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A Note on Semi-Symmetric Metric Connection in Riemannian Manifold

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Abstract In this paper we have discussed the Riemannian manifolds admitting a semi-symmetric metric connection $\overline{\nabla}$ by taking ρ as a unit parallel vector field with respect to Levi-Civita connection ∇. We found that the manifold M be concircular semi-symmetric with respect to Levi-Civita connection ∇ if and only if it is semi-symmetric with respect to ∇ and M be a quasi-Einstein manifold if it will be concircularly-flat with respect to semi-symmetric metric connection $\overline{\nabla}$. Also, we have shown that a semi-symmetric manifold M be a conformally-flat quasi-Einstein manifold under the condition $R\,.\,\overline{C}=0$ or $R\,.\,\overline{C}-C\,.\,\overline{R}=0$ for a concircular curvature tensor C.

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1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold with Riemannian connection ∇. The *torsion tensor* T of the connection ∇ in a Riemannian manifold is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.1}$$

where X and Y are vector fields.

In 1932, Hayden introduced the idea of metric connection with non-vanishing torsion tensor on a Riemannian manifold. The basic concept of a semi-symmetric linear connection in a differentiable manifold, is given by Friedmann and Shouten in 1924. A linear connection ∇ is said to be a *semi-symmetric connection* if the torsion tensor of the connection has the form

$$T(X,Y) = \omega(Y)X - \omega(X)Y. \tag{1.2}$$

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A connection is said to be a *metric connection* if the covariant derivative of a Riemannian metric g vanishes, i. e., $\nabla g = 0$.

In 1970, Yano [1] discussed the Riemannian manifold admitting a semi-symmetric metric connection with vanishing curvature tensor and proved certain results. Recently, the semi-symmetric metric connection is studied by U. C. De and S. C. Biswas [2] in a Riemannian manifold, a semi-symmetric non-metric connection by B. B. Chaturvedi and P. N. Pandey [3]. In 2008, C. Murathan and C. Özgür [4] considered the semi-symmetric metric connection with a unit parallel vector field ρ and obtained some interesting results on a Riemannian manifold. In 2014, Ahmet Yildiz and Azime cetinkaya [5] found some conditions on a Riemannian manifold equipped with a semi-symmetric metric connection to be Projectively semi-symmetric manifold, conformally flat and quasi-Einstein manifold.

If a Riemannian manifold satisfies $R.R = 0$ then it is said to be a *semi-symmetric manifold*. For a concircular tensor C, the Riemannian manifold is called *concircular semi-symmetric manifold* if $R.C = 0$.

2. Preliminaries

If the Ricci tensor S of the connection ∇ in a Riemannian manifold (M, g) satisfies

$$S = \frac{r}{n} g(X, Y), \tag{2.1}$$

then the manifold is called an *Einstein manifold*, where r is the scalar curvature tensor. A manifold is called a *quasi-Einstein manifold* if the Ricci tensor have the form

$$S(X, Y) = ag(X, Y) + b\omega(X)\omega(Y), \tag{2.2}$$

where a, b are scalars and ω is a non-zero 1-form.
Deszcz, R. [6] defined two tensor fields $R.T$ and $Q(E, T)$ for a tensor field T of type $(0, k)$ and $(0, k + 2), k\geq 1$ on (M, g) respectively by

$$\begin{aligned}(R(X, Y)T)(X_1, X_2, ..., X_k) = &- T(R(X, Y)X_1, X_2, ..., X_k) - ....\\ &- T(X_1, X_2, ..., R(X, Y)X_k),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}Q(E, T)(X_1, X_2, ...X_k, X, Y) = &- T((X \wedge_E Y)X_1, X_2, ..., X_k) - ......\\ &- T(X_1, X_2, ..., (X \wedge_E Y)X_k).\end{aligned} \tag{2.4}$$

Where $X\wedge_E Y$ defined by

$$(X \wedge_E Y)Z = E(Y, Z)X - E(X, Z)Y, \tag{2.5}$$

for a tensor field E of type $(0, 2)$.
The Weyl tensor W and the concircular tensor C of a Riemannian manifold (M, g) are defined by

$$\begin{aligned}W(X, Y, Z, T) =&R(X, Y, Z, T) - \frac{1}{n-2}[S(Y, Z)g(X, T) - S(X, Z)g(Y, T)\\ &+ g(Y, Z)S(X, T) - g(X, Z)S(Y, T)]\\ &+ \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)]\end{aligned} \tag{2.6}$$

and

$$C(X,Y,Z,T) = R(X,Y,Z,T) - \frac{r}{n(n-1)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)], \quad (2.7)$$

where r denotes the scalar curvature of M.

Deszcz, R. gave two lemmas as follows:

Lemma 2.1 ([7])**.** *Let (M,g) be an $n{\geq}3$ dimensional Riemannian manifold. Let at a point $x{\in}M$, a non-zero symmetric tensor E of type $(0,2)$ and a generalized curvature tensor B are given such that $Q(E,B)=0$. Moreover, let V be a vector at x such that the scalar $\rho = a(V)$ is non zero, where a is defined by $a(X) = E(X,V)$, $X \in T_X M$.*
i) If $E = \frac{1}{\rho} a \otimes a$, then at x we have ${}_{X,Y,Z} a(X)B(Y,Z)$ where $X,Y,Z \in T_X M$.
ii) If $E - \frac{1}{\rho}\ a \otimes a$ is non-zero, then at x we have $B = \frac{\gamma}{2} E \wedge E$, $\gamma \in R$. Moreover, in both cases, at x we have $B.B = Q(Ric(B), B)$.

Lemma 2.2 ([8])**.** *Let (M,g) be an $n{\geq}4$ dimensional semi-Riemannian manifold and E be the symmetric tensor of type $(0,2)$ at $x \in M$ defined by $E = \alpha g + \beta\omega \otimes \omega$, $\omega \in T_X M$, $\alpha, \beta \in R$. If at x, the curvature tensor R is expressed by $R = \frac{\gamma}{2} E \wedge E$, $\gamma \in R$, then the Weyl tensor vanishes at x.*

3. Semi-Symmetric Metric Connection

If ∇ be the Levi-Civita connection of a Riemannian manifold M, then we define

$$\overline{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X,Y)\rho, \quad (3.1)$$

where $\omega(X) = g(X,\rho)$ and X, Y, ρ are vector fields on M. If R and $\overline{R}$ be the Riemannian curvature tensor with respect to ∇ and $\overline{\nabla}$ respectively, then Yano K. [9] derived a relation between R and $\overline{R}$ given by

$$\begin{aligned}\overline{R}(X,Y,Z,T) =& R(X,Y,Z,T) - \theta(Y,Z)g(X,T) + \theta(X,Z)g(Y,T) \\ & - g(Y,Z)\theta(X,T) + g(X,Z)\theta(Y,T),\end{aligned} \quad (3.2)$$

where

$$\theta(X,Y) = (\nabla_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}g(X,Y). \quad (3.3)$$

Now, if ρ be a parallel unit vector field with respect to the connection ∇, then

$$\nabla\rho = 0 \text{ and } ||U|| = 1. \quad (3.4)$$

From (3.4), we can easily determine

$$(\nabla_X \omega)Y = 0. \quad (3.5)$$

Now from equation (3.3), we have θ is a symmetric tensor field of type $(0,2)$.
We know that the Kulkarni-Nomizu product $\overline{\wedge}$ of two tensors of type $(0,2)$ is defined by

$$\begin{aligned}(g\overline{\wedge}\theta)(X,Y,Z,T) =& \theta(Y,Z)g(X,T) - \theta(X,Z)g(Y,T) + \\ & g(Y,Z)\theta(X,T) - g(X,Z)\theta(Y,T).\end{aligned} \quad (3.6)$$

Then equation (3.2) can be written as

$$\overline{R}(X,Y,Z,T) = R(X,Y,Z,T) - (g\overline{\wedge}\theta)(X,Y,Z,T). \quad (3.7)$$

By straight forward calculation, we can easily write

$$R(X,Y)\rho = 0. \quad (3.8)$$

Contracting above equation, we get

$$S(Y,\rho) = \omega(LY) = 0, \tag{3.9}$$

where S denotes the Ricci tensor of the connection ∇ and L is the Ricci operator defined by $g(LX,Y) = S(X,Y)$.
Contracting (3.2), we have

$$\overline{S}(Y,Z) = S(Y,Z) - (n-2)[g(Y,Z) - \omega(Y)\omega(Z)], \tag{3.10}$$

and

$$\overline{r} = r - (n-1)(n-2). \tag{3.11}$$

Using (3.7), (3.10) and (3.11), we obtained

$$\overline{W}(X,Y,Z,T) = W(X,Y,Z,T), \tag{3.12}$$

and

$$\overline{C}(X,Y,Z,T) = C(X,Y,Z,T) - (g\overline{\wedge}\theta)(X,Y,Z,T) + G(X,Y,Z,T), \tag{3.13}$$

where

$$G(X,Y,Z,T) = \frac{(n-2)}{n}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)]. \tag{3.14}$$

From equation (3.13), we can write

$$\begin{aligned}\overline{C}(X,Y)Z = {} & C(X,Y)Z + g(Y,Z)\omega(X)\rho - g(X,Z)\omega(Y)\rho \\ & + \omega(Y)\omega(Z)X - \omega(X)\omega(Z)Y - \frac{2}{n}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.15}$$

Using (2.3) and (2.4) in (3.14), two important conditions also can be obtained easily

$$(R(X,Y)\ G)(Z,T,U,V) = 0, \tag{3.16}$$

and

$$(G(X,Y)R)(Z,T,U,V) = \frac{(n-2)}{n}Q(g,R)(Z,T,U,V,X,Y). \tag{3.17}$$

And from (2.3), (2.4) and (2.7), we have

$$\begin{aligned}(C(X,Y)R)(Z,T,U,V) = {} & (R(X,Y)R)(Z,T,U,V) \\ & - \frac{r}{n(n-1)}Q(g,R)(Z,T,U,V,X,Y).\end{aligned} \tag{3.18}$$

4. Concircular Semi-Symmetric Manifold and Quasi-Einstein Manifold

From equation (2.3), we can write

$$\begin{aligned}(R(X,Y)\,C)(Z,T,U,V) = {} & -C(R(X,Y)Z,T,U,V) - C(Z,R(X,Y)T,U,V) \\ & - C(Z,T,R(X,Y)U,T) - C(Z,T,U,R(X,Y)V).\end{aligned} \tag{4.1}$$

Using (2.7) in above equation, we have

$$\begin{aligned}(R(X,Y)\,C)(Z,T,U,V) = {} & -R(R(X,Y)Z,T,U,V) - R(Z,R(X,Y)T,U,V) \\ & - R(Z,T,R(X,Y)U,T) - R(Z,T,U,R(X,Y)V).\end{aligned} \tag{4.2}$$

Using (2.3) in (4.2), we get

$$(R(X,Y)\,C)(Z,T,U,V) = (R(X,Y)R)(Z,T,U,V). \tag{4.3}$$

The above equation shows that $R.C$ vanishes if and only if $R.R$ vanishes.
Hence we can state:

Theorem 4.1. *If M be an n-dimensional Riemannian manifold, then M be a concircular semi-symmetric manifold with respect to the the connection ∇ if and only if it is a semi-symmetric manifold with respect to ∇.*

Now we can compose:

Theorem 4.2. *Let (M, g) be a Riemannian manifold equipped with a semi-symmetric metric connection $\overline{\nabla}$. If the manifold M be a concircularly flat manifold with respect to $\overline{\nabla}$, then M is a quasi-Einstein manifold with respect to ∇.*

Proof. Let (M, g) be a Riemannian manifold equipped with a semi-symmetric metric connection $\overline{\nabla}$.
Then from equation (2.7), we have

$$\overline{C}(X,Y,Z,T) = \overline{R}(X,Y,Z,T) - \frac{\overline{r}}{n(n-1)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)]. \tag{4.4}$$

Using (3.2) and (3.11) in (4.4), we get

$$\begin{aligned}\overline{C}(X,Y,Z,T) =& R(X,Y,Z,T) + \omega(X)\omega(T)g(Y,Z) - \omega(Y)\omega(T)g(X,Z)\\ &+ \omega(Y)\omega(Z)g(X,T) - \omega(X)\omega(Z)g(Y,T)\\ &- \frac{(r+2n-2)}{n(n-1)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)].\end{aligned} \tag{4.5}$$

If the manifold M be concircularly flat with respect to semi-symmetric metric connection $\overline{\nabla}$, then from (4.5) we can write

$$\begin{aligned}R(X,Y,Z,T) =& \omega(Y)\omega(T)g(X,Z) - \omega(X)\omega(T)g(Y,Z)\\ &+ \omega(X)\omega(Z)g(Y,T) - \omega(Y)\omega(Z)g(X,T)\\ &+ \frac{(r+2n-2)}{n(n-1)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)].\end{aligned} \tag{4.6}$$

Putting $X = T = e_i$ and taking summation over i from 1 to n in (4.6), we get

$$S(Y,Z) = -\frac{(r-n^2+2n-2)}{n}g(Y,Z) - (n-2)\omega(Y)\omega(Z). \tag{4.7}$$

Hence the manifold M is a quasi-Einstein manifold. ■

5. Conformally-Flat Quasi-Einstein Manifold

Theorem 5.1. *Let (M, g) be a Riemannian manifold equipped with a semi-symmetric metric connection $\overline{\nabla}$. If ρ is a parallel unit vector field with respect to the connection ∇, then the following two relations hold:*

$$(i)\ (R(X,Y)\,\overline{C})(Z,T,U,V) = (R(X,Y)\,C)(Z,T,U,V), \tag{5.1}$$

and

$$(ii)\ (\overline{C}(X,Y)\,R)(Z,T,U,V) = (C(X,Y)\,R)(Z,T,U,V) - Q(\frac{2}{n}\,g - \omega\otimes\omega, R)(Z,T,U,V,X,Y). \tag{5.2}$$

Proof. Since ρ is a parallel unit vector field, we have

$$(R(X,Y)\,\theta)(Z,T) = 0 \text{ and } (R(X,Y)\,G)(Z,T,U,V) = 0. \tag{5.3}$$

Now, from (2.3) and (3.13), we can write

$$(R(X,Y)\,\overline{C})(Z,T,U,V) = (R(X,Y)\,C)(Z,T,U,V) - g(Z,T)\barwedge(R(X,Y)\,\theta)(U,V) - (R(X,Y)\,G)(Z,T,U,V). \tag{5.4}$$

Using (5.3) in (5.4), we get

$$(R(X,Y)\,\overline{C})(Z,T,U,V) = (R(X,Y)\,C)(Z,T,U,V). \tag{5.5}$$

Hence the first part of the theorem is completed.
Again from (2.3) and (3.13), we can write

$$(\overline{C}(X,Y)\ R)(Z,T,U,V) = -R(\overline{C}(X,Y)Z,T,U,V) - R(Z,\overline{C}(X,Y)T,U,V) - R(Z,T,\overline{C}(X,Y)U,V) - R(Z,T,U,\overline{C}(X,Y)V). \tag{5.6}$$

Using (3.15) in (5.6), we have

$$\begin{aligned}(\overline{C}(X,Y)\ R)(Z,T,U,V) &= (C(X,Y)\ R)(Z,T,U,V)\\ &+ \omega(X)\omega(Z)R(Y,T,U,V) - \omega(Y)\omega(Z)R(X,T,U,V)\\ &+ \omega(X)\omega(T)R(Z,Y,U,V) - \omega(Y)\omega(T)R(Z,X,U,V)\\ &+ \omega(X)\omega(U)R(Z,T,Y,V) - \omega(Y)\omega(U)R(Z,T,X,V)\\ &+ \omega(X)\omega(V)R(Z,T,U,Y) - \omega(Y)\omega(V)R(Z,T,U,X)\\ &- \frac{2}{n}[g(X,Z)R(Y,T,U,V) - g(Y,Z)R(X,T,U,V)\\ &+ g(X,T)R(Z,Y,U,V) - g(Y,T)R(Z,X,U,V)\\ &+ g(X,U)R(Z,T,Y,V) - g(Y,U)R(Z,T,X,V)\\ &+ g(X,V)R(Z,T,U,Y) - g(Y,V)R(Z,T,U,X)].\end{aligned} \tag{5.7}$$

Using (2.4) in (5.7), we get

$$(\overline{C}(X,Y)\ R)(Z,T,U,V) = (C(X,Y)\ R)(Z,T,U,V) - Q(g - \omega\otimes\omega, R)(Z,T,U,V,X,Y) + (G(X,Y)\ R)(Z,T,U,V). \tag{5.8}$$

Using (3.17) in (5.8), we have

$$(\overline{C}(X,Y)\ R)(Z,T,U,V) = (C(X,Y)\ R)(Z,T,U,V) - Q(\frac{2}{n}g - \omega\otimes\omega, R)(Z,T,U,V,X,Y). \tag{5.9}$$

Hence the second part of the theorem is completed. ∎

Now, from above theorem, we can state:

Theorem 5.2. *Let (M, g) be a Riemannian manifold equipped with a semi-symmetric metric connection $\overline{\nabla}$. If ρ is a parallel unit vector field with respect to the connection ∇, then M be a concircular semi-symmetric manifold if and only if*

$$(R(X,Y)\,\overline{C})(Z,T,U,V) = 0. \tag{5.10}$$

Theorem 5.3. *Let (M, g) be an $n > 3$ dimensional semi-symmetric Riemannian manifold equipped with a semi-symmetric metric connection $\overline{\nabla}$. If ρ is a parallel unit vector field with respect to the connection ∇ and $(\overline{C}(X,Y)\,R)(Z,T,U,V) = 0$ then M be a conformally flat quasi-Einstein manifold.*

Proof. Since $(\overline{C}(X,Y)\,R)(Z,T,U,V) = 0$, from equation (5.2) we can write

$$(C(X,Y)\,R)(Z,T,U,V) = Q(\frac{2}{n}g - \omega\otimes\omega, R)(Z,T,U,V,X,Y). \tag{5.11}$$

Using (3.18) in (5.11), we have

$$\begin{aligned}(R(X,Y)\,R)(Z,T,U,V) &- \frac{r}{n(n-1)}Q(g,R)(Z,T,U,V,X,Y)\\ &= Q(\frac{2}{n}g - \omega\otimes\omega, R)(Z,T,U,V,X,Y).\end{aligned} \tag{5.12}$$

Since the manifold M is semi-symmetric, from (5.12) we have

$$\frac{r}{n(n-1)}Q(g,R)(Z,T,U,V,X,Y) + Q(\frac{2}{n}g - \omega\otimes\omega, R)(Z,T,U,V,X,Y) = 0. \tag{5.13}$$

From (5.13), we can write easily

$$Q(\frac{(r+2n-2)}{n(n-1)}\,g - \omega\otimes\omega, R)(Z,T,U,V,X,Y) = 0. \tag{5.14}$$

equation (5.14) implies that one of the following two conditions may hold,
either (i) rank $(\frac{(r+2n-2)}{n(n-1)}\,g - \omega\otimes\omega)(Z,T) = 1$,
or (ii) rank $(\frac{(r+2n-2)}{n(n-1)}\,g - \omega\otimes\omega)(Z,T) \geq 1$.
But the first condition can never hold, because if (i) hold, then at a point x

$$\frac{(r+2n-2)}{n(n-1)}g - \omega\otimes\omega = \lambda z\otimes z, \tag{5.15}$$

for $z \in T_X M$ and $\lambda \in R$.
which gives contradiction due to non-zero coefficient of g.
Because if $\frac{(r+2n-2)}{n(n-1)} = 0$ then we have $r = -2(n-1)$ which gives negative values of r for $n > 3$. Hence the condition (ii) must hold and then from Lemma (2.1) we can write

$$R(X,Y,Z,T) = \frac{\mu}{2}(g - \omega\otimes\omega)\wedge(g - \omega\otimes\omega)(X,Y,Z,T), \tag{5.16}$$

for $\mu(\neq 0) \in R$.
Again using Lemma (2.2), we have

$$W(X,Y,Z,T) = 0. \tag{5.17}$$

Hence the manifold M is a conformally-flat manifold.
Now putting $Z = V = e_i$ in (5.14) and taking summation over i from 1 to n, we get

$$\begin{aligned} &\frac{(r+2n-2)}{n(n-1)}[g(X,T)S(Y,U) - g(Y,T)S(X,U) \\ &+ g(X,U)S(Y,T) - g(Y,U)S(T,X)] \\ &+ \omega(Y)\omega(T)S(X,U) - \omega(X)\omega(T)S(Y,U) \\ &+ \omega(Y)\omega(U)S(T,X) - \omega(X)\omega(U)S(T,Y) = 0 \end{aligned} \tag{5.18}$$

Again putting $X = T = e_i$ in equation (5.18) and summing over i from 1 to n, we have

$$S(Y,U) = \frac{r}{(r+n-1)}[\frac{(r+2n-2)}{n}g(Y,U) - (n-1)\omega(Y)\omega(U)]. \tag{5.19}$$

Equations (5.17) and (5.19) imply, the manifold M is a conformally-flat quasi-Einstein manifold. ∎

Theorem 5.4. *Let (M,g) be an $n > 3$ dimensional Riemannian manifold equipped with a semi-symmetric metric connection $\overline{\nabla}$. If ρ is a parallel unit vector field with respect to the connection ∇ and $(R(X,Y)\,\overline{C})(Z,T,U,V) - (\overline{C}(X,Y)\,R)(Z,T,U,V) = 0$, then M be a conformally-flat quasi-Einstein manifold.*

Proof. Since $(R(X,Y)\,\overline{C})(Z,T,U,V) - (\overline{C}(X,Y)\,R)(Z,T,U,V) = 0$,
from (5.1) and (5.2), we have

$$\begin{aligned} &(R(X,Y)\,C)(Z,T,U,V) - (C(X,Y)\,R)(Z,T,U,V) \\ &+ Q(\frac{2}{n}\,g - \omega\otimes\omega, R)(Z,T,U,V,X,Y) = 0. \end{aligned} \tag{5.20}$$

Using (3.18) and (4.3) in (5.20), we get

$$Q(\frac{(r+2n-2)}{n(n-1)}\,g - \omega\otimes\omega, R)(Z,T,U,V,X,Y) = 0. \tag{5.21}$$

Now using the method of proof of theorem (5.3), we can prove easily that the manifold M is a conformally-flat quasi-Einstein manifold. ∎

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