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Common Fixed Point Theorems for Weakly (ψ, S, C)-Contractive Mappings in Partially Ordered b-Metric Spaces

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Abstract In this paper, we introduce the notions of weakly (ψ, S, C)-contractive mappings and generalized weakly (ψ, S, C)-contractive mappings in partially ordered b-metric spaces and state some common fixed point theorems for these classes of mappings. These results generalize and improve the main results in [H.K. Nashine, Common fixed points via weakly (ψ, S, C)-contraction mappings on ordered metric spaces and application to integral equations, Thai J. Math. 12 (3) (2014) 729–747]. Also, some examples are given to illustrate the results.

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1. Introduction and Preliminaries

In 1972, Chatterjea [1] introduced the notion of a C-contraction as follows.

Definition 1.1 ([1])**.** Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping. Then T is called a *C-contraction* if there exists $\alpha \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha\big[d(x, Ty) + d(y, Tx)\big].$$

This notion was generalized to a weak C-contraction by Choudhury [2] and a (μ, ψ)-generalized f-weakly contractive mapping in metric spaces by Chandok [3].

Denote by Ψ the family of all lower semi-continuous functions $\psi : [0, \infty)^2 \longrightarrow [0, \infty)$ such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Definition 1.2 ([2], Definition 1.3)**.** Let (X, d) be a metric space and $T : X \longrightarrow X$ be a map. Then T is called a *weak C-contraction* if there exists $\psi \in \Psi$ such that for all

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$x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big] - \psi\big(d(x, Ty), d(y, Tx)\big).$$

Definition 1.3 ([4])**.** A function $\mu : [0, \infty) \longrightarrow [0, \infty)$ is called an *altering distance function* if the following properties are satisfied.

(1) μ is monotone increasing and continuous.
(2) $\mu(t) = 0$ if and only if $t = 0$.

Definition 1.4 ([3])**.** Let (X, d) be a metric space and $T, f : X \longrightarrow X$ be two mappings. Then T is called a *(μ, ψ)-generalized f-weakly contractive mapping* if there exist $\psi \in \Psi$ and an altering distance function μ such that for all $x, y \in X$,

$$\mu\big(d(Tx, Ty)\big) \leq \mu\big(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\big) - \psi\big(d(fx, Ty), d(fy, Tx)\big).$$

Remark 1.5. If f and μ are two the identify mappings, then a (μ, ψ)-generalized f-weakly contractive mapping becomes a weak C-contraction mapping.

There were some fixed point results for (μ, ψ)-generalized f-weakly contractive mappings in complete metric spaces [3, Theorem 2.1], and in complete partially ordered metric space [5, Theorem 2.1]. In 2013, Dung and Hang [6] introduced the notion of a weak C-contraction mapping in partially ordered 2-metric spaces and stated some fixed point results for this mapping in complete partially ordered 2-metric spaces [6, Theorem 2.3, Theorem 2.4, Theorem 2.5].

In 2014, Nashine [7] generalized the notion of a weak C-contraction in metric spaces to two mappings as follows. Recall that

(1) Γ_1 is the family of all strictly increasing and continuous functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that $\psi(t) \leq \frac{1}{2}t$ for all $t \geq 0$. Notice that $\psi(0) = 0$.
(2) Ω_1 is the family of all strictly increasing functions $\varphi : [0, \infty)^2 \longrightarrow [0, \infty)$ that are continuous in each variable, and $\varphi(x, y) = 0$ if and only if $x = y = 0$, and $\varphi(x, y) \leq x + y$ for all $x, y \in [0, \infty)$.
(3) Γ_2 is the family of all strictly increasing and continuous functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that $\psi(t) \leq \frac{1}{4}t$ for all $t \geq 0$. Notice that $\psi(0) = 0$.
(4) Ω_2 is the family of all strictly increasing functions $\varphi : [0, \infty)^4 \longrightarrow [0, \infty)$ that are continuous in each variable, and $\varphi(x, y, z, t) = 0$ if and only if $x = y = z = t = 0$, and $\varphi(x, y, z, t) \leq x + y + z + t$ for all $x, y, z, t \in [0, \infty)$.

Definition 1.6 ([7], Definition 3.1)**.** Let $(X, d, \preceq)$ be a partially ordered metric space and $T, S : X \longrightarrow X$ be two mappings. Then T is called a *weakly (ψ, S, C)-contractive* if there exist $\psi \in \Gamma_1$ and $\varphi \in \Omega_1$ such that for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$,

$$d(Tx, Sy) \leq \psi\big[d(x, Sy) + d(y, Tx) - \varphi(d(x, Sy), d(y, Tx))\big].$$

Definition 1.7 ([7], Definition 4.1)**.** Let $(X, d, \preceq)$ be a partially ordered metric space and $T, S : X \longrightarrow X$ be two mappings. Then T is called *a generalized weakly (ψ, S, C)-contraction* if there exist $\psi \in \Gamma_2$ and $\varphi \in \Omega_2$ such that for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$, we have

$$\begin{aligned} d(Tx, Sy) \quad &\leq \quad \psi\big[d(x, Tx) + d(y, Sy) + d(x, Sy) + d(y, Tx) \\ &\quad -\varphi(d(x, Tx), d(y, Sy), d(x, Sy), d(y, Tx))\big]. \end{aligned} \tag{1.1}$$

There were many generalizations of a metric space and many fixed point theorems on generalized metric spaces were stated [8]. The notion of a b-metric space was introduced by Bakhtin [9] and then extensively used by Czerwik [10, 11] as follows.

Definition 1.8 ([11])**.** Let X be a non-empty set and $d : X \times X \longrightarrow [0, \infty)$ be a function such that for some $s \geq 1$ and all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, y) \leq s\big(d(x, z) + d(z, y)\big)$.

Then d is called a b-*metric* on X and (X, d, s) is called a b-*metric space*.

Remark 1.9. (X, d) is a metric space if and only if $(X, d, 1)$ is a b-metric space.

The first important difference between a metric and a b-metric is that the b-metric need not be a continuous function in its two variables, see [12, Example 13]. In recent years, many fixed point theorems on b-metric spaces were stated, the readers may refer to [13–22] and references therein.

The purpose of this paper is to introduce the notions of a weakly (ψ, S, C)-contractive mapping and a generalized weakly (ψ, S, C)-contractive mapping in partially ordered b-metric spaces and to state some common fixed point theorems for these classes of mappings. Also, some examples are given to illustrate the results.

First, we recall some notions and lemmas which will be useful in what follows.

Definition 1.10 ([11])**.** Let (X, d, s) be a b-metric space. Then

(1) A sequence $\{x_n\}$ is called *convergent* to x, written as $\lim\limits_{n\to\infty} x_n = x$, if
$$\lim_{n\to\infty} d(x_n, x) = 0.$$
(2) A sequence $\{x_n\}$ is called *Cauchy* in X if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(3) (X, d, s) is called *complete* if every Cauchy sequence is a convergent sequence.

In 2014, Aghajani *et al.* [23] proved the following simple lemma about the convergence in b-metric spaces.

Lemma 1.11 ([23])**.** *Let (X, d, s) be a b-metric space and $\lim\limits_{n\to\infty} x_n = x, \lim\limits_{n\to\infty} y_n = y$. Then*

(1) $\dfrac{1}{s^2} d(x, y) \leq \liminf\limits_{n\to\infty} d(x_n, y_n) \leq \limsup\limits_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y)$. *In particular, if* $x = y$, *then* $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.
(2) For each $z \in X$, $\dfrac{1}{s} d(x, z) \leq \liminf\limits_{n\to\infty} d(x_n, z) \leq \limsup\limits_{n\to\infty} d(x_n, z) \leq s d(x, z)$.

The following lemma is a equivalent condition for the Cauchy property of $\{x_n\}$ in b-metric spaces.

Lemma 1.12. *Let (X, d, s) be a b-metric space and $\{x_n\}$ be a sequence in (X, d, s). Then the following statements are equivalent.*

(1) $\{x_n\}$ is a Cauchy sequence in (X, d, s).
(2) $\{x_{2n}\}$ is a Cauchy sequence in (X, d, s) and $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Proof. (1) $\Rightarrow$ (2). From the given assumption, we get $\{x_{2n}\}$ is a Cauchy sequence in (X, d, s) and $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.

(2) $\Rightarrow$ (1). For all $n, m \geq 0$, we only consider three following cases.

Case 1. $n = 2k+1, m = 2l$ for all $k, l \in \mathbb{N} \cup \{0\}$. Then

$$d(x_n, x_m) = d(x_{2k+1}, x_{2l}) \leq sd(x_{2k+1}, x_{2k}) + sd(x_{2k}, x_{2l}).$$

Case 2. $n = 2k, m = 2l+1$ for all $k, l \in \mathbb{N} \cup \{0\}$. Then

$$d(x_n, x_m) = d(x_{2k}, x_{2l+1}) \leq sd(x_{2k}, x_{2l}) + sd(x_{2l}, x_{2l+1}).$$

Case 3. $n = 2k+1, m = 2l+1$ for all $k, l \in \mathbb{N} \cup \{0\}$. Then

$$d(x_n, x_m) = d(x_{2k+1}, x_{2l+1}) \leq sd(x_{2k+1}, x_{2k}) + s^2 d(x_{2k}, x_{2l}) + s^2 d(x_{2l}, x_{2l+1}).$$

By the above cases, we find that $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d, s). ∎

Definition 1.13 ([24, 25])**.** Let $(X, \preceq)$ be a partially ordered set and $T, S : X \longrightarrow X$ be two mappings.

(1) The pair (T, S) is called *weakly increasing* if $Sx \preceq TSx$ and $Tx \preceq STx$ for all $x \in X$.

(2) The mapping S is called *T-weakly isotone increasing* if $Sx \preceq TSx \preceq STSx$ for all $x \in X$.

Remark 1.14 ([7])**.** If the pair (T, S) is weakly increasing, then S is T-weakly isotone increasing.

Definition 1.15 ([26])**.** Let $(X, \preceq)$ be a partially ordered set and $T, S : X \longrightarrow X$ be two mappings.

(1) For each $x_0 \in X$, put $x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Then the set $\mathcal{O}(x_0; T, S) = \{x_n : n \in \mathbb{N} \cup \{0\}\}$ is called *orbit of* (T, S) *at* x_0. If $S = T$, then $\mathcal{O}(x_0; T, S)$ is denoted by $\mathcal{O}(x_0; T)$.

(2) The space (X, d, s) is called (T, S)-*orbitally complete at* x_0 if every Cauchy sequence in $\mathcal{O}(x_0; T, S)$ converges to some $x \in X$.

(3) The mapping T is called *orbitally continuous at* x_0 if it is continuous on $\mathcal{O}(x_0; T)$.

(4) The pair (T, S) is called *asymptotically regular at* x_0 if there exists a sequence $\{x_n\}$ in X such that $x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Definition 1.16. Let $(X, d, s, \preceq)$ be a partially ordered b-metric space. Then $(X, d, s, \preceq)$ is called a *regular space* if $\{z_n\}$ is a non-decreasing sequence in X and $\lim\limits_{n\to\infty} z_n = z$, then $z_n \preceq z$ for all $n \in \mathbb{N} \cup \{0\}$.

2. Main Results

First, we introduce the notion of a weakly (ψ, S, C)-contractive in partially ordered b-metric spaces. Denote by

(1) Ψ_1 the family of all increasing functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that for all $t \geq 0$, we have $\psi(t) \leq \frac{1}{2}t$. Notice that $\psi(0) = 0$.

(2) Φ_1 the family of all lower semi-continuous functions $\varphi : [0,\infty)^2 \longrightarrow [0,\infty)$ such that $\varphi(x,y) = 0$ if and only if $x = y = 0$, and $\varphi(x,y) \leq x + y$ for all $x, y \in [0,\infty)$.

Definition 2.1. Let $(X, d, s, \preceq)$ be a partially ordered b-metric space and $T, S : X \longrightarrow X$ be two mappings. Then T is called a *weakly* (ψ, S, C)*-contraction* if there exist $\psi \in \Psi_1$ and $\varphi \in \Phi_1$ such that for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$,

$$d(Tx, Sy) \leq \psi\Big(\frac{2}{s(s^2+1)}\big[d(x, Sy) + d(y, Tx) - \varphi(d(x, Sy), d(y, Tx))\big]\Big). \tag{2.1}$$

Remark 2.2. A weakly (ψ, S, C)-contraction in Definition 1.6 is a particular case of a weakly (ψ, S, C)-contractive in Definition 2.1 for $s = 1$.

The following lemma states the relation between the fixed point of T, S and the common fixed point of T, S.

Lemma 2.3. *Let* $(X, d, s, \preceq)$ *be a partially ordered* b*-metric space and* $T, S : X \longrightarrow X$ *be two mappings satisfying the condition* (2.1)*. If* z *is a fixed point of* T *or* S*, then* z *is a common fixed point of* T *and* S*.*

Proof. Suppose that z is a fixed point of T. Form (2.1), we have

$$\begin{aligned}
d(z, Sz) &= d(Tz, Sz)\\
&\leq \psi\Big(\frac{2}{s(s^2+1)}\big[d(z, Sz) + d(z, Tz) - \varphi(d(z, Sz), d(z, Tz))\big]\Big)\\
&= \psi\Big(\frac{2}{s(s^2+1)}\big[d(z, Sz) - \varphi(0, d(z, Sz))\big]\Big)\\
&\leq \frac{1}{s(s^2+1)}\big[d(z, Sz) - \varphi(0, d(z, Sz))\big]\\
&\leq d(z, Sz) - \varphi(0, d(z, Sz)).
\end{aligned}$$

This implies that $\varphi(0, d(z, Sz)) = 0$. Therefore, $d(z, Sz) = 0$ and hence $z = Sz$, that is, z is a fixed point of S. So, z is a common fixed point of T and S.

Similarly, if z is a fixed point of S, then z is a common fixed point of T and S. ∎

The following theorem is a sufficient condition for the existence and uniqueness of the common fixed point for a weakly (ψ, S, C)-contractive in b-metric spaces.

Theorem 2.4. *Let* $(X, d, s, \preceq)$ *be a complete, partially ordered* b*-metric space and* $T, S : X \longrightarrow X$ *be two mappings such that*

(1) T *is a weakly* (ψ, S, C)*-contraction.*

(2) S *is* T*-weakly isotone increasing.*

(3) There exists x_0 *such that* $x_0 \preceq Sx_0$*.*

(4) T *or* S *is continuous, or* $(X, d, s, \preceq)$ *is a regular space.*

Then T *and* S *have a common fixed point. Moreover, the set of common fixed points of* T, S *is totally ordered if and only if* T *and* S *have a unique common fixed point.*

Proof. Define a sequence $\{x_n\}$ in X by

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where x_0 be defined by the assumption (3). Since S is T-weakly isotone increasing, we have

$$x_0 \preceq x_1 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots$$

Sine $x_{2n} \preceq x_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, from (2.1), we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n+1}, Sx_{2n}) \\
&\leq \psi\Big(\frac{2}{s(s^2+1)}\big[d(x_{2n+1}, Sx_{2n}) + d(x_{2n}, Tx_{2n+1}) \\
&\qquad -\varphi(d(x_{2n+1}, Sx_{2n}), d(x_{2n}, Tx_{2n+1})\big]\Big) \\
&= \psi\Big(\frac{2}{s(s^2+1)}\big[d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+2}) \qquad (2.2) \\
&\qquad -\varphi(d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+2}))\big]\Big) \\
&\leq \frac{1}{s(s^2+1)}\big[d(x_{2n}, x_{2n+2}) - \varphi(0, d(x_{2n}, x_{2n+2}))\big] \\
&\leq \frac{1}{s(s^2+1)} d(x_{2n}, x_{2n+2}) \\
&\leq \frac{1}{s^2+1}\big[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big] \\
&\leq \frac{1}{2}\big[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big]. \qquad (2.3)
\end{aligned}$$

This implies that

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) \tag{2.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. Similarly, we also have

$$d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2}) \tag{2.5}$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore, from (2.4) and (2.5), we have $d(x_{n+1}, x_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of non-negative real numbers. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{2.6}$$

Taking the limit as $n \to \infty$ in (2.2) and using (2.6), we get

$$r \leq \lim_{n\to\infty} \frac{d(x_{2n}, x_{2n+2})}{s(s^2+1)} \leq \frac{r+r}{2} \leq r.$$

This implies that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = rs(s^2+1). \tag{2.7}$$

Taking the limit as $n \to \infty$ in (2.2), using (2.6), (2.7) and the lower semi-continuous property of φ, we have

$$r \leq \frac{rs(s^2+1) - \varphi\big(0, rs(s^2+1)\big)}{s(s^2+1)} = r - \frac{\varphi\big(0, rs(s^2+1)\big)}{s(s^2+1)} \leq r.$$

This implies that $\varphi\big(0, rs(s^2+1)\big) = 0$ and hence $r = 0$. Then (2.6) becomes

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.8}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 1.12 and (2.8), it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{2n(k)}\}, \{x_{2m(k)}\}$ of $\{x_{2n}\}$ where $m(k)$ is a smallest integer such that $m(k) > n(k) > k$ and

$$d(x_{2n(k)}, x_{2m(k)}) \geq \varepsilon. \tag{2.9}$$

This implies that

$$d(x_{2n(k)}, x_{2m(k)-2}) < \varepsilon. \tag{2.10}$$

Then from (2.9), we have

$$\begin{aligned} \varepsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq sd(x_{2m(k)}, x_{2m(k)-2}) + sd(x_{2m(k)-2}, x_{2n(k)}) \\ &\leq sd(x_{2m(k)}, x_{2m(k)-2}) + s^2 d(x_{2m(k)-2}, x_{2n(k)-1}) \\ &\quad + s^2 d(x_{2n(k)-1}, x_{2n(k)}). \end{aligned} \tag{2.11}$$

Taking the upper limit as $k \to \infty$ in (2.11) and using (2.8), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-2}). \tag{2.12}$$

From (2.10), we have

$$\begin{aligned} d(x_{2n(k)-1}, x_{2m(k)-2}) &\leq sd(x_{2n(k)-1}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2m(k)-2}) \\ &< sd(x_{2n(k)-1}, x_{2n(k)}) + \varepsilon s. \end{aligned} \tag{2.13}$$

Taking the upper limit as $k \to \infty$ in (2.13) and using (2.8), we get

$$\limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-2}) \leq \varepsilon s. \tag{2.14}$$

Therefore, from (2.12) and (2.14), we obtain

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-2}) \leq \varepsilon s. \tag{2.15}$$

Similarly, we also have

$$\frac{\varepsilon}{s^2} \leq \liminf_{n\to\infty} d(x_{2n(k)-1}, x_{2m(k)-2}) \leq \varepsilon s. \tag{2.16}$$

Also, we have

$$d(x_{2n(k)-1}, x_{2m(k)-2}) \leq sd(x_{2n(k)-1}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2m(k)-2}). \tag{2.17}$$

Taking the upper limit as $k \to \infty$ in (2.17) and using (2.8), (2.10), (2.15), we get

$$\frac{\varepsilon}{s^3} \leq \limsup_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon. \tag{2.18}$$

Similarly, we also have

$$\frac{\varepsilon}{s^3} \leq \liminf_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon. \tag{2.19}$$

Again, we have

$$\begin{aligned} & d(x_{2m(k)-1}, x_{2n(k)-1}) \\ \leq\ & sd(x_{2m(k)-1}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2n(k)-1}) \\ \leq\ & s^2 d(x_{2m(k)-1}, x_{2m(k)-2}) + s^2 d(x_{2m(k)-2}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2n(k)-1}) \\ <\ & s^2 d(x_{2m(k)-1}, x_{2m(k)-2}) + s^2 \varepsilon + sd(x_{2n(k)}, x_{2n(k)-1}). \end{aligned} \tag{2.20}$$

Taking the upper limit as $k \to \infty$ in (2.20) and using (2.8), we get

$$\limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-1}) \leq \varepsilon s^2. \tag{2.21}$$

Also, we have

$$d(x_{2n(k)-1}, x_{2m(k)-2}) \leq sd(x_{2n(k)-1}, x_{2m(k)-1}) + sd(x_{2m(k)-1}, x_{2m(k)-2}). \tag{2.22}$$

Taking the upper limit as $k \to \infty$ in (2.22) and using (2.8), (2.12), we get

$$\frac{\varepsilon}{s^3} \leq \limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-1}). \tag{2.23}$$

Therefore, from (2.21) and (2.23), we have

$$\frac{\varepsilon}{s^3} \leq \limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-1}) \leq s^2\varepsilon. \tag{2.24}$$

Similarly, we also have

$$\frac{\varepsilon}{s^3} \leq \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-1}) \leq s^2\varepsilon. \tag{2.25}$$

Since $x_{2m(k)-2} \succeq x_{2n(k)-1}$, from (2.1), we have

$$\begin{aligned} & d(x_{2n(k)}, x_{2m(k)}) \\ \leq\ & sd(x_{2n(k)}, x_{2m(k)-1}) + sd(x_{2m(k)-1}, x_{2m(k)}) \\ =\ & sd(Tx_{2n(k)-1}, Sx_{2m(k)-2}) + sd(x_{2m(k)-1}, x_{2m(k)}) \\ \leq\ & sd(x_{2m(k)-1}, x_{2m(k)}) + s\psi\Big(\frac{2}{s(s^2+1)}\big[d(x_{2n(k)-1}, Sx_{2m(k)-2}) \\ & + d(x_{2m(k)-2}, Tx_{2n(k)-1}) - \varphi(d(x_{2n(k)-1}, Sx_{2m(k)-2}), d(x_{2m(k)-2}, Tx_{2n(k)-1}))\big]\Big) \\ \leq\ & sd(x_{2m(k)-1}, x_{2m(k)}) + s\psi\Big(\frac{2}{s(s^2+1)}\big[d(x_{2n(k)-1}, x_{2m(k)-1}) + d(x_{2m(k)-2}, x_{2n(k)}) \\ & - \varphi(d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2m(k)-2}, x_{2n(k)}))\big]\Big) \\ \leq\ & sd(x_{2m(k)-1}, x_{2m(k)}) + \frac{1}{s^2+1}\big[d(x_{2n(k)-1}, x_{2m(k)-1}) + d(x_{2m(k)-2}, x_{2n(k)}) \\ & - \varphi(d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2m(k)-2}, x_{2n(k)}))\big]. \end{aligned} \tag{2.26}$$

Taking the upper limit as $k \to \infty$ in (2.26) and using (2.8), (2.9), (2.18), (2.19), (2.24), (2.25) and the lower semi-continuous property of φ, we get

$$\begin{aligned} \varepsilon &\leq \frac{1}{s^2+1}\big[\varepsilon s^2 + \varepsilon - \liminf_{k\to\infty} \varphi(d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2m(k)-2}))\big] \\ &\leq \varepsilon - \frac{1}{s^2+1}\varphi\big[\liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-1}), \liminf_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-2})\big] \\ &< \varepsilon. \end{aligned}$$

It is a contradiction. Thus, $\{x_{2n}\}$ is a Cauchy sequence. By Lemma 1.12, $\{x_n\}$ is a Cauchy sequence in (X,d,s). Since (X,d,s) is complete, there exists $z\in X$ such that $\lim\limits_{n\to\infty} x_n = z$.

Suppose that T or S is continuous. If T is continuous, then

$$z=\lim_{n\to\infty} x_{2n+2}=\lim_{n\to\infty} Tx_{2n+1}=T(\lim_{n\to\infty} x_{2n+1})=Tz,$$

that is, z is a fixed point of T. By Lemma 2.3, z is a common fixed point of S and T. Similarly, if S is continuous, we also see that z is a common fixed point of S and T.

Suppose that $(X,d,s,\preceq)$ is a regular space. Then $x_{2n+1}\preceq z$ for all $n\geq 0$. From (2.1), we have

$$\begin{aligned}
&d(x_{2n+2},Sz)\\
=\;&d(Tx_{2n+1},Sz)\\
\leq\;&\psi\Big(\frac{2}{s(s^2+1)}\big[d(x_{2n+1},Sz)+d(z,Tx_{2n+1})-\varphi(d(x_{2n+1},Sz),d(z,Tx_{2n+1}))\big]\Big)\\
\leq\;&\psi\Big(\frac{2}{s(s^2+1)}\big[d(x_{2n+1},Sz)+d(z,x_{2n+2})-\varphi(d(x_{2n+1},Sz),d(z,x_{2n+2}))\big]\Big)\\
\leq\;&\frac{1}{s(s^2+1)}\big[d(x_{2n+1},Sz)+d(z,x_{2n+2})-\varphi(d(x_{2n+1},Sz),d(z,x_{2n+2}))\big]\\
\leq\;&\frac{1}{s(s^2+1)}\big[d(x_{2n+1},Sz)+d(z,x_{2n+2})\big].
\end{aligned}\tag{2.27}$$

Taking the upper limit as $n\to\infty$ in (2.27), using $\lim\limits_{n\to\infty} x_n=z$ and Lemma 1.11, we get

$$\frac{1}{s}d(z,Sz)\leq\frac{1}{s(s^2+1)}\big[sd(z,Sz)\big]=\frac{1}{s^2+1}d(z,Sz).$$

This implies that

$$d(z,Sz)\leq\frac{s}{s^2+1}d(z,Sz)\leq\frac{1}{2}d(z,Sz).$$

This implies that $d(z,Sz)=0$ and hence $Sz=z$, that is, z is a fixed point of S. Therefore, form Lemma 2.3, it follows that z is a common fixed point of T and S.

Now, suppose that the set of common fixed points of T and S is totally ordered. We claim that there is a unique common fixed point of T and S. If there exist $u,v\in X$ such that $Su=Tu=u$ and $Sv=Tv=v$, then, from (2.1), we have

$$\begin{aligned}
d(u,v)=d(Tu,Sv)\;&\leq\;\psi\Big(\frac{2}{s(s^2+1)}\big[d(u,Sv)+d(v,Tu)-\varphi(d(u,Sv),d(v,Tu))\big]\Big)\\
&=\;\psi\Big(\frac{2}{s(s^2+1)}\big[(2d(u,v)-\varphi(d(u,v),d(v,u))\big]\Big)\\
&\leq\;\frac{1}{s(s^2+1)}\big[2d(u,v)-\varphi(d(u,v),d(v,u))\big]\\
&=\;\frac{2}{s(s^2+1)}d(u,v)-\frac{1}{s(s^2+1)}\varphi(d(u,v),d(v,u))\\
&\leq\;d(u,v)-\frac{1}{s(s^2+1)}\varphi(d(u,v),d(v,u)).
\end{aligned}$$

This implies that $\varphi(d(u,v),d(v,u)) = 0$ and hence $d(u,v) = 0$. Therefore, $u = v$, that is, the common fixed point of T and S is unique. Conversely, if T and S have a unique common fixed point, then the set of common fixed pointsof T and S being a singleton is totally ordered. ∎

By using Remark 1.14, from Theorem 2.4, we get the following corollary.

Corollary 2.5. *Let $(X,d,s,\preceq)$ be a complete, partially ordered b-metric space and $T,S : X \longrightarrow X$ be two mappings such that*

(1) T is a weakly (ψ,S,C)-contraction.
(2) The pair (T,S) is weakly increasing.
(3) There exists x_0 such that $x_0 \preceq Sx_0$.
(4) S or T is continuous, or $(X,d,s,\preceq)$ is a regular space.

Then T and S have a common fixed point. Moreover, the set of fixed points of T,S is totally ordered if and only if T and S have a unique common fixed point.

By taking $\psi = \dfrac{x}{2}$ and $\varphi(x,y) = (1-\alpha)(x+y)$ for all $x,y \in [0,\infty)$ and for some $\alpha \in [0,1)$ in Corollary 2.5, we get the following corollary.

Corollary 2.6. *Let $(X,d,s,\preceq)$ be a complete, partially ordered b-metric space and $T,S : X \longrightarrow X$ be two mappings such that*

(1) There exists $\lambda \in [0, \frac{1}{s(s^2+1)})$ such that for all $x,y \in X$ with $x \preceq y$ or $x \succeq y$,

$$d(Tx,Sy) \leq \lambda(d(x,Sy) + d(y,Tx)).$$

(2) The pair (T,S) is weakly increasing.
(3) There exists x_0 such that $x_0 \preceq Sx_0$.
(4) T or S is continuous, or $(X,d,s,\preceq)$ is a regular space.

Then T and S have a common fixed point. Moreover, the set of fixed points of T,S is totally ordered if and only if T and S have a unique common fixed point.

In the next, we introduce the notion of a generalized weakly (ψ,S,C)-contraction in partially ordered b -metric spaces.

Denote by

(1) Ψ_2 the family of all increasing functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that for all $t > 0$, we have $\psi(t) \leq \dfrac{1}{4}t$. Notice that $\psi(0) = 0$.
(2) Φ_2 the family of all lower semi-continuous functions $\varphi : [0,\infty)^4 \longrightarrow [0,\infty)$ such that $\varphi(x,y,z,t) = 0$ if and only if $x = y = z = t = 0$, and $\varphi(x,y,z,t) \leq x+y+z+t$ for all $x,y,z,t \in [0,\infty)$.

Definition 2.7. Let $(X,d,s,\preceq)$ be a partially ordered b-metric space and $T,S : X \longrightarrow X$ be two mappings. Then T is called a *generalized weakly (ψ,S,C)-contractive* if there exist $\psi \in \Psi_2$ and $\varphi \in \Phi_2$ such that for all $x,y \in X$ with $x \preceq y$ or $y \preceq x$,

$$\begin{aligned} d(Tx,Sy) \leq \psi\Big(\frac{4}{s(s^2+1)}\big[&d(x,Tx) + d(y,Sy) + d(x,Sy) + d(y,Tx) \\ &-\varphi(d(x,Tx),d(y,Sy),d(x,Sy),d(y,Tx))\big]\Big). \end{aligned} \tag{2.28}$$

Notice that if $s = 1$, then the condition (2.28) becomes

$$\begin{aligned} d(Tx, Sy) \quad &\leq \quad \psi\Big(2\big[d(x,Tx) + d(y,Sy) + d(x,Sy) + d(y,Tx) \\ &\quad -\varphi(d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx))\big]\Big). \end{aligned} \tag{2.29}$$

The following example proves that the condition (2.29) is a proper generalization of the condition (1.1).

Example 2.8. Let $X = \big\{1, 2, 3, 4, 5\big\}$ with the usual order $\leq$. Define a metric d on X as follows.

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } (x,y) \in \{(1,2),(2,1),(1,3),(3,1)\} \\ 2 & \text{if } (x,y) \in \big\{(3,2),(2,3),(4,2),(2,4),(5,2),(2,5),(1,4),(4,1),(1,5),(5,1)\big\} \\ 3 & \text{otherwise.} \end{cases}$$

Let $T, S : X \longrightarrow X$ be defined by

$$T1 = T2 = T3 = T4 = T5 = 2 \text{ and } S1 = S2 = 2, S3 = S4 = S5 = 1.$$

Define two functions by $\varphi(x,y,z,t) = \dfrac{x+y+z+t}{2}$ and $\psi(x) = \dfrac{x}{4}$ for all $x, y, z, t \in [0, \infty)$. Then $\varphi \in \Phi_1$, $\psi \in \Psi_1$. Put

$$\begin{aligned} F \quad &= \quad \psi\Big(2[d(x,Tx) + d(y,Sy) + d(x,Sy) + d(y,Tx) \\ &\qquad -\varphi(d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx))]\Big) \\ &= \quad \frac{1}{2}\Big(d(x,Tx) + d(y,Sy) + d(x,Sy) + d(y,Tx) \\ &\qquad -\varphi(d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx))\Big) \\ &= \quad \frac{1}{4}[d(x,Tx) + d(y,Sy) + d(x,Sy) + d(y,Tx)]. \end{aligned}$$

Let $x, y \in X$ with $x \leq y$ or $y \geq x$, we have

$$d(Tx, Sy) = \begin{cases} 0 & \text{if } x, y \in \{1,2\} \text{ or } \ x \in \{3,4,5\}, y \in \{1,2\} \\ 1 & \text{if } x \in \{1,2\}, y \in \{3,4,5\} \text{ or } \ x, y \in \{3,4,5\} \end{cases}$$

and

$$F = \begin{cases} 0 & \text{if } x = y = 2 \\ \frac{1}{2} & \text{if } (x,y) \in \{(1,2),(2,1)\} \\ 1 & \text{if } x = 2, y = 3 \text{ or } \ x = 1, y \in \{1,3\} \text{ or } x \in \{3,4,5\}, y = 2 \\ \frac{5}{4} & \text{if } x \in \{1,2\}, y \in \{4,5\} \\ \frac{3}{2} & \text{if } x \in \{4,5\}, y = 1 \text{ or } \ x = 3, y \in \{1,3,4\} \\ \frac{7}{4} & \text{if } x = 3, y = 5 \text{ or } \ x \in \{4,5\}, y = 3 \\ 2 & \text{if } x, y \in \{4,5\}. \end{cases}$$

This implies that the condition (2.29) is satisfied. However, for all $\varphi \in \Omega_2$ and $\psi \in \Gamma_2$, by choosing $(x, y) = (2, 3)$, we have $d(T2, S3) = 1$ and

$$\begin{aligned}
& \psi\Big(d(2,T2) + d(3,S3) + d(2,S3) + d(3,T2) \\
& \qquad -\varphi(d(2,T2), d(3,S3), d(2,S3), d(3,T2))\Big) \\
\leq\ & \frac{1}{4}\Big(d(2,2) + d(3,1) + d(2,1) + d(3,2) - \varphi(d(2,2), d(3,1), d(2,1), d(3,2))\Big) \\
=\ & 1 - \frac{1}{4}\varphi(0,1,1,2) < 1.
\end{aligned}$$

This implies that the condition (1.1) is not satisfied.

The following example proves that the condition (2.28) is a proper generalization of the condition (2.1). Notice that we may not conclude that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ by using the similar argument as in the proof of Theorem 2.4. So, by adding the assumptions as in Theorem 2.4 to a generalized weakly (ψ, S, C)-contraction, we may not state the existence and uniqueness of the common fixed point.

Example 2.9. Let $X = \big\{1, 2, 3, 4, 5\big\}$ with the usual order $\leq$. Define a function d on X as follows.

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } (x,y) \in \big\{(1,2), (2,1), (2,3), (3,2)\big\} \\ 2 & \text{if } (x,y) \in \big\{(1,3), (3,1)\big\} \\ 22 & \text{if } (x,y) \in \big\{(1,5), (5,1), (3,5), (5,3)\big\} \\ 10 & \text{otherwise.} \end{cases}$$

Then (X, d, s) is a complete b-metric space with $s = 2$. Let $T, S : X \longrightarrow X$ be defined by $T1 = T2 = T3 = 2, T4 = T5 = 3$ and $S1 = S2 = S3 = 2, S4 = S5 = 1$. Define two functions by $\varphi(x,y,z,t) = \dfrac{x+y+z+t}{2}$ and $\psi(x) = \dfrac{x}{4}$ for all $x, y, z, t \in [0, \infty)$. Then $\varphi \in \Phi_2$, $\psi \in \Psi_2$. Put

$$\begin{aligned}
H &= \psi\Big(\frac{4}{s(s^2+1)}\big[d(x,Tx) + d(y,Sy) + d(x,Sy) + d(y,Tx) \\
&\qquad -\varphi(d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx))\big]\Big) \\
&= \psi\Big(\frac{2}{5}\big[d(x,Tx) + d(y,Sy) + d(x,Sy) + d(y,Tx) \\
&\qquad -\varphi(d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx))\big]\Big) \\
&= \frac{1}{20}[d(x,Tx) + d(y,Sy) + d(x,Sy) + d(y,Tx)].
\end{aligned}$$

Let $x, y \in X$ with $x \leq y$ or $y \geq x$, we have

$$d(Tx, Sy) = \begin{cases} 0 & \text{if } x, y \in \{1,2,3\} \\ 2 & \text{if } x, y \in \{4,5\} \\ 1 & \text{otherwise} \end{cases}$$

and

$$H = \begin{cases} 0 & \text{if } x = y = 2 \\ 2 & \text{if } x = y = 4 \\ \frac{22}{5} & \text{if } x = y = 5 \\ \frac{16}{5} & \text{if } (x,y) \in \{(4,5),(5,4)\} \\ \frac{1}{5} & \text{if } x, y \in \{1,3\} \\ \frac{1}{10} & \text{if } x = 2, y \in \{1,3\} \text{ or } \ x \in \{1,3\}, y = 2 \\ \frac{21}{20} & \text{if } (x,y) \in \{(1,4),(2,4),(4,2),(4,3)\} \\ \frac{23}{20} & \text{if } (x,y) \in \{(4,1),(3,4)\} \\ \frac{33}{20} & \text{if } (x,y) \in \{(1,5),(2,5),(5,2),(5,3)\} \\ \frac{35}{20} & \text{if } (x,y) \in \{(5,1),(3,5)\}. \end{cases}$$

This implies that the condition (2.28) is satisfied. However, for all $\varphi \in \Phi_1$ and $\psi \in \Psi_1$, by choosing $(x,y) = (1,4)$, we have $d(T1, S4) = 1$ and

$$\begin{aligned} & \psi\Big(\frac{1}{5}\big[d(1,S4) + d(4,T1) - \varphi(d(1,S4), d(4,T1))\big]\Big) \\ \leq \ & \frac{1}{10}\big[d(1,1) + d(4,2) - \varphi(d(1,1), d(4,2))\big] \\ = \ & \frac{1}{2} - \frac{1}{10}\varphi(0,10) < \frac{1}{2}. \end{aligned}$$

This implies that the condition (2.1) is not satisfied.

In what follows, we shall state the existence and uniqueness of the common fixed point for a generalized weakly (ψ, S, C)-contraction in b-metric spaces. Now, by using the similar argument as in the proof of Lemma 2.3, we get the following lemma.

Lemma 2.10. *Let $(X, d, s, \preceq)$ be a partially ordered b-metric space and $T, S : X \longrightarrow X$ be two mappings satisfying the condition (2.28). If z is a fixed point of T or S, then z is a common fixed point of T and S.*

The following result is a sufficient condition for the existence and uniqueness of the common fixed point for a generalized weakly (ψ, S, C)-contraction in b-metric spaces.

Theorem 2.11. *Let $(X, d, s, \preceq)$ be a partially ordered b-metric space and $T, S : X \longrightarrow X$ be two mappings such that*

(1) T is a generalized weakly (ψ, S, C)-contraction.
(2) S is T-weakly isotone increasing.
(3) There exists x_0 such that $x_0 \preceq Sx_0$.
(4) (T, S) is asymptotically continuous at $x_0 \in X$.
(5) X is (T, S)-orbitally complete at x_0.
(6) T or S is orbitally continuous at x_0, or $(X, d, s, \preceq)$ is a regular space.

Then T and S have a common fixed point. Moreover, the set of common fixed points of T, S is totally ordered if and only if T and S have a unique common fixed point.

Proof. Since (T, S) is asymptotically continuous at x_0 in X where x_0 defined by the assumption (3), there exists a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$$

for all $n \in \mathbb{N} \cup \{0\}$ and

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.30}$$

Since S is T-weakly isotone increasing, we also get

$$x_0 \preceq x_1 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 1.12, it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{2n(k)}\}, \{x_{2m(k)}\}$ of $\{x_{2n}\}$ where $m(k)$ is a smallest integer such that $m(k) > n(k) > k$ and

$$d(x_{2n(k)}, x_{2m(k)}) > \varepsilon. \tag{2.31}$$

This implies that

$$d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon. \tag{2.32}$$

Then from (2.31) and (2.32), by using the similar argument as in the proof of Theorem 2.4, we get

$$\frac{\varepsilon}{s^3} \leq \limsup_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon, \tag{2.33}$$

$$\frac{\varepsilon}{s^3} \leq \liminf_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon, \tag{2.34}$$

$$\frac{\varepsilon}{s^3} \leq \limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-1}) \leq s^2\varepsilon \tag{2.35}$$

and

$$\frac{\varepsilon}{s^3} \leq \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-1}) \leq s^2\varepsilon. \tag{2.36}$$

Since $x_{2m(k)-2} \succeq x_{2n(k)-1}$, from (2.28), we have

$$\begin{aligned}
& d(x_{2n(k)}, x_{2m(k)}) \\
\leq\ & sd(x_{2n(k)}, x_{2m(k)-1}) + sd(x_{2m(k)-1}, x_{2m(k)}) \\
=\ & sd(x_{2m(k)-1}, x_{2m(k)}) + sd(Tx_{2n(k)-1}, Sx_{2m(k)-2}) \\
\leq\ & sd(x_{2m(k)-1}, x_{2m(k)}) + s\psi\Big(\frac{4}{s(s^2+1)}\big[d(x_{2n(k)-1}, Tx_{2n(k)-1}) \\
& + d(x_{2m(k)-2}, Sx_{2m(k)-2}) + d(x_{2n(k)-1}, Sx_{2m(k)-2}) + d(x_{2m(k)-2}, Tx_{2n(k)-1}) \\
& - \varphi\big(d(x_{2n(k)-1}, Tx_{2n(k)-1}), d(x_{2m(k)-2}, Sx_{2m(k)-2}), d(x_{2n(k)-1}, Sx_{2m(k)-2}), \\
& \qquad d(x_{2m(k)-2}, Tx_{2n(k)-1})\big)\big]\Big) \\
\leq\ & sd(x_{2m(k)-1}, x_{2m(k)}) + s\psi\Big(\frac{4}{s(s^2+1)}\big[d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2m(k)-2}, x_{2m(k)-1}) \\
& + d(x_{2n(k)-1}, x_{2m(k)-1}) + d(x_{2m(k)-2}, x_{2n(k)}) - \varphi\big(d(x_{2n(k)-1}, x_{2n(k)}), \\
& d(x_{2m(k)-2}, x_{2m(k)-1}), d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2m(k)-2}, x_{2n(k)})\big)\big]\Big) \\
\leq\ & sd(x_{2m(k)-1}, x_{2m(k)}) + \frac{1}{s^2+1}\big[d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2m(k)-2}, x_{2m(k)-1}) \\
& + d(x_{2n(k)-1}, x_{2m(k)-1}) + d(x_{2m(k)-2}, x_{2n(k)}) - \varphi\big(d(x_{2n(k)-1}, x_{2n(k)}), \\
& d(x_{2m(k)-2}, x_{2m(k)-1}), d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2m(k)-2}, x_{2n(k)})\big)\big].
\end{aligned} \tag{2.37}$$

Taking the upper limit as $k \to \infty$ in (2.37) and using (2.30), (2.31), (2.33), (2.34) (2.35), (2.36) and the lower semi-continuous property of φ, we get

$$\begin{aligned}
\varepsilon &\leq \frac{1}{s^2+1}\Big[\varepsilon s^2 + \varepsilon - \liminf_{k\to\infty} \varphi\big(d(x_{2n(k)-1}, x_{2n(k)}),\\
&\qquad d(x_{2m(k)-2}, x_{2m(k)-1}), d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2m(k)-2}, x_{2n(k)})\big)\Big]\\
&\leq \varepsilon - \frac{1}{s^2+1}\varphi\big(\liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2n(k)}), \liminf_{k\to\infty} d(x_{2m(k)-2}, x_{2m(k)-1}),\\
&\qquad \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)-1}), \liminf_{k\to\infty} d(x_{2m(k)-2}, x_{2n(k)})\big) \qquad (2.38)\\
&< \varepsilon.
\end{aligned}$$

It is a contradiction. Thus, $\{x_{2n}\}$ is a Cauchy sequence. By Lemma 1.12, $\{x_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0; T, S)$. Since (X, d, s) is (T, S)-orbitally complete. Then there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Suppose that T or S is orbitally continuous at x_0. If T is orbitally continuous at x_0, then $z = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = T(\lim_{n\to\infty} x_{2n+1}) = Tz$, that is, z is a fixed point of T. By Lemma 2.10, z is a common fixed point of T and S. Similarly, if S is orbitally continuous at x_0, we also see that z is a common fixed point of T and S.

Suppose that $(X, d, s, \preceq)$ is a regular space. Then $x_{2n+1} \preceq z$ for all $n \in \mathbb{N} \cup \{0\}$. From (2.28), we have

$$\begin{aligned}
& d(x_{2n+2}, Sz)\\
=\ & d(Tx_{2n+1}, Sz)\\
\leq\ & \psi\Big(\frac{4}{s(s^2+1)}\Big[d(x_{2n+1}, Tx_{2n+1}) + d(z, Sz) + d(x_{2n+1}, Sz) + d(z, Tx_{2n+1})\\
& -\varphi\big(d(x_{2n+1}, Tx_{2n+1}), d(z, Sz), d(x_{2n+1}, Sz), d(z, Tx_{2n+1})\big)\Big]\Big)\\
\leq\ & \psi\Big(\frac{4}{s(s^2+1)}\Big[d(x_{2n+1}, x_{2n+2}) + d(z, Sz) + d(x_{2n+1}, Sz) + d(z, x_{2n+2})\\
& -\varphi\big(d(x_{2n+1}, x_{2n+2}), d(z, Sz), d(x_{2n+1}, Sz), d(z, x_{2n+2})\big)\Big]\Big)\\
\leq\ & \frac{1}{s(s^2+1)}\Big[d(x_{2n+1}, x_{2n+2}) + d(z, Sz) + d(x_{2n+1}, Sz) + d(z, x_{2n+2})\\
& -\varphi\big(d(x_{2n+1}, x_{2n+2}), d(z, Sz), d(x_{2n+1}, Sz), d(z, x_{2n+2})\big)\Big].
\end{aligned}$$

Suppose that $d(z, Sz) > 0$. Then by taking the limit as $n \to \infty$ in (2.39), using $\lim_{n\to\infty} x_n = z$, Lemma 1.11 and the lower semi-continuous property φ, we get

$$\begin{aligned}
\frac{1}{s}d(z, Sz) &\leq \frac{s+1}{s(s^2+1)}d(z, Sz) - \frac{1}{s(s^2+1)}\varphi\big(0, d(z, Sz), \liminf_{k\to\infty} d(x_{2n+1}, Sz), 0\big)\\
&< \frac{s+1}{s(s^2+1)}d(z, Sz).
\end{aligned}$$

This implies that

$$d(z, Sz) < \frac{s+1}{s^2+1}d(z, Sz) \leq d(z, Sz).$$

This implies that $d(z, Sz) = 0$ and hence $Sz = z$. Therefore, z is a fixed point of S. By Lemma 2.10, z is a common fixed point of T and S.

Now, suppose that the set of common fixed points of T and S is totally ordered. We claim that there is a unique common fixed point of T and S. If there exist u, v such that $Su = Tu = u$ and $Sv = Tv = v$, then from (2.28), we have

$$\begin{aligned} d(u,v) = d(Tu,Sv) &\leq \psi\Big(\frac{4}{s(s^2+1)}\big[(d(u,Tu)+d(v,Sv)+d(u,Sv)+d(v,Tu) \\ &\quad -\varphi(d(u,Tu),d(v,Sv),d(u,Sv),d(v,Tu))\big]\Big) \\ &= \psi\Big(\frac{4}{s(s+1)}\big[2d(u,v)-\varphi(0,0,d(u,v),d(v,u))\big]\Big) \\ &\leq \frac{2}{s(s^2+1)}d(u,v) - \frac{1}{s(s^2+1)}\varphi(0,0,d(u,v),d(v,u)) \\ &\leq d(u,v) - \frac{1}{s(s^2+1)}\varphi(0,0,d(u,v),d(v,u)). \end{aligned}$$

This implies that $\varphi(0,0,d(u,v),d(v,u)) = 0$ and hence $d(u,v) = 0$. Therefore, $u = v$, that is, the common fixed point of T and S is unique. Conversely, if T and S have a unique common fixed point, then the set of common fixed points of T and S being a singleton is totally ordered. ∎

By taking $T = S$ in Theorem 2.11, we get the following corollary.

Corollary 2.12. *Let $(X, d, s, \preceq)$ be a partially ordered b-metric space and $T : X \longrightarrow X$ be a mapping such that*

(1) There exist $\psi \in \Psi_2$ and $\varphi \in \Phi_2$ such that for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$,

$$\begin{aligned} d(Tx,Ty) &\leq \psi\Big(\frac{4}{s(s^2+1)}\big[d(x,Tx)+d(y,Ty)+d(x,Ty)+d(y,Tx) \\ &\quad -\varphi\big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big)\big]\Big). \end{aligned}$$

(2) $x \preceq Tx$ for all $x \in X$.
(3) T is asymptotically regular at $x_0 \in X$.
(4) X is T-orbitally complete at $x_0 \in X$.
(5) T is orbitally continuous, or $(X, d, s, \preceq)$ is a regular space.

Then T has a fixed point. Moreover, the set of fixed points of T is totally ordered if and only if T has a unique fixed point.

Remark 2.13. Since every metric space (X, d) is a b-metric space $(X, d, 1)$, our results are generalizations of the main results in [7].

Finally, we give some examples to support our results. The following example is an illustration of Theorem 2.4.

Example 2.14. Let $X = \{1, 2, 3, 4, 5\}$ with the usual order $\leq$. Define a function d on X as follows.

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } (x,y) \in \{(1,2),(2,1)\} \\ 20 & \text{if } (x,y) \in \{(3,2),(2,3),(4,2),(2,4),(5,2),(2,5)\} \\ 9 & \text{otherwise.} \end{cases}$$

Then (X,d,s) is a complete b-metric space with $s=2$. Let $T,S:X\longrightarrow X$ be defined by $T1=T2=T3=T4=T5=2$ and $S1=S2=2, S3=S4=S5=1$. Define two functions by $\varphi(x,y)=\dfrac{x+y}{2}$ and $\psi(x)=\dfrac{x}{2}$ for all $x,y\in[0,\infty)$. Then $\varphi\in\Phi_1$, $\psi\in\Psi_1$. Put

$$\begin{aligned} P &= \psi\Big(\frac{2}{s(s^2+1)}\big[d(x,Sy)+d(y,Tx)-\varphi(d(x,Sy),d(y,Tx))\big]\Big)\\ &= \psi\Big(\frac{1}{5}\big[d(x,Sy)+d(y,Tx)-\varphi(d(x,Sy),d(y,Tx))\big]\Big)\\ &= \frac{1}{20}\big[d(x,Sy)+d(y,Tx)\big]. \end{aligned}$$

Let $x,y\in X$ with $x\le y$ or $y\ge x$, we have

$$d(Tx,Sy)=\begin{cases} 0 & \text{if } x,y\in\{1,2\} \text{ or } \ x\in\{3,4,5\}, y\in\{1,2\}\\ 1 & \text{if } x\in\{1,2\}, y\in\{3,4,5\} \text{ or } \ x,y\in\{3,4,5\}\end{cases}$$

and

$$P=\begin{cases} 0 & \text{if } x=y=2\\ 1 & \text{if } x=1, y\in\{3,4,5\} \text{ or } x\in\{3,4,5\}, y=2\\ \frac{1}{10} & \text{if } x=y=1\\ \frac{1}{20} & \text{if } (x,y)\in\{(1,2),(2,1)\}\\ \frac{21}{20} & \text{if } x=2, y\in\{3,4,5\} \text{ or } x\in\{3,4,5\}, y=1\\ \frac{29}{20} & \text{if } x,y\in\{3,4,5\}.\end{cases}$$

This implies that the condition (2.1) holds and hence the assumption (1) of Theorem 2.4 is satisfied. Moreover, other assumptions of Theorem 2.4 are fulfilled. Therefore, Theorem 2.4 is applicable to T,S, φ,ψ and $(X,d,s,\le)$.

However, since $20=d(2,3)\ge d(2,1)+d(1,3)=10$, d is not a metric on X. Thus, [7, Theorem 3.2] is not applicable to (X,d,s).

The following example is an illustration of Theorem 2.11.

Example 2.15. Let (X,d,s) be a b-metric space, T and S be two mappings, and ψ,φ be two functions as in Example 2.9. It is easy to see that all assumptions of Theorem 2.11 are satisfied. Therefore, Theorem 2.11 is applicable to T, S, φ,ψ and $(X,d,s,\le)$. However, it is easy to see that d is not a metric on X and hence [7, Theorem 4.2] is not applicable to (X,d,s).

Finally, we apply Corollary 2.6 to study the existence of solutions to the system of nonlinear integral equations.

Example 2.16. Let $C[a,b]$ be the set of all continuous function on $[a,b]$, the b-metric d with $s=2^{p-1}$ be defined by

$$d(u,v)=\sup_{t\in[a,b]}|u(t)-v(t)|^p$$

for all $u,v\in C[a,b]$ and some $p>1$, and the partial order $\preceq$ be given by $u\preceq v$ if $u(t)\le v(t)$ for all $t\in[a,b]$. Consider the system of nonlinear integral equations as follows.

$$\begin{cases} u(t)=\displaystyle\int_a^b K_1(t,s,u(s))ds+g(t)\\ u(t)=\displaystyle\int_a^b K_2(t,s,u(s))ds+g(t)\end{cases}\tag{2.39}$$

where $t \in [a,b], g : [a,b] \longrightarrow \mathbb{R}, K_1, K_2 : [a,b] \times [a,b] \times u[a,b] \longrightarrow \mathbb{R}$ for each $u \in C[a,b]$. Suppose that the following statements hold.

(1) $K_1(t,s,u(s))$ and $K_2(t,s,u(s))$ are integrable with respect to s on $[a,b]$.
(2) $Tu, Su \in C[a,b]$ for all $u \in C[a,b]$, where

$$Tu(t) = \int_a^b K_1(t,s,u(s))ds + g(t),$$

$$Su(t) = \int_a^b K_2(t,s,u(s))ds + g(t)$$

for all $t \in [a,b]$.
(3) For all $t, s \in [a,b], u \in C[a,b]$,

$$K_1(t,s,u(t)) \le K_2\Big(t, s, \int_a^b K_1(s,z,u(z))dz + g(s)\Big),$$

$$K_2(t,s,u(t)) \le K_1\Big(t, s, \int_a^b K_2(s,z,u(z))dz + g(s)\Big).$$

(4) For all $s, t \in [a,b]$ and $u, v \in C[a,b]$ with $u \preceq v$ or $v \preceq u$,

$$|K_1(t,s,u(s)) - K_2(t,s,v(s))|^p \le \alpha(t,s)(|u(s) - Sv(s)|^p + |v(s) - Tu(s)|^p)$$

where $\alpha : [a,b] \times [a,b] \longrightarrow [0,\infty)$ is a continuous function satisfying

$$\sup_{t\in[a,b]} \Big(\int_a^b \alpha(t,s)ds\Big) \le \frac{1}{2^{p-1}(2^{2p-2}+1)(b-a)^{p-1}}.$$

(5) There exists $u_0 \in C[a,b]$ such that $u_0(t) \le \int_a^b K_2(t,s,u_0(s))ds + g(t)$ for all $t \in [a,b]$.

Then the system of nonlinear integral equations (2.39) has a solution $u \in C[a,b]$.

Proof. Consider $T, S : C[a,b] \longrightarrow C[a,b]$ defined by $Tu(t) = \int_a^b K_1(t,s,u(s))ds + g(t)$ and $Su(t) = \int_a^b K_2(t,s,u(s))ds + g(t)$ for all $u \in C[a,b]$ and $t \in [a,b]$. It follows from the assumptions (1) and (2) that T and S are well-defined. Notice that the existence of a solution to (2.39) is equivalent to the existence of the common fixed point of T and S. Now, we prove that all assumptions of Corollary 2.6 are satisfied.

(1). Let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. For all $u, v \in C[a, b]$ with $v \preceq u$ or $u \preceq v$, from the assumption (4), we have

$$\begin{aligned}
|Tu(t) - Sv(t)|^p &\leq \Big(\int_a^b |K_1(t, s, u(s)) - K_2(t, s, v(s))|ds\Big)^p \\
&\leq \Big[\Big(\int_a^b ds\Big)^{\frac{1}{q}}\Big(\int_a^b |K_1(t, s, u(s)) - K_2(t, s, v(s))|^p ds\Big)^{\frac{1}{p}}\Big]^p \\
&\leq (b-a)^{\frac{p}{q}}\Big(\int_a^b \alpha(t, s)(|u(s) - Sv(s)|^p + |v(s) - Tu(s)|^p)ds\Big) \\
&\leq (b-a)^{p-1}\Big(\int_a^b \alpha(t, s)(d(u, Sv) + d(v, Tu))ds\Big) \\
&\leq (b-a)^{p-1}(d(u, Sv) + d(v, Tu))\Big(\int_a^b \alpha(t, s)ds\Big) \\
&\leq \lambda(d(u, Sv) + d(v, Tu)) \\
&= d(u, Sv) + d(v, Tu) - (1-\lambda)(d(u, Sv) + d(v, Tu)).
\end{aligned}$$

where $\lambda = (b-a)^{p-1} \sup_{t\in[a,b]} \Big(\int_a^b \alpha(t, s)ds\Big)$. This implies that $0 \leq \lambda < \dfrac{1}{2^{p-1}(2^{2p-2}+1)}$ and

$$d(Tx, Sy) \leq (d(u, Sv) + d(v, Tu)) - (1-\lambda)(d(u, Sv) + d(v, Tu)).$$

Therefore, the assumption (1) in Corollary 2.6 holds with $\lambda = (b-a)^{p-1} \sup_{t\in[a,b]} \Big(\int_a^b \alpha(t, s)ds\Big)$.

(2) For all $u \in C[a, b]$ and all $t \in [a, b]$, from the assumption (3), we have

$$\begin{aligned}
Tu(t) &= \int_a^b K_1(t, s, u(s))ds \\
&\leq \int_a^b K_2\Big(t, s, \int_a^b K_1(s, z, u(z))dz + g(s)\Big)ds \\
&\leq \int_a^b K_2(t, s, Tu(s))ds \\
&= STu(t)
\end{aligned}$$

and

$$\begin{aligned}
Su(t) &= \int_a^b K_2(t, s, u(s))ds \\
&\leq \int_a^b K_1\Big(t, s, \int_a^b K_1(s, z, u(z))dz + g(s)\Big)ds \\
&\leq \int_a^b K_1(t, s, Su(s))ds \\
&= TSu(t).
\end{aligned}$$

This implies that $Tu \preceq STu$ and $Su \preceq TSu$ for all $u \in C[a, b]$. Therefore, the pair (T, S) is weakly increasing.

(3). From the assumption (5), there exits $x_0 \in C[a, b]$ such that $x_0 \preceq Sx_0$.

(4). By using the similar argument as in the proof of [27, Theorem 3.1], we also see that the space $(X, d, s, \preceq)$ is regular.

By the above, all assumptions of Corollary 2.6 are satisfied. Then T and S have a common fixed point $u \in C[a,b]$ and the system of integral equations (2.39) has a solution $u \in C[a,b]$. ∎

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