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On the Generalized HUR-Stability of Some Functional Equations

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Abstract In this paper, we prove the stability of some quadratic and cubic functional equations in random and non-Archimedean normed spaces.

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1. Introduction

In 1940, the stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. In 1941, Hyers [2] gave first an affirmative partial answer for the question of Ulam for Banach spaces. Since then, In 1978, Hyers's theorem was generalized by Th. M. Rassias [3] for linear mappings by considering the unbounded Cauchy difference as follows:

Theorem 1.1. *Let f be an approximately additive mapping from a normed vector space E into a Banach space E', i.e., f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^r + \|y\|^r)$$

for all $x, y \in E$, where ϵ and r are constants with $\epsilon > 0$ and $0 \leq r < 1$. Then the mapping $L : E \to E'$ defined by $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is the unique additive mapping which satisfies

$$\|f(x+y) - L(x)\| \leq \frac{2\epsilon}{2-2^r}\|x\|^r$$

for all $x \in E$.

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The paper of Th. M. Rassias [3] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. In 1994, a generalization of the Th. M. Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias's approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. In 1983, the generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In 1992, Czerwik [7] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8], [9], [10]–[21]).

Recently, in 2009, Gordji and Khodaei [10] introduced the quadratic functional equation

$$\begin{aligned} f(mx+ny)+f(mx-ny) \quad &= \quad \frac{n(m+n)}{2}\big(f(x+y)+f(x-y)\big) \\ &\quad +2(m^2-mn-n^2)f(x)+(n^2-mn)f(y) \end{aligned} \tag{1.2}$$

and they established the general solution of the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.2) in Banach spaces as follows:

Theorem 1.2. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation* (1.2) *if and only if $f : X \to Y$ satisfies the functional equation* (1.1).

The cubic function $f(x) = cx^3$ satisfies the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.3}$$

The equation (1.3) was solved by Jun and Kim [12]. By the similar method for a quadratic functional equation, they also proved that a function $f : X \to Y$ is a solution of the equation (1.3) if and only if there exists a function $F : X^3 \to Y$ such that $f(x) = F(x, x, x)$ for all $x \in X$ and F is symmetric for each fixed one variable and is additive for fixed two variables. Every solution of the equation (1.3) is called a *cubic function*. Also, the equation (1.3) is equivalent to the following equation:

$$f(x+2y)+f(x-2y)+f(2x)=4f(x+y)+4f(x-y)+2f(x). \tag{1.4}$$

Koh [14] introduced the following functional equation:

$$4f(x+my)+4f(x-my)+m^2f(2x)=4m^2\big(f(x+y)+f(x-y)\big)+8f(x) \tag{1.5}$$

and established the general solution for the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.5) in Banach spaces as follows:

Theorem 1.3. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation* (1.5) *if and only if f is cubic.*

It is easy to see that the function $f(x) = cx^3$ is a solution of the functional equations (1.3), (1.4) and (1.5). Thus it is natural that the functional equations (1.3), (1.4) and (1.5) are called the *cubic functional equation* and every solution of these cubic functional equations is called a *cubic function*.

Najati and Rahimi [18] introduced

$$f(rx+sy) = \frac{r+s}{2} f(x+y) + \frac{r-s}{2} f(x-y) \tag{1.6}$$

for any $r, s \in \mathbb{R}$ with $r \neq \pm s$ and investigate the Hyers-Ulam- Rassias stability of the functional equation (1.6) in Banach modules over a unital C^*-algebra.

In this paper, we prove stability of the functional equations (1.2), (1.5) and (1.6) in random and non-Archimedean normed spaces.

2. Preliminaries

In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [5].

Throughout this paper, let $\triangle^+$ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set

$$D^+ = \{F \in \triangle^+ : l^- F(-\infty) = 1\},$$

where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of $\triangle^+$. The set $\triangle^+$ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0, & \text{if} \quad t \leq a, \\ 1, & \text{if} \quad t > a. \end{cases}$$

We can easily show that the maximal element in $\triangle^+$ is the distribution function $H_0(t)$.

Definition 2.1. A function $T : [0,1]^2 \to [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(x,1) = x$ for all $x \in [0,1]$;
(d) $T(x,y) \leq T(z,w)$ whenever $x \leq z$ and $y \leq w$ for all $x, y, z, w \in [0,1]$.

Three typical examples of continuous t-norms are as follows:

$$T(x,y) = xy, \quad T(x,y) = \max\{a+b-1, 0\}, \quad T(x,y) = \min(a,b).$$

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1 = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. $T_{i=n}^\infty x_i$ is defined by $T_{i=1}^\infty x_{n+i}$.

Definition 2.2. A *random normed space* (briefly, RN-space) is a triple (X, Ψ, T), where X is a vector space, T is a continuous t-norm and $\Psi : X \to D^+$ is a mapping such that the following conditions hold:

(a) $\Psi_x(t) = H_0(t)$ for all $t > 0$ if and only if $x = 0$;
(b) $\Psi_{\alpha x}(t) = \Psi_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
(c) $\Psi_{x+y}(t+s) \geq T(\Psi_x(t), \Psi_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, Ψ, T_M), where

$$\Psi_u(t) = \frac{t}{t + \|u\|}$$

for all $t > 0$ and T_M is the minimum t-norm. This space X is called the *induced random normed space.*

If the t-norm T is such that $\sup_{0<a<1} T(a, a) = 1$, then every RN-space (X, Ψ, T) is a metrizable linear topological space with the topology τ (called the Ψ*-topology* or the (ϵ, δ)*-topology*, where $\epsilon > 0$ and $\lambda \in (0, 1)$) induced by the base $\{U(\epsilon, \lambda)\}$ of neighborhoods of θ, where

$$U(\epsilon, \lambda) = \{x \in X : \Psi_x(\epsilon) > 1 - \lambda\}.$$

Definition 2.3. Let (X, Ψ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \Psi_{x_n - x}(t) = 1$ for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n\to\infty} \Psi_{x_n - x_m}(t) = 1$ for all $t > 0$.

(3) The RN-space (X, Ψ, T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 2.1. ([26]) *If (X, Ψ, T) is RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty} \Psi_{x_n}(t) = \Psi_x(t)$.

Definition 2.4. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 2.2. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{2.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all $n_0 \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

Definition 2.5. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:

(a) $|r| = 0$ if and only if $r = 0$;

(b) $|rs| = |r||s|$;

(c) $|r + s| \leq \max\{|r|, |s|\}$.

Clearly, by (b), $|1| = |-1| = 1$ and so, by induction, it follows from (c) that $|n| \leq 1$ for all $n \geq 1$.

Definition 2.6. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$.

(1) A function $\|\cdot\|: X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:
(a) $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$;
(b) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;
(c) the strong triangle inequality (ultra-metric) holds, that is,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$.
(2) The space $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Note that

$$||x_n - x_m|| \leq max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\}$$

for all $m, n \in \mathbb{N}$ with $n > m$.

Definition 2.7. Let $(X, \|\cdot\|)$ be a non-Archimedean normed space.
(1) A sequence $\{x_n\}$ is a *Cauchy sequence* in X if $\{x_{n+1} - x_n\}$ converges to zero in X.
(2) The non-Archimedean normed space $(X, \|\cdot\|)$ is said to be *complete* if every Cauchy sequence in X is convergent.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists a positive integer n such that $x < ny$.

Example 2.1. Fix a prime number p. For any nonzero rational number x, there exists a unique positive integer n_x such that $x = \frac{a}{b}p^{n_x}$, where a and b are positive integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field.* In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is a locally compact filed.

3. Random Stability of the Functional Equation (1.2): Direct Method

Let

$$M_f(x, y) = f(mx + ny) + f(mx - ny) - \frac{n(m+n)}{2} f(x + y) - \frac{n(m+n)}{2} f(x - y) - 2(m^2 - mn - n^2) f(x) - (n^2 - mn) f(y).$$

where $m, n \in \mathbb{Z}$ with $n \neq \pm m, -3m$.

Theorem 3.1. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and $\psi : X^2 \to Z$ be a function such that there exists $0 < \alpha < m^2$ such that*

$$\Psi_{\psi(mx,0)}(t) \geq \Psi_{\alpha\psi(x,0)}(t) \tag{3.1}$$

for all $x \in X$ and $t > 0$, $f(0) = 0$ and $\lim_{n\to\infty} \Psi_{\psi(m^n x, m^n y)}(m^{2n}t) = 1$ for all $x, y \in X$ and $t > 0$. Let (Y, μ, min) be a complete RN-space. If $f : X \to Y$ is a mapping such that

$$\mu_{M_f(x,y)}(t) \geq \Psi_{\psi(x,y)}(t) \tag{3.2}$$

for all $x, y \in X$ and $t > 0$, then there is a unique quadratic mapping $C : X \to Y$ such that $C(x) = \lim_{n\to\infty} m^{-2n} f(m^n x)$ and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\psi(x,0)}\big((m^2-\alpha)t\big) \tag{3.3}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (3.2), we see that

$$\mu_{\frac{f(mx)}{m^2}-f(x)}(t) \geq \Psi_{\psi(x,0)}(m^2 t) \tag{3.4}$$

for all $x \in X$. Replacing x by $m^n x$ in (3.4) and using (3.1), we obtain

$$\mu_{\frac{f(m^{n+1}x)}{m^{2(n+1)}}-\frac{f(m^n x)}{m^{2n}}}(t) \geq \Psi_{\psi(m^n x,0)}(m^{2(n+1)}t) \geq \Psi_{\psi(x,0)}\Big(\frac{m^{2(n+1)}t}{\alpha^n}\Big) \tag{3.5}$$

and so

$$\begin{aligned} \mu_{\frac{f(m^n x)}{m^{2n}}-f(x)}\Big(\sum_{k=0}^{n-1}\frac{t\alpha^k}{m^{2(k+1)}}\Big) &= \mu_{\sum_{k=0}^{n-1}\frac{f(m^{k+1}x)}{m^{2(k+1)}}-\frac{f(m^k x)}{m^{2k}}}\Big(\sum_{k=0}^{n-1}\frac{t\alpha^k}{m^{2(k+1)}}\Big) \\ &\geq T_{k=0}^{n-1}\mu_{\frac{f(m^{k+1}x)}{m^{2(k+1)}}-\frac{f(m^k x)}{m^{2k}}}\Big(\frac{t\alpha^k}{m^{2(k+1)}}\Big) \\ &\geq T_{k=0}^{n-1}\Big(\Psi_{\psi(x,0)}(t)\Big) \\ &= \Psi_{\psi(x,0)}(t). \end{aligned} \tag{3.6}$$

This implies that

$$\mu_{\frac{f(m^n x)}{m^{2n}}-f(x)}(t) \geq \Psi_{\psi(x,0)}\Bigg(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{m^{2(k+1)}}}\Bigg). \tag{3.7}$$

Replacing x by $m^p x$ in (3.7), we obtain

$$\begin{aligned} \mu_{\frac{f(m^{n+p}x)}{m^{2(n+p)}}-\frac{f(m^p x)}{m^{2p}}}(t) &\geq \Psi_{\psi(m^p x,0)}\Bigg(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{m^{2(k+p+1)}}}\Bigg) \\ &\geq \Psi_{\psi(x,0)}\Bigg(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^{k+p}}{m^{2(k+p+1)}}}\Bigg) \\ &= \Psi_{\psi(x,0)}\Bigg(\frac{t}{\sum_{k=p}^{n+p-1}\frac{\alpha^k}{m^{2(k+1)}}}\Bigg). \end{aligned} \tag{3.8}$$

Since

$$\lim_{p,n\to\infty}\Psi_{\psi(x,0)}\Bigg(\frac{t}{\sum_{k=p}^{n+p-1}\frac{\alpha^k}{m^{2(k+1)}}}\Bigg) = 1,$$

it follows that $\left\{\frac{f(m^n x)}{m^{2n}}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete RN-space (Y, μ, min) and so there exists a point $C(x) \in Y$ such that

$$\lim_{n\to\infty} m^{-2n} f(m^n x) = C(x).$$

Fix $x \in X$ and put $p = 0$ in (3.8). Then we obtain

$$\mu_{\frac{f(m^n x)}{m^{2n}}-f(x)}(t) \geq \Psi_{\psi(x,0)}\Bigg(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{m^{2(k+1)}}}\Bigg), \tag{3.9}$$

and so, for any $\epsilon > 0$,

$$\begin{aligned} \mu_{C(x)-f(x)}(t+\epsilon) &\geq T\Big(\mu_{C(x)-\frac{f(m^n x)}{m^{2n}}}(\epsilon), \mu_{\frac{f(m^n x)}{m^{2n}}-f(x)}(t)\Big) \\ &\geq T\Big(\mu_{C(x)-\frac{f(m^n x)}{m^{2n}}}(\epsilon), \Psi_{\psi(x,0)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{m^{2(k+1)}}}\Big)\Big). \end{aligned} \tag{3.10}$$

Taking the limit as $n \to \infty$ in (3.10), we get

$$\mu_{C(x)-f(x)}(t+\epsilon) \geq \Psi_{\psi(x,0)}\big((m^2-\alpha)t\big). \tag{3.11}$$

Since ϵ is arbitrary, by taking $\epsilon \to 0$ in (3.11), we get

$$\mu_{C(x)-f(x)}(t) \geq \Psi_{\psi(x,0)}\big((m^2-\alpha)t\big). \tag{3.12}$$

Replacing x and y by $m^n x$ and $m^n y$ in (3.2), respectively, we get

$$\mu_{\frac{M_f(m^n x, m^n y)}{m^{2n}}}(t) \geq \Psi_{\psi(m^n x, m^n y)}(m^{2n}t) \tag{3.13}$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n\to\infty} \Psi_{\psi(m^n x, m^n y)}(m^{2n}t) = 1$, we conclude that

$$\begin{aligned} C(mx+ny) + C(mx-ny) &= \frac{n(m+n)}{2}\Big\{C(x+y)+C(x-y)\Big\} \\ &\quad + 2(m^2-mn-n^2)C(x) + (n^2-mn)C(y). \end{aligned}$$

To prove the uniqueness of the quadratic mapping C, assume that there exist another quadratic mapping $D : X \to Y$ which satisfies (3.3). By induction, one can easily show that

$$C(m^n x) = m^{2n}C(x), \quad D(m^n x) = m^{2n}D(x)$$

for all $n \in N$ and $x \in X$ and so

$$\begin{aligned} \mu_{C(x)-D(x)}(t) &= \lim_{n\to\infty} \mu_{\frac{C(m^n x)}{m^{2n}}-\frac{D(m^n x)}{m^{2n}}}(t) \\ &\geq \lim_{n\to\infty} min\Big\{\mu_{\frac{C(m^n x)}{m^{2n}}-\frac{f(m^n x)}{m^{2n}}}\Big(\frac{t}{2}\Big), \mu_{\frac{D(m^n x)}{m^{2n}}-\frac{f(m^n x)}{m^{2n}}}\Big(\frac{t}{2}\Big)\Big\} \\ &\geq \lim_{n\to\infty} \Psi_{\psi(m^n x,0)}\Big(\frac{m^{2n}(m^2-\alpha)}{2}\Big) \\ &\geq \lim_{n\to\infty} \Psi_{\psi(x,0)}\Big(\frac{m^{2n}(m^2-\alpha)t}{2\alpha^n}\Big). \end{aligned} \tag{3.14}$$

Since $\lim_{n\to\infty} \frac{m^{2n}(m^2-\alpha)t}{2\alpha^n} = \infty$, we get

$$\lim_{n\to\infty} \Psi_{\psi(x,0)}\Big(\frac{m^{2n}(m^2-\alpha)t}{2\alpha^n}\Big) = 1.$$

Therefore, it follows that $\mu_{C(x)-D(x)}(t) = 1$ for all $t > 0$ and so $C(x) = D(x)$. This complete the proof. ∎

Corollary 3.2. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) be a complete RN-space. Let $0 < p < 1$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping that*

$$\mu_{M_f(x,y)}(t) \geq \Psi_{||y||^p y_0}(t) \tag{3.15}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $C : X \to Y$ such that

$$C(x) = \lim_{n\to\infty} m^{-2n} f(m^n x) \tag{3.16}$$

and

$$\mu_{f(x)-C(x)}(t) \geq 1 \tag{3.17}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = m^{2p}$ and $\psi : X^2 \to Z$ be a mapping defined by $\psi(x, y) = \|y\|^p z_0$. Then, from Theorem 3.1, the conclusion follows. ■

Corollary 3.3. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) be a complete RN-space. Let $0 < p < 1$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping that*

$$\mu_{M_f(x,y)}(t) \geq \Psi_{(\|x\|^p+\|y\|^p)z_0}(t) \tag{3.18}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $C : X \to Y$ such that

$$C(x) = \lim_{n\to\infty} m^{-2n} f(m^n x) \tag{3.19}$$

and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\|x\|^p y_0}\big((m^2 - m^{2p})t\big) \tag{3.20}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = m^{2p}$ and $\psi : X^2 \to Z$ be a mapping defined by $\psi(x, y) = (\|x\|^p + \|y\|^p)z_0$. Then, from Theorem 3.1, the conclusion follows. ■

Corollary 3.4. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) a complete RN-space. Let $p, q \in \mathbb{R}^+$ with $0 < p + q < 1$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping that*

$$\mu_{M_f(x,y)}(t) \geq \Psi_{(\|x\|^{p+q}+\|y\|^{p+q}+\|x\|^p\|y\|^q)z_0}(t) \tag{3.21}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $C : X \to Y$ such that

$$C(x) = \lim_{n\to\infty} m^{-2n} f(m^n x) \tag{3.22}$$

and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\|x\|^{p+q}z_0}\big((m^2 - m^{2(p+q)})t\big) \tag{3.23}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = m^{2(p+q)}$ and $\psi : X^2 \to Z$ be a mapping defined by

$$\psi(x, y) = (\|x\|^{p+q} + \|y\|^{p+q} + \|x\|^p\|y\|^q)z_0.$$

Then, from Theorem 3.1, the conclusion follows. ■

Corollary 3.5. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) be a complete RN-space. Let $z_0 \in Z$. If $f : X \to Y$ is a mapping that*

$$\mu_{M_f(x,y)}(t) \geq \Psi_{\delta z_0}(t) \tag{3.24}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $C : X \to Y$ such that

$$C(x) = \lim_{n\to\infty} m^{-2n} f(m^n x) \tag{3.25}$$

and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\delta z_0}\big((m^2-1)t\big) \tag{3.26}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 1$ and $\psi : X^2 \to Z$ be a mapping defined by $\psi(x,y) = \delta z_0$. Then, from Theorem 3.1, the conclusion follows. ∎

Example 3.1. Let $p = 1$, $\alpha = m^4$, $\psi(x,y) = (\|x\|^2 + \|y\|^2)z_0$ and $f : X \to Y$ be a mapping satisfying

$$\mu_{M_f(x,y)}(t) \geq \Psi_{(\|x\|^2+\|y\|^2)z_0}(t). \tag{3.27}$$

As in Theorem (3.1), we obtain

$$\mu_{\frac{f(m^{n+q}x)}{m^{2(n+q)}}-\frac{f(m^q x)}{m^{2q}}}(t) \geq \Psi_{\|x\|^2 z_0}\left(\frac{t}{\sum_{k=q}^{n+q-1}\frac{m^{4k}}{m^{2k+2}}}\right).$$

Since

$$\sum_{k=q}^{n+q-1}\frac{m^{4k}}{m^{2k+2}} = \frac{1}{m^2}\sum_{k=q}^{n+q-1} m^{2k} = \frac{m^{2(n+q-1)}-m^{2q}}{1-m^2}$$

and

$$\lim_{n,q\to\infty}\frac{1-m^2}{m^{2(n+q-1)}-m^{2q}} \neq \infty,$$

we have

$$\lim_{n,q\to\infty}\mu_{\frac{f(m^{n+q}x)}{m^{3(n+q)}}-\frac{f(m^q x)}{m^{3q}}}(t) = \lim_{n,q\to\infty}\Psi_{\|x\|^2 z_0}\left(\frac{1-m^2}{m^{2(n+q-1)}-m^{2q}}\right) \neq 1.$$

This means that the sequence $\left\{\frac{f(m^n x)}{m^{2n}}\right\}_{n=1}^{\infty}$ is not a Cauchy sequence.

4. Random Stability of the Functional Equation (1.5) Fixed Point Approach

In this section, we use fixed point technique to prove the generalized Hyres-Ulam stability of the quadratic functional equations (1.5).

Let

$$\begin{aligned}\eta_f(x,y) &= 4f(x+my) + 4f(x-my) + m^2 f(2x)\\ &\quad - 8f(x) - 4m^2\big(f(x+y) + f(x-y)\big).\end{aligned}$$

where $m \in \mathbb{N}$ with $m \geq 2$.

Theorem 4.1. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Λ be a mapping from X^2 to D^+ ($\Lambda(x,y)$ is denoted by $\Lambda_{x,y}$) such that there exists $0 < \alpha < 8$ such that*

$$\Lambda_{\frac{x}{2},\frac{y}{2}}(t) \leq \Lambda_{x,y}(\alpha t) \tag{4.1}$$

for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a cubic mapping satisfying

$$\mu_{\eta_f(x,y)} \geq \Lambda_{x,y}(t) \tag{4.2}$$

for all $x, y \in X$ *and* $t > 0$. *Then*

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{8^n} \tag{4.3}$$

exists for all $x \in X$ *and there exists a unique cubic mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq \Lambda_{x,0}((8-\alpha)t) \tag{4.4}$$

for all $x \in X$ *and* $t > 0$.

Proof. Putting $y = 0$ in (4.2), we have

$$\mu_{\frac{f(2x)}{8}-f(x)}(t) \geq \Lambda_{x,0}(8t) \tag{4.5}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S := \{g : X \to Y\} \tag{4.6}$$

and the generalized metric d in S defined by

$$d(f, g) = \inf\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq \Lambda_{x,0}(t), \forall x \in X,\, t > 0\}, \tag{4.7}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete.

Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{8}h(2x) \tag{4.8}$$

for all $x \in X$. First, we prove that J is a strictly contractive mapping with the Lipschitz constant $\frac{\alpha}{8}$. In fact, let $g, h \in S$ be such that $d(g, h) < \epsilon$. Then

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \Lambda_{x,0}(t) \tag{4.9}$$

for all $x \in X$ and $t > 0$ and so

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}\left(\frac{\alpha\epsilon t}{8}\right) &= \mu_{\frac{1}{8}g(2x)-\frac{1}{8}h(2x)}\left(\frac{\alpha\epsilon t}{8}\right) \\ &= \mu_{g(2x)-h(2x)}(\alpha\epsilon t) \\ &\geq \Lambda_{2x,0}(\alpha t) \\ &\geq \Lambda_{x,0}(t) \end{aligned} \tag{4.10}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) < \epsilon$ implies that $d(Jg, Jh) < \frac{\alpha\epsilon}{8}$. This means that

$$d(Jg, Jh) \leq \frac{\alpha}{8}d(g, h) \tag{4.11}$$

for all $g, h \in S$. It follows from (4.5) that

$$d(f, Jf) \leq \frac{1}{8} < 1. \tag{4.12}$$

By Theorem 2.2, there exists a mapping $T : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A(2x) = 8A(x) \tag{4.13}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}. \tag{4.14}$$

This implies that A is a unique mapping satisfying (4.13) such that there exists $u \in (0, \infty)$ satisfying

$$\mu_{f(x)-A(x)}(ut) \geq \Lambda_{x,0}(t) \tag{4.15}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{8^n} f(2^n x) = A(x) \tag{4.16}$$

for all $x \in X$.

(3) $d(f, A) \le \frac{d(f,Jf)}{1-\frac{\alpha}{8}}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \le \frac{1}{8-\alpha} \tag{4.17}$$

and so

$$\mu_{f(x)-A(x)}\Big(\frac{t}{8-\alpha}\Big) \ge \Lambda_{x,0}(t) \tag{4.18}$$

for all $x \in X$ and $t > 0$. This implies that the inequality (4.4) holds.

Now, we have

$$\mu_{\frac{1}{8^n}\eta_f(2^n x, 2^n y)}(t) = \mu_{\eta_f(2^n x,2^n y)}(8^n t) \ge \Lambda_{2^n x,2^n y}(8^n t) \ge \Lambda_{x,y}\Big(\Big(\frac{8}{\alpha}\Big)^n t\Big) \tag{4.19}$$

for all $x, y \in X$, $t > 0$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \Lambda_{x,y}\Big(\Big(\frac{8}{\alpha}\Big)^n t\Big) = 1$ for all $x, y \in X$ and $t > 0$, then, by Theorem 2.1, we deduce that $\mu_{\eta_A(x,y)} = 1$ for all $x, y \in X$ and $t > 0$. Thus the mapping $A : X \to Y$ is cubic. This complete the proof. ∎

Corollary 4.2. *Let $\theta \ge 0$ and p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying*

$$\mu_{\eta_f(x,y)}(t) \ge \frac{t}{t + \theta\big(\|x\|^p + \|y\|^p\big)} \tag{4.20}$$

for all $x, y \in X$ and $t > 0$. Then

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n} \tag{4.21}$$

exists for all $x \in X$ and there exists a cubic mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \ge \frac{(8-8^p)t}{(8-8^p)t + \theta\|x\|^p} \tag{4.22}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 4.1 by taking

$$\Lambda_{x,y}(t) = \frac{t}{t + \theta\big(\|x\|^p + \|y\|^p\big)} \tag{4.23}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 8^p$, then we get the desired result. ∎

Similarly, we can obtain the following and so we omit the proof.

Theorem 4.3. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Λ be a mapping from X^2 to D^+ ($\Lambda(x, y)$ is denoted by $\Lambda_{x,y}$) such that there exists $0 < \alpha < \frac{1}{8}$ such that*

$$\Lambda_{2x,2y}(t) \le \Lambda_{x,y}(\alpha t) \tag{4.24}$$

for all $x, y \in X$ *and* $t > 0$. *Let* $f : X \to Y$ *be a cubic mapping satisfying*

$$\mu_{\eta_f(x,y)} \geq \Lambda_{x,y}(t) \tag{4.25}$$

for all $x, y \in X$ *and* $t > 0$. *Then*

$$A(x) := \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) \tag{4.26}$$

exists for all $x \in X$ *and there exists a unique cubic mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq \Lambda_{x,0}\left(\frac{(1-8\alpha)t}{\alpha}\right) \tag{4.27}$$

for all $x, y \in X$ *and* $t > 0$.

Corollary 4.4. *Let* $\theta \geq 0$ *and* p *be a real number with* $p > 1$. *Let* X *be a normed vector space with norm* $\|\cdot\|$. *Let* $f : X \to Y$ *be a cubic mapping satisfying*

$$\mu_{\eta_f(x,y)}(t) \geq \frac{t}{t + \theta\big(\|x\|^p + \|y\|^p\big)} \tag{4.28}$$

for all $x, y \in X$ *and* $t > 0$. *Then*

$$A(x) = \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) \tag{4.29}$$

exists for all $x \in X$ *and there exists a unique cubic mapping* $A : X \to Y$ *such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{(8^p-8)t}{(8^p-8)t + 8^{p+1}\theta\|x\|^p} \tag{4.30}$$

for all $x \in X$ *and* $t > 0$.

Proof. The proof follows from Theorem 4.3 by taking

$$\Lambda_{x,y}(t) = \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{4.31}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 8^{-p}$, then we get the desired result. ∎

5. Random Stability of the Functional Equation (1.6)

In this section, we use fixed point technique to prove the generalized Hyres-Ulam stability of the quadratic functional equations (1.6).

Theorem 5.1. *Let* $m, n \in \mathbb{N}$ *with* $m \neq n$, X *be a vector space,* (Z, Ψ, min) *be an RN-space and* $\psi : X^2 \to Z$ *be a function such that there exists* $0 < \alpha < m+n$ *such that*

$$\Psi_{\psi((m+n)x,(m+n)y)}(t) \geq \Psi_{\alpha\psi(x,y)}(t) \tag{5.1}$$

for all $x, y \in X$ *and* $t > 0$ *and*

$$\lim_{n\to\infty} \Psi_{\psi((m+n)^p x,(m+n)^p y)}((m+n)^p t) = 1$$

for all $x, y \in X$ *and* $t > 0$. *If* (Y, μ, min) *is a complete RN-space and* $f : X \to Y$ *is a mapping such that*

$$\mu_{f(mx+ny)-\frac{m+n}{2}f(x+y)-\frac{m-n}{2}f(x-y)}(t) \geq \Psi_{\psi(x,y)}(t) \tag{5.2}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $C : X \to Y$ such that

$$C(x) = \lim_{l\to\infty} (m+n)^{-l} f((m+n)^l x) \tag{5.3}$$

and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\psi(x,x)}((m+n-\alpha)t) \tag{5.4}$$

for all $x, y \in X$ and $t > 0$.

Proof. Since the proof of Theorem 5.1 is similar to the proof of Theorem 3.1 and so we omit the proof of Theorem 5.1. ■

Corollary 5.2. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) be a complete RN-space. Let $0 < p < 1$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mu_{f(mx+ny)-\frac{m+n}{2}f(x+y)-\frac{m-n}{2}f(x-y)}(t) \geq \Psi_{\|x\|^p z_0}(t) \tag{5.5}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $C : X \to Y$ such that

$$C(x) = \lim_{l\to\infty} (m+n)^{-l} f((m+n)^l x) \tag{5.6}$$

and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\|x\|^p z_0}((m+n-(m+n)^p)t) \tag{5.7}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = (m+n)^p$ and $\psi : X^2 \to Z$ be a mapping defined by $\psi(x, y) = \|x\|^p z_0$. Then, from Theorem 5.1, the conclusion follows. ■

Corollary 5.3. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) be a complete RN-space. Let $0 < p < 1$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mu_{f(mx+ny)-\frac{m+n}{2}f(x+y)-\frac{m-n}{2}f(x-y)}(t) \geq \Psi_{(\|x\|^p+\|y\|^p)z_0}(t) \tag{5.8}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $C : X \to Y$ such that

$$C(x) = \lim_{l\to\infty} (m+n)^{-l} f((m+n)^l x), \tag{5.9}$$

and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\|x\|^p}\Big(\frac{(m+n-(m+n)^p)t}{2}\Big). \tag{5.10}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = (m+n)^p$ and $\psi : X^2 \to Z$ be defined by $\psi(x, y) = (\|x\|^p + \|y\|^p)z_0$. Then, from Theorem 5.1, the conclusion follows. ■

Corollary 5.4. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) be a complete RN-space. Let $p, q \in \mathbb{R}^+$ with $0 < p + q < 1$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mu_{f(mx+ny)-\frac{m+n}{2}f(x+y)-\frac{m-n}{2}f(x-y)}(t) \geq \Psi_{(\|x\|^{p+q}+\|y\|^{p+q}+\|x\|^p\cdot\|y\|^q)z_0}(t) \tag{5.11}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $C : X \to Y$ such that

$$C(x) = \lim_{l\to\infty}(m+n)^{-l}f((m+n)^l x) \tag{5.12}$$

and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{||x||^{p+q}z_0}\left(\frac{(m+n-(m+n)^{p+q})t}{3}\right) \tag{5.13}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = (m+n)^{p+q}$ and $\psi : X^2 \to Z$ be a mapping defined by

$$\psi(x, y) = (\|x\|^{p+q} + \|y\|^{p+q} + \|x\|^p \cdot \|y\|^q)z_0.$$

Then, from Theorem 5.1, the conclusion follows. ∎

Corollary 5.5. *Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) be a complete RN-space. Let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mu_{f(mx+ny)-\frac{m+n}{2}f(x+y)-\frac{m-n}{2}f(x-y)}(t) \geq \Psi_{\delta z_0}(t) \tag{5.14}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $C : X \to Y$ such that

$$C(x) = \lim_{l\to\infty}(m+n)^{-l}f((m+n)^l x) \tag{5.15}$$

and

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\delta z_0}((m+n-1)t) \tag{5.16}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 1$ and $\psi : X^2 \to Z$ be be a mapping defined by $\psi(x, y) = \delta z_0$. Then, from Theorem 5.1, the conclusion follows. ∎

Example 5.1. Let $p = 2$, $\alpha = (m+n)^2$, $\psi(x, y) = (\|x\|^2 + \|y\|^2 + \|x\|.\|y\|)z_0$ and $f : X \to Y$ be a mapping satisfying

$$\mu_{f(mx+ny)-(\frac{m+n}{2})f(x+y)-(\frac{m-n}{2})f(x-y)}(t) \geq \Psi_{(\|x\|^2+\|y\|^2+\|x\|.\|y\|)z_0}(t). \tag{5.17}$$

As in Theorem 3.1, we obtain

$$\mu_{\frac{f((m+n)^{l+q}x)}{(m+n)^{l+q}}-\frac{f((m+n)^q x)}{(m+n)^q}}(t) \geq \Psi_{\|x\|^2 z_0}\left(\frac{t}{3\sum_{k=q}^{l+q-1}(m+n)^{k-1}}\right).$$

Since

$$\sum_{k=q}^{l+q-1}(m+n)^{k-1} = \frac{(m+n)^{l+q-2}-(m+n)^{q-1}}{(m+n)(1-m-n)}$$

and

$$\lim_{l,q\to\infty}\frac{(m+n)(1-m-n)}{(m+n)^{l+q-2}-(m+n)^{q-1}}\neq\infty,$$

we have

$$\lim_{l,q\to\infty}\mu_{\frac{f((m+n)^{l+q}x)}{(m+n)^{l+q}}-\frac{f((m+n)^{q}x)}{(m+n)^{q}}}(t)=\lim_{n,q\to\infty}\Psi_{\|x\|^2z_0}\left(\frac{(m+n)(1-m-n)}{3((m+n)^{l+q-2}-(m+n)^{q-1})}\right)\neq 1.$$

This means the sequence $\left\{\frac{f((m+n)^l x)}{(m+n)^l}\right\}_{l=1}^{\infty}$ is not a Cauchy sequence.

6. Stability of the Functional Equation (1.2) in Non-Archimedean Normed Spaces

In this section, we solve the stability problem of the functional equation (1.2) in non-Archimedean normed spaces.

Throughout this section, let G be an additive semigroup and X be a complete non-Archimedean space.

Theorem 6.1. *Let $\psi : G^2 \to [0,+\infty)$ be a function such that*

$$\lim_{n\to\infty}|m|^{-2n}\psi(m^n x, m^n y)=0 \tag{6.1}$$

for all $x, y \in G$. Suppose that, for all $x \in G$, the limit

$$\Psi(x) = \lim_{n\to\infty}\max\left\{\frac{1}{|m|^{2k}}\psi(m^k x,0); 0\le k<n\right\}$$

exists and $f : G \to X$ is a mapping satisfying the inequality

$$\|M_f(x,y)\|\le\psi(x,y) \tag{6.2}$$

for all $x, y \in G$. Then the limit

$$T(x):=\lim_{n\to\infty}m^{-2n}f(m^n x)$$

exist for all $x \in G$ and $T : G \to X$ is a mapping satisfying

$$\|f(x)-T(x)\|\le\frac{1}{|m|^2}\Psi(x) \tag{6.3}$$

for all $x \in G$. Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\frac{1}{|m|^{2j}}\psi(m^j x,0): k\le j<n+k\right\}=0, \tag{6.4}$$

then T is the unique mapping satisfying (6.3).

Proof. Putting $y = 0$ in (6.2), we have

$$\left\|\frac{f(mx)}{m^2}-f(x)\right\|\le\frac{1}{|m|^2}\psi(x,0). \tag{6.5}$$

Replacing x by $m^n x$ in (6.5) and dividing both sides by m^{2n}, we get

$$\left\|\frac{f(m^{n+1}x)}{m^{2(n+1)}}-\frac{f(m^n x)}{m^{2n}}\right\| \le \frac{1}{|m|^{2n+2}}\psi(m^n x,0). \tag{6.6}$$

Thus it follows from (6.1) and (6.6) that the sequence $\left\{\frac{f(m^n x)}{m^{2n}}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, $\left\{\frac{f(m^n x)}{m^{2n}}\right\}_{n=1}^{\infty}$ is convergent and so set

$$T(x) := \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}.$$

By induction, we can see that

$$\left\|\frac{f(m^n x)}{m^{2n}} - f(x)\right\| \quad \le \quad \frac{1}{|m|^2} \max\left\{\frac{1}{|m|^{2k}}\psi(m^k x, 0) : 0 \le k < n\right\}. \tag{6.7}$$

Indeed, (6.7) holds for $n = 1$ by (6.5). Now, if (6.7) holds for all $0 \le k < n$, then, by (6.6), we obtain

$$\begin{aligned}
& \left\|\frac{f(m^{n+1}x)}{m^{2(n+1)}} - f(x)\right\| \\
= & \left\|\frac{f(m^{n+1}x)}{m^{2(n+1)}} \pm \frac{f(m^n x)}{m^{2n}} - f(x)\right\| \\
\le & \max\left\{\left\|\frac{f(m^{n+1}x)}{m^{2(n+1)}} - \frac{f(m^n x)}{m^{2n}}\right\|, \left\|\frac{f(m^n x)}{m^{2n}} - f(x)\right\|\right\} \qquad (6.8)\\
\le & \frac{1}{|m|^2} \max\left\{\frac{1}{|m|^{2n}}\psi(m^n x, 0), \max\left\{\frac{1}{|m|^{2k}}\psi(m^k x, 0) : 0 \le k < n\right\}\right\} \\
= & \frac{1}{|m|^2} \max\left\{\frac{1}{|m|^{2k}}\psi(m^k x, 0) : 0 \le k < n+1\right\}.
\end{aligned}$$

So, (6.7) holds for all $n \in \mathbb{N}$ and $x \in G$. By taking $n \to \infty$ in (6.8), we can obtain (6.3).

If S is another mapping satisfies (6.3), then we get

$$\begin{aligned}
\|T(x) - S(x)\| &= \lim_{k\to\infty} |m|^{-2k} \|T(m^k x) - S(m^k x)\| \\
&\le \lim_{k\to\infty} |m|^{-2k} \max\left\{\|T(2^k x) - f(2^k x)\|, \|f(2^k x) - S(2^k x)\|\right\} \\
&\le \frac{1}{|m|^2} \lim_{k\to\infty} \lim_{n\to\infty} \max\left\{\frac{1}{|m|^{2j}}\psi(m^j x, 0) : k \le j < n+k\right\} = 0
\end{aligned}$$

for all $x \in G$. Therefore, we have $T = S$. This completes the proof. ■

Theorem 6.2. *Let $\psi : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |m|^{2n} \psi\Big(\frac{x}{m^{2n}}, \frac{y}{m^{2n}}\Big) = 0 \tag{6.9}$$

for all $x, y \in G$. Suppose that, for all $x \in G$, the limit

$$\Psi(x) \quad = \quad \lim_{n\to\infty} \max\left\{|m|^{2k}\psi\Big(\frac{x}{2^k}, 0\Big) : 0 \le k < n\right\}$$

exists and $f : G \to X$ is a mapping satisfying the inequality

$$\big\|M_f(x, y)\big\| \le \psi(x, y) \tag{6.10}$$

for all $x, y \in G$. Then the limit

$$T(x) := \lim_{n\to\infty} m^{2n} f\Big(\frac{x}{m^n}\Big)$$

exist for all $x \in G$ and $T : G \to X$ is a mapping satisfying

$$\big\|f(x) - T(x)\big\| \leq \frac{1}{|m|^2}\Psi(x) \tag{6.11}$$

for all $x \in G$. Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty}\max\Big\{|m|^{2j}\psi\Big(\frac{x}{2^j},0\Big) : k \leq j < n+k\Big\} = 0 \tag{6.12}$$

for all $x \in G$, then T is the unique mapping satisfying (6.11).

Proof. As in the proof of Theorem (6.1), we obtain

$$\Big\|\frac{f(mx)}{m^2} - f(x)\Big\| \leq \frac{1}{|m|^2}\psi(x,0). \tag{6.13}$$

Replacing x by $\frac{x}{m^n}$ in (6.13) , we get

$$\Big\|m^{2(n-1)}f\Big(\frac{x}{m^{n-1}}\Big) - m^{2n}f\Big(\frac{x}{m^n}\Big)\Big\| \quad \leq \quad \frac{|m|^{2n}}{|m|^2}\psi\Big(\frac{x}{m^n},0\Big) \tag{6.14}$$

and so it follows from (6.9) and (6.14) that the sequence $\Big\{m^{2n}f\Big(\frac{x}{m^n}\Big)\Big\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, $\Big\{m^{2n}f\Big(\frac{x}{m^n}\Big)\Big\}_{n=1}^{\infty}$ is convergent and so it follows from (6.14) that

$$\begin{aligned}
&\Big\|m^{2n}f\Big(\frac{x}{m^n}\Big) - m^{2p}f\Big(\frac{x}{m^p}\Big)\Big\| \\
= &\Big\|\sum_{k=p+1}^{n} m^{2(k-1)}f\Big(\frac{x}{m^{k-1}}\Big) - m^{2k}f\Big(\frac{x}{m^k}\Big)\Big\| \\
\leq &\max\Big\{\Big\|m^{2(k-1)}f\Big(\frac{x}{m^{k-1}}\Big) - m^{2k}f\Big(\frac{x}{m^k}\Big)\Big\| : p+1 \leq k < n\Big\} \\
\leq &\frac{1}{|m|^2}\max\Big\{|m|^{2k}\psi\Big(\frac{x}{m^k},0\Big) : p+1 \leq k < n\Big\}
\end{aligned} \tag{6.15}$$

for all $x \in G$ and $m, n \in \mathbb{N}$ with $n - 1 > p \geq 0$. Letting $p = 0$ and $n \to \infty$ in (6.15), we obtain (6.11).

The rest of the proof is similar to the proof of Theorem 6.1. This complete the proof. ∎

Corollary 6.3. *Let $\lambda : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\lambda(|m|t) \leq \lambda(|m|)\lambda(t), \quad \lambda(|m|) < |m|^2$$

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ be a mapping satisfying the inequality

$$\big\|M_f(x,y)\big\| \leq \delta\big(\lambda(\|x\|) + \lambda(\|y\|)\big)$$

for all $x, y \in G$. Then there exists a unique mapping $T : G \to X$ such that

$$\big\|f(x) - T(x)\big\| \leq \frac{\delta\lambda(\|x\|)}{|m|^2} \tag{6.16}$$

for all $x \in G$.

Proof. By induction, wee can show that

$$\lambda(|m|^n t) \leq (\lambda(|m|)^n \lambda(|t|)) \leq |m|^{2n}\lambda(|t|) \tag{6.17}$$

for all $n \in \mathbb{N}$. If we Define a mapping $\psi : G^2 \to [0, \infty)$ by

$$\psi(x, y) := \delta\Big(\lambda(\|x\|) + \lambda(\|y\|)\Big)$$

for all $x, y \in G$, then, from $\frac{\lambda(|m|)}{|m|^2} < 1$, it follows that

$$\lim_{n\to\infty} \frac{\psi(m^n x, m^n y)}{|m|^{2n}} \leq \lim_{n\to\infty} \Big(\frac{\lambda(|m|)}{|m|^2}\Big)^n \psi(x, y) = 0 \tag{6.18}$$

for all $x, y \in G$. Also, for all $x \in G$, the limit

$$\Psi(x) \quad = \quad \lim_{n\to\infty} \max\Big\{\frac{1}{|m|^{2k}}\psi(m^k x, 0) : 0 \leq k < n\Big\} = \psi(x, 0) = \delta\lambda(\|x\|)$$

exists and

$$\lim_{k\to\infty} \lim_{n\to\infty} max\Big\{\frac{1}{|m|^{2j}}\psi(m^j x, 0) : k \leq j < n + k\Big\} = 0$$

for all $x \in G$. Therefore, the result follows by Theorem 6.1. This completes the proof. ■

Corollary 6.4. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\lambda(|m|^{-1}t) \leq \lambda(|m|^{-1})\lambda(t), \quad \lambda(|m|^{-1}) < |m|^{-2}$$

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ be a mapping satisfying the inequality

$$\big\|M_f(x, y)\big\| \leq \delta\big(\lambda(\|x\|) + \lambda(\|y\|)\big)$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $T : G \to X$ such that

$$\big\|f(x) - T(x)\big\| \leq \frac{\delta\lambda(||x||)}{|m|^2} \tag{6.19}$$

for all $x \in G$.

However, the following example shows that Theorem 1.1 is not true in non-Archimedean normed spaces.

Example 6.1. Let $p > 2$ and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be a mpping defined by $f(x) = 2$ for all $x \in \mathbb{Q}_p$. Then, for $\epsilon = 1$,

$$|f(x + y) - f(x) - f(y)| = 1 \leq \epsilon$$

for all $x, y \in \mathbb{Q}_p$. However, neither $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ nor $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence.

In fact, by using the fact that $|2| = 1$, we have

$$\Big|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\Big| = |2^{-n} \cdot 2 - 2^{-(n+1)} \cdot 2| = |2^{-n}| = 1$$

and

$$\Big|2^n f\Big(\frac{x}{2^n}\Big) - 2^{n+1} f\Big(\frac{x}{2^{n+1}}\Big)\Big| = |2^n \cdot 2 - 2^{(n+1)} \cdot 2| = |2^{n+1}| = 1$$

for all $x, y \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence these sequences are not convergent in $\mathbb{Q}_p$.

7. Stability of the Functional Equation (1.5) in Non-Archimedean Normed Spaces

In this section, we solve the stability problems of the functional equation (1.5) in non-Archimedean normed spaces.

Throughout this section, let G be an additive semigroup and X be a complete non-Archimedean space.

Theorem 7.1. *Let $\psi : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |8|^{-n}\psi(2^n x, 2^n y) = 0 \tag{7.1}$$

for all $x, y \in G$. Suppose that, for all $x \in G$, the limit

$$\Psi(x) = \lim_{n\to\infty} \max\left\{\frac{1}{|8|^k}\psi(2^k x, 0) : 0 \le k < n\right\}$$

exists and $f : G \to X$ is a mapping satisfying the inequality

$$\|M_f^*(x, y)\| \le \psi(x, y) \tag{7.2}$$

for all $x, y \in G$. Then the limit

$$T(x) := \lim_{n\to\infty} 8^{-n} f(2^n x)$$

exist for all $x \in G$ and $T : G \to X$ is a cubic mapping satisfying

$$\|f(x) - T(x)\| \le \frac{1}{|8|}\Psi(x) \tag{7.3}$$

for all $x \in G$. Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\left\{\frac{1}{|8|^j}\psi(2^j x, 0) : k \le j < n + k\right\} = 0 \tag{7.4}$$

for all $x \in G$, then T is the unique mapping satisfying (7.3).

Proof. Putting $y = 0$ in (7.2), we have

$$\left\|\frac{f(2x)}{8} - f(x)\right\| \le \frac{1}{|8|}\psi(x, 0). \tag{7.5}$$

Replacing x by $2^n x$ in (7.5) and dividing both sides by 8^n, we get

$$\left\|\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}\right\| \le \frac{1}{|8|^{n+1}}\psi(2^n x, 0). \tag{7.6}$$

Thus it follows from (7.1) and (7.6) that the sequence $\left\{\frac{f(2^n x)}{8^n}\right\}_{n=1}^{\infty}$ is convergent and so set

$$T(x) := \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}.$$

The rest of the proof is similar to proof of the Theorem 6.1. ■

Theorem 7.2. *Let $\psi : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |8|^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{7.7}$$

for all $x, y \in G$. Suppose that, for all $x \in G$, the limit

$$\Psi(x) = \lim_{n\to\infty} \max\left\{|8|^k \psi\left(\frac{x}{2^k}, 0\right) : 0 \le k < n\right\}$$

exists and $f : G \to X$ *is a mapping satisfying the inequality*

$$\left\|M_f^*(x, y)\right\| \le \psi(x, y) \tag{7.8}$$

for all $x, y \in G$*. Then the limit*

$$T(x) := \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$$

exist for all $x \in G$ *and* $T : G \to X$ *is a cubic mapping satisfying*

$$\left\|f(x) - T(x)\right\| \le \frac{1}{|8|}\Psi(x) \tag{7.9}$$

for all $x \in G$*. Moreover, if*

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{|8|^j \psi\left(\frac{x}{2^j}, 0\right) : k \le j < n + k\right\} = 0 \tag{7.10}$$

for all $x \in G$*, then* T *is the unique mapping satisfying* (7.9)*.*

Corollary 7.3. *Let* $\lambda : [0, \infty) \to [0, \infty)$ *be a function satisfying*

$$\lambda(|2|t) \le \lambda(|2|)\lambda(t), \quad \lambda(|2|) < |8|$$

for all $t \ge 0$*. Let* $\delta > 0$ *and* $f : G \to X$ *be a mapping satisfying the inequality*

$$\left\|M_f(x, y)\right\| \le \delta\big(\lambda(\|x\|) + \lambda(\|y\|)\big)$$

for all $x, y \in G$*. Then there exists a unique cubic mapping* $T : G \to X$ *such that*

$$\left\|f(x) - T(x)\right\| \le \frac{\delta\lambda(\|x\|)}{|8|} \tag{7.11}$$

for all $x \in G$*.*

Corollary 7.4. *Let* $\lambda : [0, \infty) \to [0, \infty)$ *be a function satisfying*

$$\lambda(|2|^{-1}t) \le \lambda(|2|^{-1})\lambda(t), \quad \lambda(|2|^{-1}) < |8|^{-1}$$

for all $t \ge 0$*. Let* $\delta > 0$ *and* $f : G \to X$ *be a mapping satisfying the inequality*

$$\left\|M_f(x, y)\right\| \le \delta\big(\lambda(\|x\|) + \lambda(\|y\|)\big)$$

for all $x, y \in G$*. Then there exists a unique cubic mapping* $T : G \to X$ *such that*

$$\|f(x) - T(x)\| \le \frac{\delta\lambda(\|x\|)}{|8|} \tag{7.12}$$

for all $x \in G$*.*

8. Stability of the Functional Equation (1.6) in Non-Archimedean Normed Spaces

In this section, we solve the stability problems of the functional equation (1.6) in non-Archimedean normed spaces.

Throughout this section, let G be an additive semigroup and X be a complete non-Archimedean space.

Theorem 8.1. *Let $\psi : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{p\to\infty} |m+n|^{-p}\psi((m+n)^p x, (m+n)^p y) = 0 \tag{8.1}$$

for all $x, y \in G$. Suppose that, for all $x \in G$, the limit

$$\Psi(x) = \lim_{p\to\infty} \max\left\{|m+n|^{-k+1}\psi((m+n)^k x, (m+n)^k y) : 0 \le k < p\right\} \tag{8.2}$$

exists and $f : G \to X$ is a mapping satisfying

$$\left\|f(mx+ny) - \frac{m+n}{2}f(x+y) - \frac{m-n}{2}f(x-y)\right\|_X \le \psi(x,y) \tag{8.3}$$

for all $x, y \in G$. Then the limit

$$T(x) = \lim_{l\to\infty}(m+n)^{-l}f((m+n)^l x)$$

exists for all $x \in G$ and $T : G \to X$ is a mapping satisfying

$$\|f(x) - T(x)\|_X \le \frac{1}{|m+n|}\Psi(x) \tag{8.4}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty}\lim_{p\to\infty} \max\left\{|m+n|^{-k+1}\psi((m+n)^k x, (m+n)^k y) : j \le k < p+j\right\} = 0 \tag{8.5}$$

for all $x \in G$, then T is the unique mapping satisfying (8.4).

Proof. Putting $y = x$ in (8.3), we get

$$\left\|\frac{f((m+n)x)}{m+n} - f(x)\right\|_X \le \frac{1}{|m+n|}\psi(x,x). \tag{8.6}$$

Replacing x by $(m+n)^{p-1}x$ in (8.6) and dividing both sides by $(m+n)^{p-1}$, we get

$$\left\|\frac{f((m+n)^p x)}{(m+n)^p} - \frac{f((m+n)^{p-1}x)}{(m+n)^{p-1}})\right\|_X \le |m+n|^{-p}\psi((m+n)^{p-1}x, (m+n)^{p-1}x), \tag{8.7}$$

for all $x \in H$. It follows from (8.1) and (8.7) that the sequence $\left\{\frac{f((m+n)^p x)}{(m+n)^p}\right\}_{p=1}^{+\infty}$ is a Cauchy sequence. Since X is complete, so the sequence $\left\{\frac{f((m+n)^p x)}{(m+n)^p}\right\}_{p=1}^{+\infty}$ is convergent. Set

$$T(x) := \lim_{p\to\infty}\frac{f((m+n)^p x)}{(m+n)^p}.$$

Using induction we see that

$$\left\|\frac{f((m+n)^p x)}{(m+n)^p} - f(x)\right\|_X \le \frac{1}{|m+n|}\max\left\{|m+n|^{-k+1}\psi((m+n)^k x, (m+n)^k x)\ ;\ 0 \le k < p\right\}.$$

The rest of the proof is similar to proof of the Theorem 6.1. ∎

Theorem 8.2. *Let $\psi : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |m+n|^p \psi\Big(\frac{x}{(m+n)^p}, \frac{y}{(m+n)^p}\Big) = 0 \tag{8.8}$$

for all $x, y \in G$. Suppose that, for all $x \in G$, the limit

$$\Psi(x) = \lim_{n\to\infty} max\Big\{|m+n|^{k-1}\psi\Big(\frac{x}{(m+n)^k}, \frac{y}{(m+n)^k}\Big) = 0 : 0 \le k < p\Big\} \tag{8.9}$$

exists and $f : G \to X$ is a mapping satisfying

$$\Big\|f(mx+ny) - \frac{m+n}{2} f(x+y) - \frac{m-n}{2} f(x-y)\Big\|_X \le \psi(x, y) \tag{8.10}$$

for all $x, y \in G$. Then the limit

$$T(x) = \lim_{l\to\infty} (m+n)^l f\Big(\frac{x}{(m+n)^l}\Big)$$

exists for all $x \in G$ and $T : G \to X$ is a mapping satisfying

$$\|f(x) - T(x)\|_X \le \frac{1}{|m+n|}\Psi(x) \tag{8.11}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty}\lim_{p\to\infty} \max\Big\{|m+n|^{k-1}\psi\Big(\frac{x}{(m+n)^k}, \frac{y}{(m+n)^k}\Big) : j \le k < n+j\Big\} = 0 \tag{8.12}$$

for all $x \in G$, then T is the unique mapping satisfying (8.11).

Proof. Letting $y = x$ in (8.10), we get

$$\Big\|f((m+n)x) - (m+n)f(x)\Big\|_X \le \psi(x, x), \tag{8.13}$$

for all $x \in G$. If we replace x by $\frac{x}{(m+n)^p}$ in (8.13), then we have

$$\Big\|(m+n)^{p-1} f\Big(\frac{x}{(m+n)^{p-1}}\Big) - (m+n)^p f\Big(\frac{x}{(m+n)^p}\Big)\Big)\Big\|_X \le |m+n|^{p-1}\psi\Big(\frac{x}{(m+n)^p}, \frac{x}{(m+n)^{p-1}}\Big), \tag{8.14}$$

for all $x \in G$ and all non-negative integer n. It follows from (8.14) and (8.8) that the sequence $\Big\{(m+n)^p f\Big(\frac{x}{(m+n)^p}\Big)\Big\}_{p=1}^{\infty}$ is a Cauchy sequence in X for all $x \in G$. Since X is complete, the sequence $\Big\{(m+n)^p f\Big(\frac{x}{(m+n)^p}\Big)\Big\}_{n=1}^{\infty}$ converges for all $x \in G$. On the other hand, it follows from (8.15) that

$$\begin{aligned}
&\Big\|(m+n)^p f\Big(\frac{x}{(m+n)^p}\Big) - (m+n)^q f\Big(\frac{x}{(m+n)^q}\Big)\Big\|_X \\
&= \Big\|\sum_{k=p}^{q-1} (m+n)^{k+1} f\Big(\frac{x}{(m+n)^{k+1}}\Big) - (m+n)^k f\Big(\frac{x}{(m+n)^k}\Big)\Big\|_X \qquad (8.15)\\
&\le \max\Big\{\Big\|(m+n)^{k+1} f\Big(\frac{x}{(m+n)^{k+1}}\Big) - (m+n)^k f\Big(\frac{x}{(m+n)^k}\Big)\Big\|_X \;;\; p \le k < q\Big\} \\
&\le \frac{1}{|m+n|} max\Big\{|m+n|^{k-1}\psi\Big(\frac{x}{(m+n)^k}, \frac{x}{(m+n)^k}\Big); p \le k < q\Big\},
\end{aligned}$$

for all $x \in G$ and all non-negative integers p, q with $q > p \geq 0$. Letting $p = 0$ and passing the limit $q \to \infty$ in the last inequality, we obtain (8.11).
The rest of the proof is similar to the proof of Theorem (6.1). ∎

Corollary 8.3. *Let $\gamma : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\gamma\Big(\frac{t}{|m+n|}\Big) \leq \gamma\Big(\frac{1}{|m+n|}\Big)\gamma(t), \quad \gamma\Big(\frac{1}{|m+n|}\Big) < \frac{1}{|m+n|} \tag{8.16}$$

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ be a mapping satisfying

$$\Big\|f(mx+ny) - \frac{m+n}{2}f(x+y) - \frac{m-n}{2}f(x-y)\Big\|_X \leq \delta\big(\gamma(|x|) + \gamma(|y|)\big) \tag{8.17}$$

for all $x, y \in G$. Then there exists a unique mapping $T : G \to X$ such that

$$\|f(x) - T(x)\|_X \leq \frac{2\delta\gamma(|x|)}{|m+n|} \tag{8.18}$$

for all $x \in G$.

Corollary 8.4. *Let $\gamma : [0, \infty) \to [0, \infty)$ is a function satisfying*

$$\gamma(|m+n|t) \leq \gamma(|m+n|)\gamma(t), \quad \gamma(|m+n|) < |m+n| \tag{8.19}$$

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ be a mapping satisfying

$$\Big\|f(mx+ny) - \frac{m+n}{2}f(x+y) - \frac{m-n}{2}f(x-y)\Big\|_X \leq \delta\big(\gamma(|x| + \gamma(|y|)\big) \tag{8.20}$$

for all $x, y \in G$. Then there exists a unique mapping $T : H \to X$ such that

$$\|f(x) - T(x)\|_X \leq \frac{2\delta\gamma(|x|)}{|m+n|} \tag{8.21}$$

for all $x \in G$.

Proof. If $\psi : G^2 \to [0, \infty)$ is a mapping defined by $\psi(x, y) := \delta\big(\gamma(|x| + \gamma(|y|)\big)$ for all $x, y \in G$, then, from Theorem 8.1, the result follows. ∎

Corollary 8.5. *Let $\gamma : [0, \infty) \to [0, \infty)$ is a function satisfying*

$$\gamma\Big(\frac{t}{|m+n|}\Big) \leq \gamma\Big(\frac{1}{|m+n|}\Big)\gamma(t), \quad \gamma\Big(\frac{1}{|m+n|}\Big) < \frac{1}{|m+n|} \tag{8.22}$$

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ be a mapping satisfying

$$\Big\|f(mx+ny) - \frac{m+n}{2}f(x+y) - \frac{m-n}{2}f(x-y)\Big\|_X \leq \delta\big(\gamma(|x|) \cdot \gamma(|y|)\big) \tag{8.23}$$

for all $x, y \in G$. Then there exists a unique mapping $T : G \to X$ such that

$$\|f(x) - T(x)\|_X \leq \frac{\delta\gamma^2(|x|)}{|m+n|} \tag{8.24}$$

for all $x \in G$.

Proof. Define a mapping $\psi : G^2 \to [0, \infty)$ by $\psi(x, y) = \delta\big(\gamma(|x|) \cdot \gamma(|y|)\big)$ for all $x, y \in G$. Then, from Theorem 8.1, the conclusion follows. ∎

Corollary 8.6. *Let* $\gamma : [0,\infty) \to [0,\infty)$ *is a function satisfying*

$$\gamma(|m+n|t) \leq \gamma(|m+n|)\gamma(t), \gamma(|m+n|) < |m+n| \tag{8.25}$$

for all $t \geq 0$. *Let* $\delta > 0$ *and* $f : G \to X$ *be a mapping satisfying*

$$\left\| f(mx+ny) - \frac{m+n}{2} f(x+y) - \frac{m-n}{2} f(x-y) \right\| \leq \delta\big(\gamma(|x|)\cdot\gamma(|y|)\big) \tag{8.26}$$

for all $x, y \in G$. *Then there exists a unique mapping* $T : G \to X$ *such that*

$$\big\| f(x) - T(x) \big\|_X \leq \frac{\delta\gamma^2(|x|)}{|m+n|} \tag{8.27}$$

for all $x \in G$.

Proof. Define a mapping $\psi : G^2 \to [0,\infty)$ by $\psi(x,y) = \delta\big(\gamma(|x|)\cdot\gamma(|y|)\big)$ for all $x, y \in G$. Then, from Theorem 8.1, the conclusion follows. ∎

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