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Representable Autometrized Semialgebra

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Abstract In this paper, two new concepts, namely, semilattice ordered Autometrized Algebra and Representable Autometrized Semialgebra are introduced. We characterized the Representable Autometrized Algebras among the Representable Autometrized Semialgebras.

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1. Introduction

Swamy [1] introduced Autometrized Algebras in general and, DRl Semigroups, in particular. The class of all commutative DRl Semigroups includes the classes of Birkhoff,G [2], Autometrized Boolean Algebras [3–5], commutative l-groups as proper subclasses. Subba Rao [6–11] introduced and studied the class of Representable Autometrized Algebras which includes the class of all commutative DRl Semigroups [1] as a proper subclass.

By a Representable Autometrized Algebra we mean a lattice ordered Autometrized Algebra $A = (A, +, \leq, *)$ satisfying the conditions.
$(L1)$: $*$ is semiregular i.e., $a \geq 0$ implies $a * 0 = a$, and
$(L2)$: for each a in A, the translations $x \mapsto a + x$, $x \mapsto a \vee x$, $x \mapsto a \wedge x$ and $x \mapsto a * x$ are contractions.

Here $(A, \leq)$ is a Lattice with the Lattice operations $\wedge$, $\vee$. Making use of both of the operations $\wedge$, $\vee$. Subba Rao studied the structure and the geometry of Representable Autometrized Algebras.

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In the present paper, we first introduced Semilattice ordered Autometrized Algebra by assuming that $(A, \leq)$ is just a meet Semi Lattice (taking a Semi lattice instead of a lattice).

Using this definition, we introduced the new concept Representable Autometrized Semialgebra i.e. $A = (A, +, \leq, *)$ where $(A, \leq)$ is a Semilattice with operation $\wedge$ satisfying the conditions (L_1) and (L_2) (deleting the contraction of $x \mapsto a \vee x$).

Every Representable Autometrized Algebra is clearly a Representable Autometrized Semialgebra. We characterized the class of Representable Autometrized Algebras among the class of Representable Autometrized Semialgebras, as in the main theorem (3.5).

This class of all Representable Autometrized Semialgebras is a more wider class than the class of all Representable Autometrized Algebras which includes the classes of above mentioned Autometrized algebras as proper subclasses.

2. Preliminaries

We recall the following definitions from Swamy [1] and Subba Rao [6, 7, 11].

Definition 2.1. (Swamy [1]) A system $A = (A, +, \leq, *)$ is called a "Lattice Ordered Autometrized Algebra", if and only if, A satisfies following conditions.
i) $(A, +, \leq)$ is a commutative lattice ordered semigroup with $'0'$, and
ii) $*$ is a metric operation on A i.e., $*$ is a mapping from $A \times A$ into A satisfying the formal properties of distance, namely,
$(M1)$ $a * b \geq 0$ for all a, b in A, equality, if and only if $a = b$,
$(M2)$ $a * b = b * a$ for all a, b in A, and
$(M3)$ $a * b \leq a * c + c * b$ for all a, b, c in A.

Definition 2.2. (Subba Rao [6, 7, 11]) A lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is called a " Representable Autometrized algebra", if and only, if A satisfies the following conditions:
$(L1)$ $A = (A, +, \leq, *)$ is a semiregular autometrized algebra, i.e., $a \in A$ and $a \geq 0$ implies $a * 0 = a$, and
$(L2)$ For every a in A, all the mappings $x \mapsto a + x$, $x \mapsto a \vee x$, $x \mapsto a \wedge x$ and $x \mapsto a * x$ are contractions (i.e., if θ denotes any one of the operations $+$,$\vee$, $\wedge$ and $*$,then, for each a in A, $(a\theta x) * (a\theta y) \leq x * y$ for all x, y in A).

3. Main Results

In this section, we introduce the new notion of a Representable Autometrized Semialgebra in the following.

Definition 3.1. An algebraic system $(A, +, \leq, *)$is said to be a semilattice ordered Autometrized Algebra, if and only if, A satisfies the following axioms.
(i) $(A, +)$is a commutative semigroup with identity 0.
(ii) $(A, \leq)$ is a meet semilattice i.e., $(A, \leq)$ is poset in which every pair of elements
a, b has a greatest lower bound denoted by $a \wedge b$ such that $a + (b \wedge c) = (a + b) \wedge (a + c)$ for all a,b,c in A.
(iii) $* : A \times A \to A$ is a mapping satisfying the properties of a distance function, namely
$M(1) : a * b \geq 0$ for all a, b in A equality,if and only if, $a = b$ (non negativity),

$M(2) : a * b = b * a$ for all a, b in A (Symmetry), and
$M(3)$ $a * c \leq (a * b) + (b * c)$ for all a, b, c in A(Triangle inequality).

Definition 3.2. A semilattice ordered Autometrized Algebra $(A, +, \leq, *)$ is said to be a Representable Autometrized Semialgebra, if and only if, the following conditions are satisfied.
$i)$ $*$ is semiregular i.e., $a \geq 0$ in $A \Rightarrow a * 0 = a$
$ii)$ for every a in A, all the mappings $x \mapsto a + x$, $x \mapsto a \wedge x$ and $x \mapsto a * x$ are contractions with respect to $*$, i.e.,
$(a + x) * (a + y) \leq x * y$ for all x,y in A,
$(a \wedge x) * (a \wedge y) \leq x * y$ for all x,y in A,and
$(a * x) * (a * y) \leq x * y$ for all x,y in A.

Note 3.3. Every Representable Autometrized algebra $(A, +, \leq, *)$ [11] is clearly a Representable Autometrized semialgebra.

Note 3.4. Clearly, the class of all Representable Autometrized Semialgebras is a more wider class than the class of all Representable Autometrized Algebras (Subba Rao [11]), which includes the class of all commutative DRl - semigroups (Swamy [1]) as a special subclass, In turn, the class of all DRl - semigroups includes the class of all Boolean algebras (Blumenthal, L.M [12], Ellis, D [4]), the class of all Brouwerian algebras (Nordhaus E.A., and L. Lapidaus [13]) and the class of all commutative l - groups, as subclasses.

The following theorem characterizes the class of all Representable Autometrized Algebras satisfying certain natural conditions, among the class of Representable Autometrized Semialgebras.

Theorem 3.5. *Let $(A, +, \leq, *)$ be a Representable Autometrized Semialgebra satisfying the conditions.*
(R) $a * (a \wedge b) + a \wedge b = a$ *for all* $a, b \in A$
(C) *Cancellation laws hold good in A w.r.t $'+'$, and*
(U) $(a * x + a \wedge x) * (a * y + a \wedge y) \leq x * y$ *for all* $x, y, a \in A$
Then,$(A, +, \leq, *)$ is a Representable Autometrized algebra, if and only if, A satisfies the condition.
(T) $a \leq b \Rightarrow b = b * a + a$ *for all* $a, b \in A$.

Proof. First, assume that $(A, +, \leq, *)$ is a Representable Autometrized Semialgebra satisfying the conditions (R), (C) and (U) mentioned in the above hypothesis. Now, let us assume that A satisfies the following condition also
(T) $a \leq b \Rightarrow b = b * a + a$ for all a, b in A
clearly, $(A, +, \leq, *)$ is an algebraic system, where
$i)$ $(A, +)$ is a commutative semigroup with identity 0.
$ii)$ $(A, \leq)$ is a meet semilattice,
where $a \wedge b = a \Rightarrow a \leq b$ and $a + (b \wedge c) = (a + b) \wedge (a + c)$ for all a, b, c in A.
$iii)$ $*$ is a semiregular metric operation in A such that for each a in A,all the mappings $x \mapsto a + x$, $x \mapsto a \wedge x$, $x \mapsto a * x$, are contractions w.r.t. $*$in A.
In order to prove that $(A, +, \leq, *)$ is a Representable Autometrized algebra satisfying (R), it is enough if we prove that $(A, \leq)$ is a lattice such that
$a + (b \vee c) = (a + b) \vee (a + c)$ for all a,b,c in A and $x \mapsto a \vee x$, is a contraction w.r.t $*$ in A for all a in A. So,let us define for any a, b in A, $a \vee b = a * b + a \wedge b$.

Let $a \in A$ and $b \in A$ we have
$a \vee a = a * a + a \wedge a = 0 + a = a$ (Idempotency),
$a \vee b = a * b + a \wedge b = b * a + b \wedge a = b \vee a$ (Commutativity),
$a = a * (a \wedge b) + a \wedge b$ (by (T)) $\leq a * b + a \wedge b = a \vee b$
(since $x \mapsto a \wedge x$ is contraction and $a \wedge a = a$),
$b = b * (a \wedge b) + a \wedge b$ (by (T)$\leq b * a + b \wedge a = b \vee a = a \vee b$.
Thus, $a \vee b$ is an upper bound of $\{a, b\}$. Let $c \in A$ such that $a \leq c$, $b \leq c$.
Now,

$$\begin{aligned} c &= c * a + a \text{ (by (T))} \\ &\geq (c \wedge b) * (a \wedge b) + a \text{ since } \wedge \text{ satisfies contraction property} \\ &\geq b * (a \wedge b) + a * (a \wedge b) + a \wedge b \text{ from (R)} \\ &\geq b * a + a \wedge b \text{(by (M3))} \\ &= a \vee b \text{ by the definition of } \vee . \end{aligned}$$

Therefore $a \vee b$ is the lub of $\{a, b\}$ for all a, b in A. Since $*$ is symmetric and $\wedge$ is commutative in A it follows that $a \vee b = a * b + a \wedge b = b * a + b \wedge a = b \vee a$.
Thus $\vee$ is commutative also in A.
Further

$$\begin{aligned} a \vee (b \vee c) &= lub\{a, b \vee c\} \\ &= lub\{a, lub\{b, c\}\} \\ &= lub\{a, b, c\} \end{aligned}$$

Similarly $(a \vee b) \vee c = lub\{a, b, c\}$.
Therefore $a \vee (b \vee c) = (a \vee b) \vee c$ for all a, b, c in A.
Thus, $\vee$ is idempotent, commutative, and associative in A.
Further, $a \wedge (a \vee b) = a$ (since $a \leq a \vee b$) for all a,b in A
and

$$\begin{aligned} a \vee (a \wedge b) &= a * (a \wedge b) + a \wedge (a \wedge b) \text{ (by (T))} \\ &= (a * (a \wedge b)) + a \wedge b \\ &= a \text{ (by (T) and } a \wedge b \leq a). \end{aligned}$$

Therefore $\wedge$ and $\vee$ satisfy the absorption laws also in A.
Thus, $(L, \leq)$ is a Lattice also. We also have the following properties.

(1) $x * (x \wedge y) = (x \vee y) * y$ for all x,y in A.
For,

$$\begin{aligned} (x \vee y) * y &\geq [(x \vee y) \wedge x] * (y \wedge x) \text{ (by the contraction property of } \wedge\, w.r.t *) \\ &\geq x * (x \wedge y) \text{(since } y \vee x = x \vee y) \\ &\geq (x * y) * [(x \wedge y) * y] \text{ (by the contraction property of } * w.r.t *) \\ &\geq (x * y + x \wedge y) * [(y * (x \wedge y)) + x \wedge y] \text{(since} * \textit{ is symmetric}) \\ &= (x \vee y) * y \text{ (by (T))} \end{aligned}$$

Thus, $(x \vee y) * y = (x \wedge y) * x$.

(2) $(A, \wedge, \vee)$ is a Distributive lattice
For this, it is enough (page 39, Birkhoff, G [2]), if we prove that
$a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for a, x, y in $A \Rightarrow x = y$.
Assume that $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for a, x, y in A.

Now, we have

$$\begin{aligned} x*y &= [x*(x\wedge a)+x\wedge a]*[y*(y\wedge a)+y\wedge a] \\ &= [a*(a\vee x)+a\wedge x]*[a*(a\vee y)+a\wedge x] \text{ (from (1) above)} \\ &\leq [a*(a\vee x)]*[a*(a\vee y)] \text{ (by contraction property of + w.r.t } *) \\ &\leq (a\vee x)*(a\vee y) \text{ (by contraction property of } *w.r.t*) \\ &= (a\vee x)*(a\vee x) (\text{since a} \vee y = a\vee x) \\ &= 0. \end{aligned}$$

i.e., $x*y \leq 0$, but $x*y \geq 0$.
Therefore, $x=y$. Thus, $a\wedge x = a\wedge y$ and $a\vee x = a\vee y \Rightarrow x=y$ for any a,x,y in A.
Therefore, $(A,\wedge,\vee)$ is a Distributive lattice.

(3) $a\vee b = a*(a\vee b)+b = b*(a\wedge b)+a$ for all a, b in A
since

$$\begin{aligned} a\vee b &= [(a\vee b)*b]+b (\text{by (T)}) \\ &= [(a\wedge b)*a]+b(\text{ by (1) above }) \\ &= [a*(a\wedge b)]+b \end{aligned}$$

and

$$\begin{aligned} a\vee b &= [(a\vee b)*a]+a \text{ (by (T))} \\ &= [b*(a\wedge b)]+a \text{ (by (1) above)} \end{aligned}$$

(4) $a+b=(a\vee b)+(a\wedge b)$ for a, b in A.
For,

$$\begin{aligned} a\vee b + a\wedge b &= [(a\vee b)*b+b]+(a\wedge b) \text{ (since } a\vee b\geq b \text{ (from(3) above)} \\ &= [a*(a\wedge b)+a\wedge b]+b \\ &\quad \text{(from (1) above and by the associativity and commutativity of} +) \\ &= a+b, \text{(by (T)) since } a\geq a\wedge b. \end{aligned}$$

(5) $a+(b\vee c)=(a+b)\vee(a+c)$ for all a,b, c in A.
For, let a,b, c in A.
we have

$$\begin{aligned} (a+b)\vee(a+c)+(a+b)\wedge(a+c) &= (a+b)+(a+c) \text{ from (4) above} \\ &= 2a+(b+c) --------(P) \end{aligned}$$

and

$$\begin{aligned} (a+(b\vee c))+(a+b)\wedge(a+c) &= [a+(b\vee c)]+[a+(b\wedge c)] \\ &= 2a+(b\vee c+b\wedge c) \\ &= 2a+(b+c)----(Q) \text{ (from (4) above)} \end{aligned}$$

from (P) and (Q) above, we have
$[a+(b\vee c)]+(a+b)\wedge(a+c) = (a+b)\vee(a+c)+(a+b)\wedge(a+c)$. Therefore, by cancellation laws (C) w.r.t $+$ in A, we have $a+(b\vee c)=(a+b)\vee(a+c)$ for all a,b,c in A. To complete the proof, it is enough if we prove that $x\mapsto a\vee x$ is a contraction w.r.t $*$ for each a in A. By hypothesis, A satisfies the condition (U)$(a*x+a\wedge x)*(a*y+a\wedge y)\leq x*y$ for all a, x, y in A but $a*x+a\wedge x = a\vee x$ (by definition of $\vee$) $a*y+a\wedge y=a\vee y$ (by definition of $\vee$). Therefore by (U) above $(a\vee x)*(a\vee y)\leq x*y$ for all x,y in A. Thus, for each a in A, the mapping $x\mapsto a\vee x$ is also a contraction w.r.t $*$. Hence, $A=(A,+,\leq,*)$ is a Representable Autometrized algebra satisfying the condition(R). Conversly, assume that $A=(A,+,\leq,*)$ is a Representable Autometrized algebra satisfying (R).

Clearly, A is a Representable Autometrized Semialgebra. Now we have to prove that A satisfies the condition
(T) $a \leq b \Rightarrow b = b * a + a$ for any a,b in A.

For, assume that $a \leq b$ for some a, b in A but by (R), we have $b = b * (b \wedge a) + b \wedge a = (b * a) + a$ (since $a \leq b$). Thus, A satisfies the condition (T). Hence the Theorem. ∎

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