).
To prove the reverse inclusion, let $x \in \mathcal{D}(A \cup B)$ and $\lim d_s(x_\alpha, x) = 0$; where $\{x_\alpha/\alpha \in \Delta\}$ is a net in $A \cup B$. If there exists $\lambda \in \Delta$ such that $x_\alpha \in A$. For $\alpha \in \Delta$ and $\alpha = \lambda$, then $\{x_\alpha/\alpha = \lambda, \alpha \in \Delta\}$ is a *co-final* sub net of $\{x_\alpha/\alpha \in \Delta\}$ and $\lim_{\alpha \geq \lambda} d_s(x, x_\alpha) = \lim_{\alpha \in \Delta} d_s(x, x_\alpha) = 0$.

If no such λ exists in Δ, then for every $\alpha \in \Delta$, choose $\beta(\alpha) \in \Delta$ such that $x_{\beta(\alpha)} \in B$, then $\{x_{\beta(\alpha)}/\alpha \in \Delta\}$ is a co-final subnet in B of $\{x_\alpha/\alpha \in \Delta\}$ and $\lim_{\alpha \in \Delta} d(x_{\beta(\alpha)}, x) = \lim_{\alpha \in \Delta} d_s(x_\alpha, x) = 0$. So that $x \in \mathcal{D}(A)$. It now follows that $\mathcal{D}(A \cup B) \subseteq \mathcal{D}(A) \cup \mathcal{D}(B)$. Hence (iii) holds.

To prove (iv), Let $x \in \mathcal{D}(\mathcal{D}(A)), x = \lim_{\alpha \in \Delta}(x_\alpha)$ where $x_\alpha \in \mathcal{D}(A)$ and for all $\alpha \in \Delta$ and let $\{x_{\alpha(\beta)}/\beta \in \Delta(\alpha)\}$ be a net A such that $x_\alpha = \lim_{\beta \in \Delta(\alpha)} x_{\alpha(\beta)}$.

For each positive integer i there exists $\alpha_i \in \Delta$ such that $d_s(x_{\alpha_i}, x) < \frac{1}{i}$ and $\beta_i \in \Delta(\alpha_i)$ such that $d_s(x_{\alpha_i \beta_i}, x_{\alpha_i}) < \frac{1}{i}$. If we write $\alpha_{i \beta_i} = \gamma_i$ for all i, then $\{\gamma_1, \gamma_2, \gamma_3, ...\}$ is directed set with $\gamma_i < \gamma_j$ if $i < j$ and $\lim d_s(x_{\gamma_i}, x_{\alpha_i}) = 0$. From (C_1), $\lim d(x_{\gamma_i}, x) = 0$. This implies that $x \in \mathcal{D}(A)$. ∎

Corollary 2.6. *If we write $\overline{A} = A \cup \mathcal{D}(A)$, for $A \subset X$, the operation $A \to \overline{A}$ satisfies Kuratawski's closure axioms [11] so that set $\Im = \{A/A \subset X \text{ and } \overline{A^c} = A^c\}$ is a topology on X.*

Definition 2.7. Let (X, d_s) be a d-symmetric space. Then

(1) Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is d_s-*Cauchy sequence* if $\lim\limits_{m,n \to \infty} d_s(x_n, x_{n+m}) = 0$.
(2) It is said to be d_s-*complete* if every $d_s d_s$-Cauchy sequence in X is d_s-convergent to some element in X.

Proposition 2.8. *In a d-symmetric space (X, d_s), $(C_1) \Rightarrow (C_5)$.*

Proposition 2.9. *$x \in X$ is a limit point of $A \subset X$ iff for every $r > 0, A \cap V_r(x) \neq \phi$.*

Proof. Suppose x is a limit point of $A \subset X$, then there exists a sequence $\{x_n\}$ in A such that $\lim\limits_{n \to \infty} x_n = x$.

If $r > 0$, there exists $\alpha_0 \in \Delta$ such that $x_n \in V_r(x) \cap A$ for $n \geq n_0$.

Conversely suppose that for every $r > 0, V_r(x) \cap A \neq \phi$.Then for every positive integer n,there exists $x_n \in V_{\frac{1}{n}}(x) \cap A$, so that $d_s(x, x_n) < \frac{1}{n}$ and $x_n \in A$. Hence $\lim_{n\to\infty} d_s(x_n, x) = 0$ and x is a limit point of A. ■

Corollary 2.10. *$x \in \overline{A}$ iff for every $r > 0, V_r(x) \cap A \neq \phi$.*

Example 2.11. Let $X = \mathbb{R}^+$. Define

$$d_s(x,y) = \begin{cases} \frac{1}{x} + \frac{1}{y}, \text{ if } x \neq 0 \neq y \\ 1, \text{ if } x = y \end{cases} \tag{2.4}$$

$d_s(x, 0) = d_s(0, x) = x;\ if\ x \neq 0$. Since $d_s(x, y) \neq 0\ \forall x, y$, (X, d_s) is a d-symmetric space. We show that (X, d_s) satisfies (C_5). By definition,

$$d_s(x_n, x) = \begin{cases} \frac{1}{x_n} + \frac{1}{x}, \text{ if } x_n \neq 0 \neq x \\ 1, \text{ if } x_n = x \end{cases} \tag{2.5}$$

$d_s(x_n, 0) = x_n$ if $x_n \neq 0$.

Case(i): If $x = 0$. Let us suppose that $x_n = 0 \Rightarrow d_s(x_n, 0) = 1$, which is not possible. So $x_n \neq 0$ after certain stage and $\lim x_n = 0$.

Case(ii): If $x \neq 0$. Suppose $x_n = 0$ for infinitely many n, $d_s(x_n, x) = d_s(0, x) = x \neq 0$. So $x_n \neq 0$ after certain stage.

$\Rightarrow \lim d(x_n, x) = \lim(\frac{1}{x_n} + \frac{1}{x}) = 0$

$\Rightarrow \lim \frac{1}{x_n} = -\frac{1}{x}, x_n > 0, x > 0$. This can not happen.

So the only possibility is that $x = 0$ and $x_n \neq 0$ after certain stage $\lim x_n = 0$.

Now $\lim d_s(x_n, y) = 0$, then $y = 0$. Hence $x = y = 0$,which satisfies (C_5).

Since $\lim d_s(x_n, x) = 0 \Rightarrow x = 0$. Thus (X, d_s) satisfies (C_5) which is a d-symmetric space. But 3rd condition of KuratowsKi axiom does not satisfy, i.e., $\mathcal{D}(\mathcal{D}(A)) \nsubseteq \mathcal{D}(A)$.

Let $x \in \mathcal{D}(\mathcal{D}(A))$, then there exists a sequence $\{x_n\}$ in $\mathcal{D}(A)$ such that $\lim d_s(x_n, x) \to 0$. Since $\{x_n\}$ is in $\mathcal{D}(A)$, there exists a sequence $\{x_{n_k}\}$ in A such that $\lim_k d_s(x_{n_k}, x_n) \to 0$. From (C_5), $x = x_{n_k}$.

Now we shall prove $x \in \mathcal{D}(A)$, which implies that it is enough to prove $\lim d_s(x_{n_k}, x) \to 0$. But $\lim d_s(x_{n_k}, x) \neq 0$, since $d_s(x_{n_k}, x) = 1$. Thus $\mathcal{D}(\mathcal{D}(A)) \nsubseteq \mathcal{D}(A)$.

Remark: From the above example it is clear that a d-symmetric function on a space X satisfying (C_5) may not yield a topology on X .

Proposition 2.12. *Let (X, d_s) be a d-symmetric space. If $\epsilon > 0$ and $d_s(x, y) < \epsilon, d_s(y, z) < \epsilon \Rightarrow d_s(x, z) < \epsilon$, then $(C_1), (C_2), (C_3), (C_4) and (C_5)$ holds.*

Proof. Take (C_1), i.e., $\lim d_s(x_n, x) = 0, \lim d_s(y_n, x_n) = 0 \Rightarrow \lim d_s(y_n, x) = 0$, i.e., $\lim d_s(x_n, x) = 0, \lim d_s(y_n, x_n) = 0$

$\Rightarrow$ for $\epsilon > 0$, there exist N such that $\lim d_s(x_n, x) < \epsilon$ and $\lim d_s(y_n, x_n) < \epsilon, \forall n \geq N$.

$\Rightarrow \lim d_s(y_n, x) < \epsilon, \forall n \geq N$

$\Rightarrow \lim d_s(y_n, x) = 0$.

The arguments for the validity of $(C_2), (C_3)$ and (C_5) are similar, hence omitted. Now take (C_4), i.e., $\lim d_s(x_n, x) = 0 \Rightarrow \lim d_s(x_n, y) = d_s(x, y), \forall y$.

$\lim d_s(x_n, x) = 0 \Rightarrow$ for every $\epsilon > 0$, there exist N such that $\lim d_s(x_n, x) < d_s(x, y) + \epsilon, \forall n \geq N$.
Also $d_s(y, x) < d_s(x, y) + \epsilon$
$\Rightarrow d_s(x_n, y) < d_s(x, y) + \epsilon, \forall n \geq N$.
$\Rightarrow d_s(x_n, y) - d_s(x, y) < \epsilon, \forall n \geq N$.
We prove there exist N_1 such that $d_s(x, y) - d_s(x_n, y) < \epsilon \ \forall n \geq N_1$.
If $d_s(x, y) - \epsilon < 0$, this holds trivially. Suppose $d_s(x, y) - \epsilon > 0$, there exist N_1 such that $d_s(x_n, x) < d_s(x, y) - \epsilon \ \forall n \geq N_1$. Since $d_s(x, y) > d_s(x, y) - \epsilon$ either $d_s(x, x_n) > d_s(x, y) - \epsilon$ or $d_s(y, x_n) > d_s(x, y) - \epsilon$.
Since $d_s(x, x_n) < d_s(x, y) - \epsilon \ \forall n \geq N_1$. We must have $d_s(y, x_n) > d_s(x, y) - \epsilon \ \forall n \geq N_1$.
$\epsilon > d_s(x, y) - d_s(x_n, y) \ \forall n \geq N_1$.
Hence $|d_s(x_n, y) - d_s(x, y)| < \epsilon$, for $n \geq max\{N, N_1\}$. Thus $\lim d_s(x_n, y) = d_s(x, y)$. ∎

Proposition 2.13. *If (X, d_s) is a d-symmetric space which satisfies (C_4), then balls are open.*

Proof. Let $x \in X, \delta > 0$ and $y \in \overline{V^c_\delta(x)}$. We show that $y \in V^c_\delta(x)$.
Since $y \in \overline{V^c_\delta(x)}$, there exist a sequence $\{y_n\}$ in $V^c_\delta(x)$ such that $\lim d_s(y_n, y) = 0$.
From (C_4), it follows that $\lim d_s(y_n, x) = d_s(y, x)$. Since $\{y_n\}$ in $V^c_\delta(x), d_s(y_n, x) \geq \delta \forall n$.
Hence $d_s(y, x) \geq \delta$, so $y \in V^c_\delta(x)$. Hence $V^c_\delta(x)$ is closed so that $V_\delta(x)$ is open. ∎

Corollary 2.14. *If (X, d_s) is a d-symmetric space which satisfies (C_4), then d_s induces a topology on X.*

Proof. Follows from proposition 2.13. ∎

Proposition 2.15. *$(X, d_s, \Im)$ is Haussdorff space.*

Proof. Suppose $x \neq y$. Claim that $V_\delta(x) \cap V_\delta(x) = \phi$.
Let us suppose that $V_\delta(x) \cap V_\delta(x) \neq \phi$. For each positive integer n, choose $x_n \in V_{\frac{1}{n}}(x) \cap V_{\frac{1}{n}}(x)$. Thus $d_s(x_n, x) < \frac{1}{n}$ and $d_s(x_n, y) < \frac{1}{n}$, which implies $\lim d_s(x_n, x) \to 0$ and $d_s(x_n, y) \to 0$. Since $(C_1) \Rightarrow (C_5)$, $x = y$; which is a contradiction. Hence $V_\delta(x) \cap V_\delta(x) = \phi$. ∎

Note: If $x \in X$, the collection $\{V_r(x)/x \in X\}$ is an open basis at x for $(X, d_s, \Im)$. Hence $(X, d_s, \Im)$ is first countable.

Proposition 2.16. *Let (X, d_s) be a d-symmetric space with induced topology $\Im$ and $A \subset X$*
$(i).A \in \Im$ iff A^c is closed.
$(ii).A$ is closed iff $d(x, A) = 0 \Rightarrow x \in A$.

Proof. Proof of (i): Suppose that A is open. Claim: A^c is closed.
Let $\{x_n\}$ be a sequence in $X - A$ and $\lim d_s(x_n, x) = 0$. Then we have to prove that $x \in X - A$. Suppose $x \notin X - A \Rightarrow x \in A$.
Since A is open, there exist $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$. From above, $x_n \in B_\epsilon(x) \subseteq A$. Which is a contradiction.

Conversely suppose that $X - A$ is closed. Claim:A is open, i.e., for $x \in A$ there exist $r > 0$ such that $B_r(x) \subseteq A$.
Suppose for some $x \in A$ this does not hold.Then for all $n \geq 1, B_{\frac{1}{n}}(x) \nsubseteq A$.

So there exist $x_n \in B_{\frac{1}{n}}(x) \cap X - A \Rightarrow d_s(x, x_n) < \frac{1}{n}$. Since $X - A$ is closed, $x \in X - A$. Which is a contradiction. Hence A is open.

Proof of (ii): Suppose A is closed. Claim: $d_s(x, A) = 0 \Rightarrow x \in A$.
Since $d_s(x, A) = 0$,there exist a sequence $\{x_n\}$ in A such that $\lim d_s(x, x_n) \to 0$.
Since A is closed,$x \in A$. Conversely suppose that $d_s(x, A) = 0 \Rightarrow x \in A$.
Let $\{x_n\}$ be a sequence in A such that $\lim d_s(x, x_n) \to 0$.
$\Rightarrow \forall n$, there exist k_n such that $d_s(x, x_{k_n}) < \frac{1}{n}$.
$\Rightarrow \inf\{d_s(x, y)/y \in A\} = 0$.
$\Rightarrow d_s(x, A) = 0$.
$\Rightarrow x \in A$.
Hence A is closed. ■

3. Continuity

Definition 3.1. Let (X, d_s) and (Y, d'_s) be d-symmetric spaces with (C_1) and (C_4). $f : X \to Y$ is said to be *d-symmetric continuous at $x \in X$* if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(V_\delta(x)) \subseteq V_\epsilon(f(x))$. f is *d-symmetric continuous* if f is d-symmetric continuous at every x in X.

Theorem 3.2. *Let (X, d_s) and (Y, d'_s) be d-symmetric spaces with (C_1) and (C_4). $f : (X, d_s) \to (Y, d'_s)$ be continuous at x iff $f : (X, d_s, \Im) \to (Y, d'_s, \Im')$ is continuous at x where $\Im$ and $\Im'$ are the corresponding induces topologies.*

Proof. Assume that f is d-symmetric continuous at x.
Let V be an open set in $(Y, d'_s, \Im')$. Then there exists $\epsilon > 0$ such that $V_\epsilon(f(x)) \subseteq V$.
By hypothesis, there exists $\delta > 0$ such that $f(V_\delta(x)) \subseteq V_\epsilon(f(x))$. Therefore $f(V_\delta(x)) \subseteq V$. Which implies $V_\delta(x) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is an open set in $(X, d_s, \Im)$. It follows that f is continuous from $(X, d_s, \Im)$ to $(Y, d'_s, \Im')$.

Conversely suppose that $f : (X, d_s, \Im) \to (Y, d'_s, \Im')$ is continuous. Claim: f is d-symmetric continuous. Since $V_\epsilon(f(x)) \subset \Im'$, $V_\epsilon(f(x))$ is open in $(Y, d'_s, \Im')$,
which implies $f^{-1}(V_\epsilon(f(x)))$ is open in $(X, d_s, \Im)$. Thus there exists $\delta > 0$ such that $V_\delta(x) \subset f^{-1}(V_\epsilon(f(x)))$. Which implies $f(V_\delta(x)) \subseteq V_\epsilon(f(x))$.
Hence f is d-symmetric continuous. ■

Theorem 3.3. *Let (X, d_s) and (Y, d'_s) be d-symmetric spaces (C_1) and (C_4). Then f is d-symmetric continuous iff for every convergent sequence $\{x_n\} \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$.*

Proof. Suppose that f is d-symmetric continuous. Claim: Suppose $\{x_n\} \to x$ in X. Which implies $\lim d_s(x_n, x) \to 0$. For every $\delta > 0$ there exists N_δ such that $d_s(x, x_n) < \delta$, for all $n \geq N_\delta$. Which implies that

$$x_n \in V_\delta(x), \text{for all } n \geq N_\delta. \tag{3.1}$$

Since f is d-symmetric continuous, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(V_\delta(x)) \subseteq V_\epsilon(f(x)).$$

For this δ applying (6), then there exists N such that (6) holds for $n \geq N$.
Which implies $x_n \in V_\delta(x)$ for $n \geq N$.
$\Rightarrow f(x_n) \in f(V_\delta(x)) \subseteq V_\epsilon(f(x))$ for $n \geq N$.

$\Rightarrow d_s(f(x_n, f(x)) < \epsilon$.
$\Rightarrow f(x_n) \to f(x)$. ∎

To prove reverse inclusion first we shall prove the following lemma.

Lemma 3.4. *Let (X, d_s) be a d-symmetric space (C_1) and (C_4). Then following are equivalent.*

(1) f is d-symmetric continuous.
(2) For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.
(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.
(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof. (1) $\Rightarrow$ (2) : Assume that f is d-symmetric continuous.
Let A be a set of X. We prove that if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$.
Let $V_\delta(f(x))$ be a neighborhood of $f(x)$. Then $f^{-1}(V_\delta(f(x)))$ is an open set of X containing x. Which implies $f^{-1}(V_\delta(f(x))) \cap A \neq \emptyset$. Then $V_\delta(f(x)) \cap f(A) \neq \emptyset$. Thus $f(x) \in \overline{f(A)}$.

(2) $\Rightarrow$ (3) : Let B be closed in Y and let $A = f^{-1}(B)$. We have to prove that A is closed in X. So it is enough to show $\overline{A} = A$.
Take $f(A) = f(f^{-1}(B)) \subseteq B$. If $x \in \overline{A}$ implies $f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$.
$\Rightarrow x \in f^{-1}(B) = A$
$\Rightarrow \overline{A} \subset A$.
Thus $A = \overline{A}$.

(3) $\Rightarrow$ (1) : Let V be an open set of V. The set $B = Y - V$. Then $f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$. Now B is a closed set of Y. Then $f^{-1}(B)$ is closed in X by hypothesis, so that $f^{-1}(V)$ is open in X.

(1) $\Rightarrow$ (4) : Let $x \in X$ and let V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subset V$.

(4) $\Rightarrow$ (1) : Let V be an open set of Y. Let x be a point of $f^{-1}(V)$. Then $f(x) \in V$, by hypothesis, there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that, it is open. Which ends the proof of the lemma. ∎

Now coming to the proof of the theorem.
Let us assume that $x_n \to x$ implies $f(x_n) \to f(x)$. Claim:f is d-symmetric continuous. To prove this it is enough to show $f(\overline{A}) \subset \overline{f(A)}$. If $x \in \overline{A}$, then there exist $\{x_n\}$ in A such that $x_n \to x$. Which implies $f(x_n) \to f(x)$. Since $f(x_n) \in f(A)$, $f(x) \in \overline{f(A)}$.
Hence $f(\overline{A}) \subset \overline{f(A)}$.

Theorem 3.5. *$f : (X, d_s, \Im) \to (Y, d'_s, \Im')$ is d-symmetric continuous iff given $x \in X$ and $\epsilon > 0$, there exist $\delta > 0$ such that $d_s(x, y) < \delta$ implies $d'_s(f(x), f(y)) < \epsilon$.*

Proof. Suppose that f is continuous. Consider the set $f^{-1}(V(f(x), \epsilon))$ which is open in X and contains the point x. Which implies there exists $\delta > 0$ such that $V_\delta(x) \subset f^{-1}(V(f(x)))$.
$\Rightarrow f(V_\delta(x)) \subset V_\delta(f(x))$.
Let $y \in V_\delta(x)$ implies $d_s(x, y) < \delta$. Thus $d'_s(f(x), f(y)) < \epsilon$.

Conversely, suppose that for given $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \epsilon$. Claim: f is d-symmetric continuous.

Let V be open in Y. We show that $f^{-1}(V)$ is open in X. Let x be a point of the set $f^{-1}(V)$. Since $f(x) \in V$, there exists $\epsilon > 0$ such that $f(V_\delta(x)) \subseteq V_\epsilon(f(x)) \subset V$.
Which implies $V_\delta(x) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open in X. ∎

4. A Fixed Point Theorem

Definition 4.1 ([10]). Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$. T is called a *cyclic map* iff $T(A) \in B$ and $T(B) \in A$.

Definition 4.2 ([10]). Let A and B be nonempty subsets of a metric space (X, d). A cylic map $T : A \cup B \to A \cup B$ is said to be a *cyclic contraction* if there exist $k \in (0, 1)$ such that, for all $x \in A$ and $y \in B$,

$$d(Tx, Ty) \leq k(d(x, y)).$$

There is a huge literature pertinent to cyclic map and generalizations of metric space. The reader can refer to [12]-[21] for more informations. Motivated by above, we introducing the concept of a *d_s-cyclic-Banach contraction.*

Definition 4.3. Let A and B be nonempty subsets of a d-symmetric space. A cyclic map $T : A \cup B \to A \cup B$ is said to be a *d_s-cyclic-Banach contraction* if there exist $k \in (0, 1)$ such that

$$d_s(Tx, Ty) \leq k(d_s(x, y)),$$

for all $x \in A$ and $y \in B$.

Proposition 4.4. *Suppose that T is a d_s*-cyclic-Banach contraction. *Then any fixed point $w \in X$ of T satisfies $d_s(w, w) < \infty \Rightarrow d_s(w, w) = 0$.*

Note that, for every $x \in X$, let $\delta(\mathcal{D}, f, x) = \sup\{\mathcal{D}(f^i(x), f^j(x)) : i, j \in \mathbb{N}\}$.

Theorem 4.5. *Let A and B be nonempty closed subsets of a complete d-symmetric space (X, d_s) with P.S property. Let T be a cyclic mapping that satisfies the condition of a d_s*-cyclic-Banach contraction. *Further, there exists $x_0 \in X$ such that $\delta(d_s, T, x_0) < \infty$. Then $\{T^n(x_0)\}$ d_s-converges to $w \in A \cap B$, a fixed point of T.*
Moreover, if $w^* \in X$ is another fixed point of T such that $d_s(w, w^*) < \infty$, then $w = w^*$.

Proof. Fix $x_0 \in A$ and $n \in \mathbb{N}(n \geq 1)$. Since T is *d_s-cyclic-Banach contraction,* for all $i, j \in \mathbb{N}$, we have

$$d_s(T^{n+i}(x_0), T^{n+j}(x_0)) \leq k d_s(T^{n-1+i}(x_0), T^{n-1+j}(x_0)),$$

which implies that

$$\delta(d_s, T, T^n(x_0)) \leq k\delta(d_s, T, T^{n-1}(x_0)).$$

Then, for every $n \in \mathbb{N}$, we have

$$\delta(d_s, T, T^n(x_0)) \leq k^n \delta(d_s, T, x_0).$$

By using the above inequality, for every $n, m \in \mathbb{N}$, we get

$$d_s(T^n(x_0), T^{n+m}(x_0)) \leq \delta(d_s, T, T^n(x_0)) \leq k^n \delta(d_s, T, x_0).$$

Since $\delta(d_s, T, x_0) < \infty$ and $k \in (0, 1)$, we obtain

$$\lim_{n,m\to\infty} d_s(T^n(x_0), T^{n+m}(x_0)) = 0.$$

Which implies that $\{T^n(x_0)\}$ is a d_s-Cauchy sequence.
Since (X, d_s) is d_s-complete, there exists some $w \in X$ such that $\{T^n(x_0)\}$ is d_s-convergent to w.

We shall prove that $\{T^{2n}(x_0)\}, \{T^{2n-1}(x_0)\}$ are sequences in A and B which tends to the same limit point w. From the $P.S$-property we have

$$d_s(T^{2n}(x_0), w) \leq \mathcal{C} \lim_{n\to\infty} sup d_s(T^{2n}(x_0, T^n(x_0))$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} d_s(T^{2n}(x_0), w) = 0.$$

Similarly,

$$\lim_{n\to\infty} d_s(T^{2n-1}(x_0), w) = 0.$$

Since A and B are closed, we have $w \in A \cap B$, and then $A \cap B \neq \emptyset$.
Now we will show that $Tw = w$. Since T is a d_s-*cyclic-Banach contraction,* for all $n \in \mathbb{N}$, we have

$$d_s(T^{n+1}(x_0), T(w) \leq k(d_s(T^n(x_0), w)).$$

Applying limits as $n \to \infty$, we get

$$\lim_{n\to\infty} d_s(T^{n+1}(x_0), T(w) = 0.$$

$$d_s(T^n(x_0), T(w)) \leq \mathcal{C} \limsup_{n\to\infty} d_s(T^n(x_0), T^n(x_0)) \leq \mathcal{C} \sup \lim_{n\to\infty} d_s(T^n(x_0), T^n(x_0)).$$

Thus, $\lim_{n\to\infty} d_s(T^n(x_0), T(w)) = 0$.
Hence, $\{T^n(x_0)\}$ is d_s-convergent to $T(w)$. By the uniqueness of the limit, we get $w = T(w)$, that is, w is a fixed point of T. Finally, to prove the uniqueness of a fixed point, let $w^* \in X$ be another fixed point of T such that $Tw^* = w^*$ and $d_s(w, w^*) < \infty$. Then we have,

$$d_s(w, w^*) = d_s(T(w), T(w^*)) \leq k d_s(w, w^*),$$

which implies $w = w^*$. Hence w is a unique fixed point of T. ■

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