Thai Journal of Mathematics
Volume 19 Number 4 (2021)
Pages 1257–1266

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Existence Theorems for Coincidence Points of f-Contractive Mappings in Cone b-Metric Spaces

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Abstract In this paper, two f-contractive mappings are provided and studied in cone b-metric spaces. The first one is called *f-contractive mapping of type B* and the second one is called *f-contractive mapping of type KC*, respectively. Under some available properties on a complete cone b-metric space, some suitable conditions on scalars and the considered two new f-contractive mappings T, we can generate a Cauchy sequence to find and confirm the existence of a coincidence point of f and T without the assumption of normality. The obtained results not only directly improve and generalize some fixed point results in metric spaces and b-metric spaces but also expand and complement some previous results in cone metric spaces.

MSC: 47H09; 47H10; 55M20; 54H25

Keywords: cone b-metric space; f-contractive mapping; coincidence point; fixed point

Submission date: 09.03.2018 / Acceptance date: 10.12.2019

1. Introduction

It is well known that fixed point theory plays an important role in applications of many branches of mathematics. For guaranteeing or finding a solution of some optimization problems, it can be able to simulate to some fixed point problems. Finding a fixed point of contractive mappings becomes the center of strong research activity. There are many works about the fixed point of contractive maps; see, for example, [1, 2]. The Banach's fixed point theorem first appeared in explicit form in Banach's thesis in 1922 [1], where it was used to establish the existence of a solution for an integral equation. In the general setting of complete metric space (M, d), Banach [1] provided the definition of a contraction as follows: A mapping $T : M \to M$ is called a Banach contraction (or contraction) if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

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for all $x, y \in M$. He proved that a mapping $T : M \to M$ that satisfies (1.1) has a unique fixed point.

In 1968, Kannan [3] established a fixed point theorem which has the following details: If $T : M \to M$ where (M, d) is a complete metric space, satisfies the inequality:

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty)) \tag{1.2}$$

for all $x, y \in M$ where $0 \leq k < \frac{1}{2}$, then T has a unique fixed point. A mapping that satisfies (1.2) is called a Kannan mapping.

In 1972, Chatterjea [4] established a fixed point theorem which has the following details: If $T : M \to M$ where (M, d) is a complete metric space, satisfies the inequality:

$$d(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx)) \tag{1.3}$$

for all $x, y \in M$ where $0 \leq k < \frac{1}{2}$, then T has a unique fixed point. A mapping that satisfies (1.3) is called a Chatterjea mapping.

Subsequently, many mathematicians have become interested in investigating the existence of fixed points of generalized forms of mappings related to (1.1), (1.2) and (1.3); see, for instance, [5–8] and references cited therein.

On the other hand, In 1989, Bakhtin [9] presented a generalization of metric space called b-metric spaces. He established the contraction mapping principle in b-metric spaces which is an extension of Banach contraction principle in metric space. Later, there are many papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces; see [10–12] and the references cited therein. In recent explorations, the interest on non-convex analysis related to fixed points, particularly in an ordered normed space, is considered by several authors; see [13–16]. Cone metric spaces are generalizations of metric spaces, where each pair of points is assigned to a member of a real Banach space with a cone. This cone naturally induces a partial order in the Banach space. The concept of cone metric space was introduced in the work of Huang and Zhang [13] where they also proved the Banach's contraction mapping principle in such spaces. In [14], Hussain and Shah established cone b-metric spaces as a generalization of both b-metric spaces and cone metric spaces. They introduced some topological properties in such spaces and enhanced some recent results about KKM mappings in cone b-metric spaces. Afterwards, several authors have studied fixed point problems in cone metric spaces. Some of these works are noted in [17–22]. In 2013, Huang and Xu [7] introduced two contractive mappings related to (1.1), (1.2) and (1.3) and then studied some fixed point theorems of such mappings in cone b-metric spaces.

As far as we know from searching for information from the history of mathematical writings about coincidence points, Eilenberg and Montgomery [23, Theorem 3], Gorniewicz [24], Singh and Kulshrestha [25] and Rhoades et al. [26] are probably at the top of the list who were the first to define and study the concept of coincidence points in metric spaces and some related spaces. Since then, there have been many mathematicians interested in the study of coincidence points; see, for instant, Singh et al. [27], Abbas and Jungck [17], Berinde [5, 6] and references cited therein.

Inspired and motivated by above research works, the aim of this paper is to provide some existence theorems of coincidence points which can be viewed as the general ideas from fixed point theorems of contractive mappings to the sense of coincidence point theorems of f-contractive mappings in the frame work of cone b-metric spaces.

2. Preliminaries

In this section, we provide some definitions, notations, and some useful results that will be used in the next section.

Let E be a real Banach space and P be a subset of E. The notations θ and $\text{int}P$ are denoted by zero element of E and the interior of P, respectively. The subset P is called cone if and only if:

(i) P is closed, nonempty, and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.

By using the above cone P, we can define a partial ordering $\leq$ with respect to P by the meaning that $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$. Write $\|\cdot\|$ as the norm on E. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying the above is called the normal constant of P. It is well known that $K \geq 1$.

Throughout this paper, unless otherwise specified, we let E be a real Banach space, P is a cone in E with $\text{int}P \neq \varnothing$ and $\leq$ is a partial ordering with respect to P.

Definition 2.1 ([13])**.** Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 ([14])**.** Let X be a nonempty subset and $s \geq 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be cone b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

($\hat{\text{d}}$1) $\theta < d(x, y)$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
($\hat{\text{d}}$2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
($\hat{\text{d}}$3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a cone b-metric space.

Remark 2.3. It is obvious from the above definitions that the class of cone b-metric spaces is larger than the class of cone metric spaces. Therefore, it is clearly that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

The following examples (see [7]) indicate that there are cone b-metric spaces which are not cone metric spaces.

Example 2.4 ([7, Example 1.4])**.** Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq E$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that

$$d(x, y) = \left(|x - y|^p, \alpha|x - y|^p\right),$$

where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space, but not a cone metric space.

Example 2.5 ([7, Example 1.5])**.** Let $X = l^p$ with $0 < p < 1$, where $l^p = \left\{ \{x_n\} \subset \mathbb{R} \,\middle|\, \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$. Let $d : X \times X \to \mathbb{R}$,

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = \{x_n\}, y = \{y_n\} \in l^p$. Then (X,d) is a b-metric space (see [10]). Put $E = l^p$, $P = \{\{x_n\} \in E : x_n \geq 0, \text{ for all } n \geq 1\}$. Letting the mapping $\widehat{d} : X \times X \to E$ be defined by $\widehat{d}(x,y) = \left\{ \frac{d(x,y)}{2^n} \right\}_{n \geq 1}$. Then $(X, \widehat{d})$ is a cone b-metric space with the coefficient $s = 2^{\frac{1}{p}} > 1$, but it is not a cone metric.

Example 2.6 ([7, Example 1.6])**.** Let $X = \{1,2,3,4\}$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\}$. Define $d : X \times X \to E$ by

$$d(x,y) = \begin{cases} \left(|x - y_n|^{-1}, |x - y|^{-1} \right), & \text{if } \quad x \neq y, \\ \theta, & \text{if } \quad x = y. \end{cases}$$

Then (X,d) is a cone b-metric space with the coefficient $s = \frac{6}{5}$. But it is not a cone metric space.

Inspired by Banach [1], Kannan [3], Chatterjea [4], Huang and Xu [7] and the concept of coincidence points, the following two types of f-contractive mappings are defined.

Definition 2.7. Let (X,d) be a cone b-metric space with $f, T : X \to X$. The mapping T is said to be an *f-contractive mapping of type B* if

$$d(Tx,Ty) \leq \lambda d(fx,fy)$$

for all $x, y \in X$.

Definition 2.8. Let (X,d) be a cone b-metric space with $f, T : X \to X$. The mapping T is said to be an *f-contractive mapping of type KC* if

$$d\left(Tx,Ty\right) \leq \lambda_1 d\left(fx,Tx\right) + \lambda_2 d\left(fy,Ty\right) + \lambda_3 d\left(fx,Ty\right) + \lambda_4 d\left(fy,Tx\right)$$

for all $x, y \in X$.

Definition 2.9. Let (X,d) be a cone b-metric space with $f, T : X \to X$. Then,

(a) a point $x \in X$ is said to be a fixed point of T if $Tx = x$. The set of all fixed points of T is denoted by $Fix(T) = \{x \in X : x = Tx\}$.

(b) a point $x \in X$ is said to be a coincidence point of f and T if $fx = Tx$. The set of all coincidence points of f and T is denoted by $C(f,T) = \{x \in X : fx = Tx\}$.

It is obvious that, $Fix(T) = C(I,T)$ where I is an identity mapping.

Definition 2.10 ([14])**.** Let (X,d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ converges to x whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(2) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_m, x_n) \ll c$ for all $n, m \geq N$.

(3) (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

The following lemmas are often used without the need of the normality of the cone P.

Lemma 2.11 ([15]). *Let P be a cone and $\{a_n\}$ be a sequence in E. If $c \in intP$ and $\theta \leq a_n \to \theta$ (as $n \to \infty$), then there exists N such that for all $n > N$, we have $a_n \ll c$.*

Lemma 2.12 ([15]). *Let $x, y, z \in E$, if $x \leq y$ and $y \ll z$, then $x \ll z$.*

Lemma 2.13 ([14]). *Let P be a cone and $\theta \leq u \ll c$ for each $c \in intP$, then $u = \theta$.*

Lemma 2.14 ([28]). *Let P be a cone. If $u \in P$ and $u \leq ku$ for some $0 \leq k < 1$, then $u = \theta$.*

Lemma 2.15 ([15]). *Let P be a cone and $a \leq b + c$ for each $c \in intP$, then $a \leq b$.*

3. Main Results

In this section, we will prove some coincidence point theorems of f-contractive mappings in the setting of cone b-metric spaces.

Theorem 3.1. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ is an f-contractive mapping of type B, that is,*

$$d(Tx, Ty) \leq \lambda d(fx, fy) \quad \text{for all } x, y \in X,$$

where $\lambda \in [0, 1)$ is a constant. If $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then f and T have a point of coincidence in X.

Proof. Let $x_0 \in X$. Since $T(X) \subseteq f(X)$, we can construct the sequence $\{fx_n\}$ where $fx_n = Tx_{n-1}$, $n \in \mathbb{N}$.

If $fx_{n+1} = fx_n$, for some $n \in \mathbb{N} \cup \{0\}$, then trivially f and T have a coincidence point x_n in X.

If $fx_{n+1} \neq fx_n$, for all $n \in \mathbb{N} \cup \{0\}$, then let us consider the following way

$$d(fx_n, fx_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda d(fx_{n-1}, fx_n) \leq \ldots \leq \lambda^n d(fx_1, fx_0).$$

For any $m \geq 1$, $p \geq 1$, we find that

$$\begin{aligned}
& d(fx_{m+p}, x_m) \\
& \leq s\left[d(fx_{m+p}, fx_{m+p-1}) + d(fx_{m+p-1}, fx_m)\right] \\
& = sd(fx_{m+p}, fx_{m+p-1}) + sd(fx_{m+p-1}, fx_m) \\
& \leq sd(fx_{m+p}, fx_{m+p-1}) + s^2\left[d(fx_{m+p-1}, fx_{m+p-2}) + d(fx_{m+p-2}, fx_m)\right] \\
& = sd(fx_{m+p}, fx_{m+p-1}) + s^2 d(fx_{m+p-1}, fx_{m+p-2}) + s^2 d(fx_{m+p-2}, fx_m) \\
& \leq sd(fx_{m+p}, fx_{m+p-1}) + s^2 d(fx_{m+p+1}, fx_{m+p-2}) + s^3 d(fx_{m+p-2}, fx_{m+p-3}) \\
& \quad + \ldots + s^{p-1}\lambda^{m+1} d(fx_1, fx_0) + s^{p-1}\lambda^m d(fx_1, fx_0) \\
& = \left(s\lambda^{m+p-1} + s^2\lambda^{m+p-2} + s^3\lambda^{m+p-3} + \cdots + s^{p-1}\lambda^{m+1}\right) d(fx_1, fx_0) \\
& \quad + s^{p-1}\lambda^m d(fx_1, fx_0) \\
& = \frac{s\lambda^{m+p}\left[\left(s\lambda^{-1}\right)^{p-1} - 1\right]}{s - \lambda} d(fx_1, fx_0) + s^{p-1}\lambda^m d(fx_1, fx_0)
\end{aligned}$$

$$\leq \frac{s^p\lambda^{m+1}}{s-\lambda}d\left(fx_1,fx_0\right)+s^{p-1}\lambda^m d\left(fx_1,fx_0\right).$$

Let $c \gg \theta$ be given. Notice that $\frac{s^p\lambda^{m+1}}{s-\lambda}d\left(fx_1,fx_0\right)+s^{p-1}\lambda^m d\left(fx_1,fx_0\right) \to \theta$ as $m \to \infty$ for any p. By employing Lemma 2.11, we can find $m_0 \in \mathbb{N}$ such that

$$\frac{s^p\lambda^{m+1}}{s-\lambda}d\left(fx_1,fx_0\right)+s^{p-1}\lambda^m d\left(fx_1,fx_0\right) \ll c,$$

for each $m > m_0$. Thus,

$$d\left(fx_{m+p},fx_m\right) \leq \frac{s^p\lambda^{m+1}}{s-\lambda}d\left(fx_1,fx_0\right)+s^{p-1}\lambda^m d\left(fx_1,fx_0\right) \ll c$$

for all $m > m_0$ and any p. Applying Lemma 2.12, $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Since $f(X)$ is a complete subspace of X, there exists $z \in f\left(X\right)$ such that $fx_n \to z = fx^*$ for some $x^* \in X$. We can find $n_0 \in \mathbb{N}$ such that

$$d\left(fx_n,fx^*\right) \ll \frac{c}{s\left(\lambda+1\right)}$$

for all $n > n_0$. Hence,

$$\begin{aligned} d\left(Tx^*,fx^*\right) &\leq s\left[d\left(Tx^*,Tx_n\right)+d\left(Tx_n,fx^*\right)\right] \\ &\leq s\left[\lambda d\left(fx^*,fx_n\right)+d\left(fx_{n+1},fx^*\right)\right] \ll c, \end{aligned}$$

for each $n > n_0$. Then, by Lemma 2.13, we deduce that $d\left(Tx^*,fx^*\right)=\theta$, i.e., $Tx^* = fx^*$. That is, x^* is a coincidence point of f and T. ■

If $f : X \to X$ is an identity mapping, that is, $fx = Ix = x$ for all $x \in X$, then we have the following corollary.

Corollary 3.2 ([7, Theorem 2.1]). *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx,Ty) \leq \lambda d(x,y) \in X,$$

for all $x, y \in X$, where $\lambda \in (0,1]$ is a constant. Then T has a fixed point in X.

Next, we prove a coincidence point theorem of contractive mapping of type KC.

Theorem 3.3. *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ is and f-contractive mapping of type KC, that is,*

$$d\left(Tx,Ty\right) \leq \lambda_1 d\left(fx,Tx\right)+\lambda_2 d\left(fy,Ty\right)+\lambda_3 d\left(fx,Ty\right)+\lambda_4 d\left(fy,Tx\right),$$

for all $x, y \in X$, where the constant $\lambda_i \in [1,0)$ and $\lambda_1+\lambda_2+s\left(\lambda_3+\lambda_4\right) < \min\left\{1,\frac{2}{s}\right\}, i = 1,2,3,4$. If $f(X)$ is a complete subspace of X, then f and T have a point of coincidence in X.

Proof. Let $x_0 \in X$. Then, since $T(X) \subseteq f(X)$, we can produce a sequence $\{fx_n\}$ such that $fx_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

If $fx_{n+1} = fx_n$, for some $n \in \mathbb{N}\cup\{0\}$, then obviously that f and T have a coincidence point x_n in X.

If $fx_{n+1} \neq fx_n$ for all $n \in \mathbb{N}\cup\{0\}$, then let us consider the following

$$\begin{aligned} &d\left(fx_{n+1},fx_n\right) \\ &= d\left(Tx_n,Tx_{n-1}\right) \\ &\leq \lambda_1 d\left(fx_n,Tx_n\right)+\lambda_2 d\left(fx_{n-1},Tx_{n-1}\right)+\lambda_3 d\left(fx_n,Tx_{n-1}\right)+\lambda_4 d\left(fx_{n-1},Tx_n\right) \end{aligned}$$

$$\leq \lambda_1 d(fx_n, fx_{n+1}) + \lambda_2 d(fx_{n-1}, fx_n) + \lambda_4 s[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})]$$
$$= (\lambda_1 + \lambda_4 s) d(fx_n, fx_{n+1}) + (\lambda_2 + \lambda_4 s) d(fx_n, fx_{n-1}).$$

It follows that

$$(1 - \lambda_1 - \lambda_4 s) d(fx_{n+1}, fx_n) \leq (\lambda_2 + \lambda_4 s) d(fx_n, fx_{n-1}). \tag{3.1}$$

Secondly,

$$\begin{aligned}
&d(fx_{n+1}, fx_n)\\
&= d(Tx_n, Tx_{n-1}) = d(Tx_{n-1}, Tx_n)\\
&\leq \lambda_1 d(fx_{n-1}, Tx_{n-1}) + \lambda_2 d(fx_n, Tx_n) + \lambda_3 d(fx_{n-1}, Tx_n) + \lambda_4 d(fx_n, Tx_{n-1})\\
&\leq \lambda_1 d(fx_{n-1}, fx_n) + \lambda_2 d(fx_n, fx_{n+1}) + \lambda_3 s[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})]\\
&= (\lambda_2 + \lambda_3 s) d(fx_n, fx_{n+1}) + (\lambda_1 + \lambda_3 s) d(fx_n, fx_{n-1}).
\end{aligned}$$

This implies that

$$(1 - \lambda_2 - \lambda_3 s) d(fx_{n+1}, fx_n) \leq (\lambda_1 + \lambda_3 s) d(fx_n, fx_{n-1}) \tag{3.2}$$

Adding up (3.1) and (3.2) yields

$$d(fx_{n+1}, fx_n) \leq \frac{\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4)}{2 - \lambda_1 - \lambda_2 - s(\lambda_3 + \lambda_4)} d(fx_n, fx_{n-1}).$$

Put $\lambda = \frac{\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4)}{2 - \lambda_1 - \lambda_2 - s(\lambda_3 + \lambda_4)}$, it is easy to see that $0 \leq \lambda < 1$. Thus,

$$d(fx_{n+1}, fx_n) = d(fx_n, fx_{n-1}) \leq \ldots \leq \lambda^n d(fx_1, fx_0).$$

Following an argument similar to that given in Theorem 3.1, there exists $fx^* \in X$ such that $fx_n \to fx^*$.

Let $c \gg \theta$ be arbitrary. Since $fx_n \to fx^*$, there exists N such that

$$d(fx_n, fx^*) \ll \frac{2 - s\lambda_1 + s\lambda_2 + s^2\lambda_3 + s^2\lambda_4}{2s^2 + 2s} c$$

for all $n > N$.

Next we claim that x^* is a coincidence point of f and T. Actually, on the one hand

$$\begin{aligned}
&d(Tx^*, fx^*)\\
&\leq s[d(Tx^*, Tx_n) + d(Tx_n, fx^*)] = sd(Tx^*, Tx_n) + sd(fx_{n+1}, fx^*)\\
&\leq s[\lambda_1 d(fx^*, Tx^*) + \lambda_2 d(fx_n, Tx_n) + \lambda_3 d(fx^*, Tx_n) + \lambda_4 d(fx_n, Tx^*)]\\
&\quad + sd(fx_{n+1}, fx^*)\\
&\leq s\lambda_1 d(fx^*, Tx^*) + s^2\lambda_2 d(fx_n, fx^*) + s^2\lambda_2 d(fx^*, fx_{n+1}) + s\lambda_3 d(fx^*, fx_{n+1})\\
&\quad + s^2\lambda_4 d(fx_n, fx^*) + s^2\lambda_4 d(fx^*, Tx^*) + sd(fx_{n+1}, fx^*)\\
&= \left(s\lambda_1 + s^2\lambda_4\right) d(fx^*, Tx^*) + \left(s^2\lambda_2 + s^2\lambda_4\right) d(fx_n, fx^*)\\
&\quad + \left(s^2\lambda_2 + s\lambda_3 + s\right) d(fx^*, fx_{n+1}).
\end{aligned}$$

Which implies that

$$\begin{aligned}
&\left(1 - s\lambda_1 - s^2\lambda_4\right) d(fx^*, Tx^*)\\
&\leq \left(s^2\lambda_2 + s^2\lambda_4\right) d(fx_n, fx^*) + \left(s^2\lambda_2 + s\lambda_3 + s\right) d(fx^*, fx_{n+1}).
\end{aligned}$$

On the other that

$$\begin{aligned}
&d\left(fx^*,Tx^*\right)\\
&\leq s\left[d\left(fx^*,Tx_n\right)+d\left(Tx_n,Tx^*\right)\right]=sd\left(fx^*,fx_{n+1}\right)+sd\left(Tx_n,Tx^*\right)\\
&\leq sd\left(fx^*,fx_{n+1}\right)\\
&\quad+s\left[\lambda_1 d\left(fx_n,Tx_n\right)+\lambda_2 d\left(fx^*,Tx^*\right)+\lambda_3 d\left(fx_n,Tx^*\right)+\lambda_4 d\left(fx^*,Tx_n\right)\right]\\
&=sd\left(fx^*,fx_{n+1}\right)\\
&\quad+s\left[\lambda_1 d\left(fx_n,fx_{n+1}\right)+\lambda_2 d\left(fx^*,Tx^*\right)+\lambda_3 d\left(fx_n,Tx^*\right)+\lambda_4 d\left(fx^*,fx_{n+1}\right)\right]\\
&\leq sd\left(fx^*,fx_{n+1}\right)+s^2\lambda_1 d\left(fx_n,fx^*\right)+s^2\lambda_1 d\left(fx^*,Tx_{n+1}\right)+s\lambda_2 d\left(fx^*,Tx^*\right)\\
&\quad+s^2\lambda_3 d\left(fx_n,fx^*\right)+s^2\lambda_3 d\left(fx^*,Tx^*\right)+s\lambda_4 d\left(fx^*,Tx_{n+1}\right)\\
&=\left(s\lambda_2+s^2\lambda_3\right)d\left(fx^*,Tx^*\right)+\left(s^2\lambda_1+s^2\lambda_3\right)d\left(fx_n,fx^*\right)\\
&\quad+\left(s^2\lambda_1+s\lambda_4+s\right)d\left(fx^*,fx_{n+1}\right).
\end{aligned}$$

Which means that

$$\begin{aligned}
&\left(1-s\lambda_2-s^2\lambda_3\right)d\left(fx^*,Tx^*\right)\\
&\leq\left(s^2\lambda_1+s^2\lambda_3\right)d\left(fx_n,fx^*\right)+\left(s^2\lambda_1+s\lambda_4+s\right)d\left(fx^*,fx_{n+1}\right).
\end{aligned}$$

Combining and yields

$$\begin{aligned}
&\left(2-s\lambda_1-s\lambda_2-s^2\lambda_3-s^2\lambda_4\right)d\left(fx^*,Tx^*\right)\\
&\leq s^2\left(\lambda_1-\lambda_2-\lambda_3-\lambda_4\right)d\left(fx_n,fx^*\right)\\
&\quad+\left(s^2\lambda_1-s^2\lambda_2-s\lambda_3-s\lambda_4+2s\right)d\left(fx^*,fx_{n+1}\right)\\
&\leq s^2 d\left(fx_n,fx^*\right)+\left(s^2+2s\right)d\left(fx^*,fx_{n+1}\right).
\end{aligned}$$

Simple calculations ensure that

$$d\left(fx^*,Tx^*\right)\leq\frac{s^2 d\left(fx_n,fx^*\right)+\left(s^2+2s\right)d\left(fx^*,fx_{n+1}\right)}{2-s\lambda_1-s\lambda_2-s^2\lambda_3-s^2\lambda_4}\ll c.$$

It is easy to see from Lemma 2.13 that $d\left(fx^*,Tx^*\right)=\theta$, i.e., x^* is coincidence point of f and T. ■

If $f:X\to X$ is an identity mapping, then we have the following corollary.

Corollary 3.4 ([7, Theorem 2.3]). *Let (X,d) be a complete cone b-metric space with the coefficient $s\geq 1$. Suppose the mapping $T:X\to X$ satisfies the contractive condition*

$$d\left(Tx,Ty\right)\leq\lambda_1 d\left(x,Tx\right)+\lambda_2 d\left(y,Ty\right)+\lambda_3 d\left(x,Ty\right)+\lambda_4 d\left(y,Tx\right),$$

for all $x,y\in X$, where the constant $\lambda_i\in[1,0)$ and $\lambda_1+\lambda_2+s\left(\lambda_3+\lambda_4\right)<\min\left\{1,\frac{2}{s}\right\},i=1,2,3,4$. Then T has a fixed point in X.

4. Conclusion

Base on the concept of coincidence points and the inspiration of the considered mappings form [7], in this reserch paper we have established the two new mappings on cone b-metric spaces, the first one is an *f-contractive mapping of type B* (see Defintion 2.7) defined by

$$d(Tx,Ty)\leq\lambda d(fx,fy)$$

for all $x, y \in X$ and the second one is an *f-contractive mapping of type KC* (see Defintion 2.8) defined by

$$d(Tx, Ty) \leq \lambda_1 d(fx, Tx) + \lambda_2 d(fy, Ty) + \lambda_3 d(fx, Ty) + \lambda_4 d(fy, Tx)$$

for all $x, y \in X$. Each new mapping can be viewed as a generalization of the previous one proposed by [7]. Under some available properties on a complete cone b-metric space, some suitable conditions on scalars and the considered two new mappings, these allow us to generate a Cauchy sequence which acts as a tool that we used it to find and confirm the existence of a coincidence point of f and T as in Theorem 3.1 and Theorem 3.3, respectively.

Acknowledgements

This research was partially supported by the Research Fund for DPST Graduate with First Placement, Institute for the Promotion of Teaching Science and Technology (IPST). The second author would like to thank Naresuan University for financial support.

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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