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A Common Fixed-Point Iterative Process with Errors for Quasi-Nonexpansive Nonself-Mappings in Banach Spaces

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Abstract : A new common fixed-point iterative process with errors for quasi-nonexpansive nonself-mappings in Banach spaces is introduced and studied. The strong convergence theorems of such iterative process are established.

Keywords : quasi-nonexpansive nonself-mappings, Banach space, closed convex subset, retract, common fixed-point, necessary and sufficient condition.
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1 Introduction

In 1972, Goebel and Kirk [1] introduced the concept of asymptotically nonexpansive self-mappings, which generalizes the class of nonexpansive self-mappings. They proved that every asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space has a fixed point. Since then, the weak and strong convergence problems of iterative sequences (with errors) for asymptotically nonexpansive type have been studied by many authors (see, for examples, [3-13]). In 2006, Lan [2] introduced a new class of iterative procedures with errors to approximate the common fixed point of two generalized asymptotically quasi-nonexpansive self-mappings and proved some strong convergence results for the iterative sequence with errors in Banach spaces. For nonexpansive nonself-mappings, some authors, (see, for examples, [14-18]), have studied the strong and weak convergence theorems for such mappings in Hilbert spaces or uniformly convex Banach spaces. The concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume, Ofoedu and Zegeye [19] in 2003. Such a nonself-mapping is defined as follows. Let X be a real normed linear space, C be a nonempty subset of X. Let $P : X \to C$ be the nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called *asymptotically nonexpansive* if there exists a sequence $\{r_n\} \subset [0, 1)$ with $r_n \to 0$ as $n \to \infty$ such

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that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1 + r_n)\|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$.

Wang [20] proved strong and weak convergence theorems for common fixed points of two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces. In this paper, we construct an iterative procedure to approximate common fixed points of two quasi-nonexpansive nonself-mappings and prove some strong convergence theorems for such mappings in arbitrary real Banach spaces.

In sequel, we need the following definitions and lemmas for the main results in this paper.

Definition 1.1. *Let X be a real Banach space, and let C be a nonempty (closed) convex nonexpansive retract of X with P as a nonexpansive retraction. A nonself-mapping $T : C \to X$ is called* quasi-nonexpansive *if*

$$\|Tx - p\| \leq \|x - p\|, \tag{1.1}$$

for all $x \in C, p \in F(T)$ and $n \geq 1$ where $F(T)$ denote the set of fixed points of T.

Lemma 1.2. (See [12, Lemma 2.1].) Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \ \ \forall n = 1, 2, ...,$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(1) $\lim_{n\to\infty} a_n$ exists .
(2) $\lim_{n\to\infty} a_n = 0$ if $\{a_n\}$ has a subsequence converging to zero.

Lemma 1.3. Let C be a nonempty closed subset of a Banach space X and $T : C \to X$ be a quasi-nonexpansive nonself-mapping with the fixed point set $F(T) \neq \emptyset$. Then $F(T)$ is a closed subset in C.

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. Since C is closed and $\{p_n\}$ is a sequence in C, we must have $p \in C$. Since $T : C \to X$ is a quasi-nonexpansive nonself-mapping, we obtain

$$\|Tp - p_n\| \leq \|p - p_n\|.$$

Taking limit as $n \to \infty$, we get

$$\|Tp - p\| \leq 0,$$

which implies that $Tp = p$. The proof is complete. □

2 Main Results

We are interested in sequences in the following process. For $x_1 \in C$, define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} y_n &= P(a_nT_2x_n + (1-a_n-\beta_n)x_n + \beta_n u_n) \\ x_{n+1} &= P(b_nT_1y_n + c_nT_1x_n + (1-b_n-c_n-\alpha_n)x_n + \alpha_n v_n), \end{aligned} \tag{2.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C.

Remark 2.1. *If T is a self-mapping, then P becomes the identity mapping so that T is a quasi-nonexpansive self-mapping. Also the iterative sequences (2.1) become*

$$\begin{aligned} y_n &= a_nT_2x_n + (1-a_n-\beta_n)x_n + \beta_n u_n \\ x_{n+1} &= b_nT_1y_n + c_nT_1x_n + (1-b_n-c_n-\alpha_n)x_n + \alpha_n v_n. \end{aligned} \tag{2.2}$$

Theorem 2.2. *Let X be a real arbitrary Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. For $i = 1, 2$, let $T_i : C \to X$ be a quasi-nonexpansive nonself-mapping such that $F(T_1) \cap F(T_2) \neq \emptyset$ in C. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$, and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + \beta_n$ and $b_n + c_n + \alpha_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \alpha_n < \infty$, and $\sum_{n=1}^{\infty} \beta_n < \infty$. Assume that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C. For arbitrary $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the iterative sequences defined in 2.1*
Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 if and only if

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0, \tag{2.3}$$

where $d(x_n, F(T_1) \cap F(T_2))$ is the distance between x and the set $F(T_1) \cap F(T_2)$.

Proof. The necessity is obvious, so it is omitted.
We now proof the sufficiency. Let $p \in F(T_1) \cap F(T_2))$. By the boundedness of $\{u_n\}$ and $\{v_n\}$, we let

$$M = \max\{sup_{n\geq 1}\|u_n - p\|, sup_{n\geq 1}\|v_n - p\|\}.$$

Since $T_i : C \to X$ is a quasi-nonexpansive mapping for $i = 1, 2$, and C is a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, we have

$$\begin{aligned} \|y_n - p\| &= \|P(a_nT_2x_n + (1-a_n-\beta_n)x_n + \beta_n u_n) - Pp\| \\ &\leq \|a_nT_2x_n + (1-a_n-\beta_n)x_n + \beta_n u_n - p\| \\ &\leq a_n\|T_2x_n - p\| + (1-a_n-\beta_n)\|x_n - p\| + \beta_n\|u_n - p\|\} \\ &\leq a_n\|x_n - p\| + (1-a_n-\beta_n)\|x_n - p\| + \beta_n M \\ &= (1-\beta_n)|x_n - p\| + \beta_n M, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\|x_{n+1}-p\| &= \|P(b_nT_1y_n+c_nT_1x_n+(1-b_n-c_n-\alpha_n)x_n+\alpha_nv_n)-Pp\|\\ &\le \|b_nT_1y_n+c_nT_1x_n+(1-b_n-c_n-\alpha_n)x_n+\alpha_nv_n-p\|\\ &\le b_n\|T_1y_n-p\|+c_n\|T_1x_n-p\|+(1-b_n-c_n-\alpha_n)\|x_n-p\|+\alpha_n\|v_n-p\|\\ &\le b_n\|y_n-p\|+c_n\|x_n-p\|+(1-b_n-c_n-\alpha_n)\|x_n-p\|+\alpha_nM\\ &\le b_n\{(1-\beta_n)|x_n-p\|+\beta_nM\}+(1-b_n-\alpha_n)\|x_n-p\|+\alpha_nM\\ &= \{b_n(1-\beta_n)+(1-b_n-\alpha_n)\}\|x_n-p\|+b_n\beta_nM+\alpha_nM\\ &\le (1-b_n\beta_n-\alpha_n)\|x_n-p\|+(b_n\beta_n+\alpha_n)M\\ &\le \|x_n-p\|+(\beta_n+\alpha_n)M\\ &\le \|x_n-p\|+d_n, \qquad (2.5)\end{aligned}$$

where $d_n=\beta_n+\alpha_n$. Now by the assumptions that $\sum_{n=1}^{\infty}\alpha_n<\infty$ and $\sum_{n=1}^{\infty}\beta_n<\infty$, we have that $\sum_{n=1}^{\infty}d_n<\infty$. Then Lemma 1.2 implies that $\lim_{n\to\infty}\|x_n-p\|$ exists. From (2.5) and by induction, for $m,n\ge 1$ and $p\in F(T_1)\cap F(T_2))$, we have

$$\|x_{n+m}-p\|\le\|x_n-p\|+\sum_{i=n}^{n+m-1}d_i. \qquad (2.6)$$

From (2.5), we obtain

$$d(x_{n+1},F(T_1)\cap F(T_2))\le d(x_n,F(T_1)\cap F(T_2))+d_n.$$

But, the assumption $\liminf_{n\to\infty}d(x_n,F(T_1)\cap F(T_2))=0$ implies that there exists a subsequence of $\{d(x_n,F(T_1)\cap F(T_2))\}$ converging to zero. Therefore Lemma 1.2 tells us that

$$\lim_{n\to\infty}d(x_n,F(T_1)\cap F(T_2))=0. \qquad (2.7)$$

We now show that $\{x_n\}$ is a Cauchy sequence in X. Let $\epsilon>0$. From (2.7) and $\sum_{n=1}^{\infty}d_n<\infty$, there exists n_0 such that, for $n\ge n_0$, we have

$$d(x_n,F(T_1)\cap F(T_2))<\epsilon/4, \text{ and } \sum_{n=n_0}^{\infty}d_n<\epsilon/2. \qquad (2.8)$$

By the first inequality in (2.8) and the definition of infimum, there exists $p_0\in F(T_1)\cap F(T_2)$ such that

$$\|x_{n_0}-p_0\|<\epsilon/4. \qquad (2.9)$$

Combining (2.6),(2.8) and (2.9), we obtain

$$\begin{aligned}\|x_{n_0+m}-x_{n_0}\| &\le \|x_{n_0+m}-p_0\|+\|x_{n_0}-p_0\|\\ &\le 2\|x_{n_0}-p_0\|+\sum_{i=n_0}^{n_0+m-1}d_i\\ &< \epsilon/2+\epsilon/2=\epsilon,\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. But X is a Banach space, so there must exist $p \in X$ such that $x_n \to p$. Since C is closed and $\{x_n\}$ is a sequence in C converging to p, we have that $p \in C$. Also, by Lemma 1.3, $F(T_1) \cap F(T_2)$ is closed. Now $d(x_n, F(T_1) \cap F(T_2)) \to 0$ and $x_n \to p$ as $n \to \infty$, the continuity of $d(x, F(T_1) \cap F(T_2))$ implies that $d(p, F(T_1) \cap F(T_2)) = 0$. Thus $p \in F(T_1) \cap F(T_2)$. Therefore $\{x_n\}$ converges to a common fixed point of T_1 and T_2, as desired. □

If $T_1 = T_2 = T$, we have the following result.

Theorem 2.3. *Let X be a real arbitrary Banach space, and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a quasi-nonexpansive nonself-mapping such that $F(T) \neq \emptyset$ in C. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$, and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + \beta_n$ and $b_n + c_n + \alpha_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \alpha_n < \infty$, and $\sum_{n=1}^{\infty} \beta_n < \infty$. Assume that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C. For arbitrary $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the iterative sequences defined by*

$$\begin{aligned} y_n &= P(a_n T x_n + (1 - a_n - \beta_n) x_n + \beta_n u_n) \\ x_{n+1} &= P(b_n T y_n + c_n T x_n + (1 - b_n - c_n - \alpha_n) x_n + \alpha_n v_n) \end{aligned}$$

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0, \tag{2.10}$$

where $d(x_n, F(T))$ is the distance between x and the set $F(T)$.

We also have the following theorems as in Lan [2].

Theorem 2.4. *Let $X, C, T_i (i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem 2.2. Suppose that the conditions in Theorem 2.2 hold and*
(1) the mapping $T_i (i = 1, 2)$ is asymptotically regular in x_n ,i.e.,

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad i = 1, 2;$$

(2) $\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0$ *implies that*

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0.$$

Then the sequences $\{x_n\}$ converges to a common fixed point of T_1 and T_2.

Theorem 2.5. *Let $X, C, T_i (i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem 2.2. Suppose that the conditions (i) and (ii) in Theorem 2.2 hold, the mapping T_i is asymptotically regular in x_n, and there exists an increasing function $f : R^+ \to R^+$ with $f(r) > 0$ for all $r > 0$ such that for $i = 1, 2$,*

$$\|x_n - T_i x_n\| \geq f(d(x_n, F(T_1) \cap F(T_2))), \quad \forall n \geq 1.$$

Then the sequences $\{x_n\}$ converges to a common fixed point of T_1 and T_2.

As a result of Theorem 2.2 and Remark 2.1, we obtain the following theorem for a generalized asymptotically quasi-nonexpansive self-mapping T and the iterative sequences defined in (2.2).

Theorem 2.6. *Let X be a real arbitrary Banach space, and let C be a nonempty closed convex subset of X. For $i = 1, 2$, let $T_i : C \to C$ be a quasi-nonexpansive self-mapping such that $F(T_1) \cap F(T_2) \neq \emptyset$ in C. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$, and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + \beta_n$ and $b_n + c_n + \alpha_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \alpha_n < \infty$, and $\sum_{n=1}^{\infty} \beta_n < \infty$. Assume that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C. For arbitrary $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the iterative sequences defined in (2.2)*
Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 if and only if

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0, \tag{2.11}$$

where $d(x, F(T_1) \cap F(T_2))$ is the distance between x and the set $F(T_1) \cap F(T_2)$.

If $T_1 = T_2 = T$, we have the following result.

Theorem 2.7. *Let X be a real arbitrary Banach space, and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a quasi-nonexpansive self-mapping such that $F(T) \neq \emptyset$ in C. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$, and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + \beta_n$ and $b_n + c_n + \alpha_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \alpha_n < \infty$, and $\sum_{n=1}^{\infty} \beta_n < \infty$. Assume that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C. For arbitrary $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the iterative sequences defined by*

$$\begin{aligned} y_n &= a_n T x_n + (1 - a_n - \beta_n) x_n + \beta_n u_n \\ x_{n+1} &= b_n T y_n + c_n T x_n + (1 - b_n - c_n - \alpha_n) x_n + \alpha_n v_n. \end{aligned} \tag{2.12}$$

Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0, \tag{2.13}$$

where $d(x, F(T))$ is the distance between x and the set $F(T)$.

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