Thai Journal of Mathematics
Volume 19 Number 4 (2021)
Pages 1187–1197

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Approximation of Jain Operators by Statistical Convergence

Nurhayat İspir[1], **Naokant Deo**[2] **and Neha Bhardwaj**[3,*]

[1] *Gazi University, Sciences & Arts Faculty Department of Mathematics, 06500 Teknikokullar/Ankara, Turkey*
e-mail : nispir@gazi.edu.tr

[2] *Department of Applied Mathematics, Delhi Technological University*
(Formerly Delhi College of Engineering), Bawana Road, Delhi 110042, India
e-mail : naokantdeo@dce.ac.in

[3] *Department of Mathematics, School of Basic Sciences and Research*
Sharda University, Greater Noida 201310, India.
e-mail : neha_bhr@yahoo.co.in

Abstract In this paper, we consider a positive linear operators $P_n^{[\beta]}$ introduced by Jain [G.C. Jain, Approximation of functions by a new class of linear positive operators, Jour. Austral. Math. Soc. 13 (3) (1972) 271–276] with the help of Poisson type distribution and study the Voronovskaya type result of the operator then obtain an error estimate in terms of the higher order modulus of continuity of the function being approximated and its A-statistical convergence. We also compute the corresponding rate of A-statistical convergence for these operators.

MSC: 41A10; 41A25; 41A30; 41A63; 26A15

Keywords: Poisson distribution; Voronovskaya; A-statistical convergence

Submission date: 18.02.2017 / Acceptance date: 25.10.2021

1. Introduction and Definitions

Several extension and generalization of Bernstein polynomials have been given by various mathematician like Szâsz [1], Meyer-König and Zeller [2], Meir and Sharma [3], Stancu [4] and Balázs [5]. In [6], Kajla and Acar have studied the blending type approximation properties by generalized Bernstein-Durrmeyer type operators. Deo and Bhardwaj [7] considered the multidimensional Bernstein operators and its Durrmeyer variants on simplex. Mirakyan [8] has also given another modification with the help of the Poisson distribution.

Later on in the same way with the help of a Poisson type distribution,

$$w_\beta(k,\alpha) = \frac{\alpha}{k!}(\alpha + k\beta)^{k-1} e^{-(\alpha+k\beta)},\ k \in N_0 = \{0\} \cup \mathbb{N},$$

*Corresponding author.

Published by The Mathematical Association of Thailand.

for $0 < \alpha < \infty$ and $|\beta| < 1$, Jain [9] defined the following class of positive linear operators,

$$P_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} w_\beta(k;nx) f\left(\frac{k}{n}\right), \; x \geq 0, \tag{1.1}$$

where $\beta \in [0,1)$ and $f \in C(\mathbb{R}_+)$, the space of all real valued continuous functions defined on $\mathbb{R}_+ = [0,\infty)$. Notice that the parameter β may depend on the natural number n. Original Szâsz-Mirakyan operator can easily be obtained for $\beta = 0$. Deo et al. [10, 11] studied another modification of Bernstein operators and Gupta [12] introduced q analogue of Bernstein operator. For better approximation of well-known Szâsz-Mirakyan operators [1], Jain and Pethe [13] generalized the operators as:

$$S_n^{[\gamma]}(f;x) = (1+n\gamma)^{-x/\gamma} \sum_{k=0}^{\infty} \left(\gamma + \frac{1}{n}\right)^{-k} \frac{x^{(k,-\gamma)}}{k!} f\left(\frac{k}{n}\right),$$

where $x^{(k,-\gamma)} = x(x+\gamma)\ldots(x+(k-1)\gamma), x^{(0,-\gamma)} = 1$ and f is any function of exponential type such that $|f(t)| \leq Ke^{Mt}\,(t \geq 0)$ for some finite constants $K, M > 0$ and $\gamma = (\gamma_n)_{n\in\mathbb{N}}$.

Very recently, Bodur [14] introduced a new modification of modified Lupas-Jain operators based on function ϕ having some properties. Various researchers worked on Jain's operators, one can refer ([15], [16], [17]).

Now the following lemmas follow from [9], for the operators $P_n^{[\beta]}$ mentioned by (1.1).

Lemma 1.1 ([9]). *Let $e_i(x) = x^i$, $i = 0,1,2$, then for fixed $x \in [0,\infty)$, $n \in \mathbb{N}$ and $\beta \in [0,1)$, we have*

(i) $P_n^{[\beta]}(e_0;x) = 1$,
(ii) $P_n^{[\beta]}(e_1;x) = \frac{1}{1-\beta}x$,
(iii) $P_n^{[\beta]}(e_2;x) = \frac{1}{(1-\beta)^2}x^2 + \frac{1}{n(1-\beta)^3}x$.

Lemma 1.2. *For $x \in [0,\infty)$, $n \in \mathbb{N}$, $\beta \in [0,1)$ and $\varphi_x(t) = t - x$, we have*

(i) $P_n^{[\beta]}(\varphi_x;x) = \frac{\beta}{1-\beta}x$,
(ii) $P_n^{[\beta]}(\varphi_x^2;x) = \frac{\beta^2}{(1-\beta)^2}x^2 + \frac{1}{n(1-\beta)^3}x$.

This paper is organised in four sections. The first section is devoted to definitions and auxiliary properties of operators $P_n^{[\beta]}$ given by (1.1). In the second section, we obtain the Voronovskaya type result of the operator and an error estimate of the function being approximated. In the third section, we give some definitions and properties of statistical and A-statistical convergence followed by statistical approximation theorems. In the last section, we compute the corresponding A-statistical rates of approximation and its order.

2. Voronovskaya Type Results & Error Estimation

In this section we compute the Voronovskaya type results of these operators $P_n^{[\beta]}$ given by (1.1).

Let $f \in C_B[0,\infty)$ be the space of all real valued functions with bounded and uniformly continuous on $[0,\infty)$, equipped with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. The Peetre's K_2-functional is defined by

$$K_2 = \inf\left\{\|f-g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\right\}, \quad \delta > 0$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$ and the norm

$$\|f\|_{C_B^2[0,\infty)} = \|f\|_{C_B} + \|f'\|_{C_B} + \|f''\|_{C_B}. \tag{2.1}$$

From [18], there exists a positive constant C such that

$$K_2(f,\delta) \le C\omega_2\left(f,\sqrt{\delta}\right) \tag{2.2}$$

and

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Theorem 2.1. *Let $f \in C_B[0,\infty)$, and $\beta \to 0$ as $n \to \infty$, then for every $x \in [0,\infty)$, $\beta \in [0,1)$ and for $C > 0$, we have*

$$\left|P_n^{[\beta]}(f;x) - f(x)\right| \le C\omega_2\left(f, \sqrt{\frac{\beta}{1-\beta}x}\right).$$

Proof. Let $g \in C_B^2$. Using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

From Lemma 1.2, we have

$$P_n^{[\beta]}(g;x) - g(x) = P_n^{[\beta]}(g'(x)(t-x);x) + P_n^{[\beta]}\left(\int_x^t (t-u)g''(u)du; x\right).$$

We know that

$$\left|\int_x^t (t-u)g''(u)du\right| \le (t-u)^2\|g''\|.$$

Therefore

$$\begin{aligned}\left|P_n^{[\beta]}(g;x) - g(x)\right| &\le P^{[\beta]}((t-x);x)\|g'\| + P_n^{[\beta]}\left((t-u)^2;x\right)\|g''\| \\ &= \frac{\beta}{1-\beta}x\|g'\| + \left(\frac{\beta^2x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3}\right)\|g''\| \\ &\le \frac{\beta}{1-\beta}x\{\|g'\| + \|g''\|\} \\ &\le \frac{\beta}{1-\beta}x\|g''\|.\end{aligned}$$

By Lemma 1.1, we have

$$\left|P_n^{[\beta]}(g;x)\right| \le \sum_{k=0}^{\infty}\frac{\alpha}{k!}(\alpha+k\beta)^{k-1}e^{-(\alpha+k\beta)}g\left(\frac{k}{n}\right) \le \|g\|.$$

Hence

$$\begin{aligned}\left|P_n^{[\beta]}(g;x)-g(x)\right| &\leq \left|P_n^{[\beta]}((g-f);x)-(g-f)(x)\right|+\left|P_n^{[\beta]}(g;x)-g(x)\right| \\ &\leq 2\|g-f\|+\left(\frac{\beta}{1-\beta}x\right)\|g''\|.\end{aligned}$$

Taking the infimum on the right side over all $g \in C_B^2$ and using (2.2), we get the required result. ∎

3. A-Statistical Convergence

In this section of the paper, we use concept of statistical convergence and study the Korovkin type approximation theorem for the operators $P_n^{[\beta]}$. Before we present the main results, we shall recall some definitions and results on the statistical and A-statistical convergence. In the year 1951, Fast [19] studied the definition of statistical convergence for sequences of real numbers as:

Let K be a subset of natural numbers $\mathbb{N}$ and $K_j = \{n \leq j : n \in \mathbb{N}\}$. If limit exists, then the natural density of K is defined by

$$\delta(K) = \lim_j \frac{1}{j}|K_j|,$$

where vertical bars indicating the number of elements in the enclosed set. A sequence $(x_n)_n$ is said to be statistically convergent to a number L, if for every $\varepsilon > 0$, the natural density of the set

$$K_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$$

is zero, i.e.,

$$\lim_n \frac{1}{n}|\{n \leq j : |x_n - L| \geq \varepsilon\}| = 0.$$

We denote this statistical limit by $st - \lim_n x_n = L$ (see [19–22]).

Let $A = (a_{jn})$, $j, n = 1, 2, ...$, be a non-negative regular summability matrix. A sequence $(x_n)_n$ is said to be $A-$statistically convergent to a number L, if for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0,$$

holds. We denote this limit by $st_A - lim_n x_n = L$. We note that by taking $A = C_1$, the Cesàro matrix, $A-$statistical convergence reduce to ordinary convergence. Kolk [21] studied $A-$statistical convergence is stronger than ordinary convergence.

$A-$statistical convergence may also be given in normed spaces. Suppose $(X, \|.\|)$ is a normed space and $v = (v_k)$ is a sequence. This sequence is $A-$statistically convergent to $v_0 \in X$, if for every $\varepsilon > 0, \delta_A \{k \in \mathbb{N} : \|v_k - v_0\| \geq \varepsilon\} = 0$ (see [23, 24]).

Now $A = [a_{jn}]$ $(j, n \in \mathbb{N})$ is a non-negative regular summability matrix. Assume that for each $t \in [0, \infty)$, $(\alpha_n^*(t))_{n\in\mathbb{N}}$ is a sequence in $[0, \infty)$ satisfying

$$st_A - \lim_n \alpha_n^*(t) = t$$

then we have

$$st_A - \lim_n (t - \alpha_n^*(t)) = 0.$$

Gadjiev and Orhan [25] have explored a new research area, the use of statistical convergence in approximation theory and they investigated the Korovkin type approximation theorems via statistical convergence. Later on Korovkin type approximation theorems are introduced for A-statistical convergence by Duman et al. [26]. İspir and Gupta [27] studied approximation properties and estimate the rates of $A-$statistical convergence of Kantorovich variant of generalized Bernstein type operators. Using A-statistical and statistical convergence some approximating operators are studied by several mathematicians [28, 29].

In this section $A = [a_{jn}]$ is a non-negative regular summability matrix and β, given in definition of (1.1), will denote the sequence $\beta := \{\beta_n\}$ such that

$$st_A - \lim_n \beta_n = 0 \tag{3.1}$$

with $\beta_n \in [0, 1)$. Notice that, the present condition (3.1) includes the given condition for β of Theorem (2.1) in [9], its converse is not true. For example, if we define a sequence (β_n) such that
$\beta_n := \begin{cases} \sqrt{n}, & \text{if } n \text{ is a square,} \\ n & \text{otherwise} \end{cases}$ then observe that (3.1) holds true with the choice of $A = C_1$, the Cesaro matrix; but β_n is a non-convergent sequence in the usual sense.

Theorem 3.1. *Let $A = [a_{jn}]$ is a non-negative regular summability matrix and let $\beta := \{\beta_n\}$ be a sequence satisfying the condition (3.1) with $\beta_n \in [0, 1)$ for all $n \in \mathbb{N}$. Then, for every $f \in C[0, \infty)$ and for each compact interval $[0, b] \subset [0, \infty)$ uniformly*

$$st_A - \lim_n \left| P_n^{[\beta]}(e_i; x) - e_i(x) \right| = 0; \; e_i(t) = t^i, i = 0, 1, 2.$$

Proof. From Lemma 1.1, obviously $st_A - \lim_n \left| P_n^{[\beta]}(e_0; x) - 1 \right| = 0.$

$$\left| P_n^{[\beta]}(e_1; x) - x \right| = \left| \frac{\beta_n x}{1 - \beta_n} \right| \leq b \frac{\beta_n}{1 - \beta_n}, \quad x \in [0, b] \subset [0, \infty), \tag{3.2}$$

and from hypothesis we get $st_A - \lim_n \frac{\beta_n}{1-\beta_n} = 0.$
Now, for a given $\varepsilon > 0$, we define

$$V = \left\{ n : \frac{\beta_n}{1 - \beta_n} \geq \frac{\varepsilon}{b} \right\},$$

Therefore, by (3.2), we obtain

$$\sum_{n: \left| P_n^{[\beta]}(e_1, x) - x \right| \geq \varepsilon} a_{jn} \leq \sum_{n \in V} a_{jn}.$$

Taking limit as $j \to \infty$ and from the above inequality we get the result.
Similarly, for each $x \in [0,b] \subset [0,\infty)$

$$\begin{aligned}
\left|P_n^{[\beta]}(e_2;x) - x^2\right| &= \left|\frac{x^2}{(1-\beta_n)^2} + \frac{x}{n(1-\beta_n)^3} - x^2\right| \\
&= \left|\frac{x^2\beta_n(2-\beta_n)}{(1-\beta_n)^2} + \frac{x}{n(1-\beta_n)^3}\right| \\
&\leq b^2\frac{\beta_n(2-\beta_n)}{(1-\beta_n)^2} + b\frac{1}{n(1-\beta_n)^3} \\
&\leq A\left\{\frac{\beta_n(2-\beta_n)}{(1-\beta_n)^2} + \frac{1}{n(1-\beta_n)^3}\right\}, \qquad (3.3)
\end{aligned}$$

where $A = \max\left\{b^2, b\right\}$. Now, for a given $\varepsilon > 0$, define

$$\begin{aligned}
T &= \left\{n : \frac{\beta_n(2-\beta_n)}{(1-\beta_n)^2} + \frac{1}{n(1-\beta_n)^3} \geq \frac{\varepsilon}{A}\right\}, \\
T_1 &= \left\{n : \frac{\beta_n(2-\beta_n)}{(1-\beta_n)^2} \geq \frac{\varepsilon}{2b^2}\right\} \\
T_2 &= \left\{n : \frac{1}{n(1-\beta_n)^3} \geq \frac{\varepsilon}{2b}\right\}.
\end{aligned}$$

It is easy to see that $T \subseteq T_1 \cup T_2$. Since $st_A - \lim_n \beta_n = 0$ we get $st_A - \lim_n \frac{\beta_n(2-\beta_n)}{(1-\beta_n)^2}$, $st_A - \lim_n \frac{1}{n(1-\beta_n)^3}$. Therefore by (3.3), we obtain

$$\sum_{n:\left|P_n^{[\beta]}(e_2,x)-x^2\right|\geq\varepsilon} a_{jn} \leq \sum_{n\in T} a_{jn} \leq \sum_{n\in T_1} a_{jn} + \sum_{n\in T_2} a_{jn}.$$

Taking limit as $j \to \infty$, we reach the result. ∎

Similarly, from Lemma 1.1 and by $st_A - \lim_n \beta_n = 0$ we have for each compact interval $[0,b] \subset [0,\infty)$

$$st_A - \lim_n \left\|P_n^{[\beta]}(e_1 - xe_0)^j\right\|_{C[0,b]} = 0, \ \ j = 1,2.$$

Now we give a Korovkin type theorem for the operators $P_n^{[\beta]}(f;x)$ via A-statistical convergence.

Theorem 3.2. *Let $A = [a_{jn}]\,(j,n \in N)$ is a non-negative regular summability matrix and let us suppose that $\beta := \{\beta_n\}$ is a sequence satisfying the condition (3.1) with $\beta_n \in [0,1)$ for all $n \in N$. Then, for each $x \in [0,b] \subset [0,\infty)$ and for every $f \in C[0,\infty)$, we have*

$$st_A - \lim_n \left\|P_n^{[\beta]}(f;x) - f(x)\right\|_{C[0,b]} = 0.$$

Proof. The result follows by Theorem 3.1 and from Theorem 1 in [25] (see also [26]). We recall that Theorem 1 in [25] is given for statistical convergence but the proof also works for A-statistical convergence. ∎

If we take $A = I$, identity matrix, then we have Theorem 2.1 in [9].

Remark. *We now present an example of a sequence of positive linear operators satisfying the conditions of Theorem 3.2 but that does not satisfy the conditions of Theorem 2.1 of* [9].

Assume now that $\{u_n\}$ *is an* A*-statistically null sequence but not convergent. Notice that, if* $A = (a_{jn})$ *is a non-negative regular matrix such that* $\lim_j \max_n \{a_{jn}\} = 0$, *then* A*-statistical convergence is stronger than convergence* [21]. *Without loss of generality we may assume that* $\{u_n\}$ *is a non-negative; otherwise we would replace* $\{u_n\}$ *by* $\{|u_n|\}$. *Now define* $\{L_n\}$ *on* $C[0,\infty)$ *by*

$$L_n(f;x) = (1+u_n)P_n^{[\beta]}(f;x),$$

where $\left\{P_n^{[\beta]}\right\}$ *is the sequence of the operators given with* (1.1). *Now observe that* $\left\{P_n^{[\beta]}\right\}$ *being convergent and* $\{u_n\}$ *being* A*-statistical null, their product will also be* A*-statistical null. Hence* $\{L_n\}$ *will not be convergent to* f *but will be* A*-statistically convergent to* f.

4. Rate of A-Statistical Convergence

Now we give the rate of A-statistical convergence for the operators $P_n^{[\beta]}(f;x)$ by using the Peetre's K-functional in the space $C_B^2[0,\infty)$.

Theorem 4.1. *Let* $A = [a_{jn}]\,(j,n \in N)$ *is a non-negative regular summability matrix and let* $\beta := \{\beta_n\}$ *is a sequence satisfying the condition* (3.1) *with* $\beta_n \in [0,1)$ *for all* $n \in N$. *For each* $f \in C_B[0,\infty)$

$$\left\|P_n^{[\beta]}(f;x) - f(x)\right\|_{C_B} \leq \kappa\left(f;\gamma_{n,x}\right),$$

where $\kappa\left(f;\gamma_{n,x}\right)$ *is the sequence of Peetre's* K*-functional and*

$$\gamma_{n,x} = \left\|P_n^{[\beta]}((e_1-x);x)\right\|_{C_B} + \left\|P_n^{[\beta]}((e_1-x)^2;x)\right\|_{C_B}$$

and $st_A - \lim_n \gamma_{n,x} = 0$ *for each fixed* $x \in [0,\infty)$.

Proof. Applying Taylor expansion to the function $f \in C_B^2[0,\infty)$, we get

$$P_n^{[\beta]}(f,x) - f(x) = f'(x)P_n^{[\beta]}((e_1-x);x) + \frac{1}{2}f''(\xi)P_n^{[\beta]}((e_1-x)^2;x),\ \ \xi \in (t,x).$$

Hence

$$\left\|P_n^{[\beta]}(f;x) - f(x)\right\|_{C_B}$$
$$\leq \|f'\|_{C_B}\left\|P_n^{[\beta]}((e_1-x);x)\right\|_{C[0,\infty)} + \|f''\|_{C_B}\left\|P_n^{[\beta]}((e_1-x)^2;x)\right\|_{C[0,\infty)}. \tag{4.1}$$

Using (2.1) and (4.1), for each $g \in C_B^2[0,\infty)$,

$$\left\|P_n^{[\beta]}(g;x)-g(x)\right\|_{C_B^2} = \left(\left\|P_n^{[\beta]}((e_1-x);x)\right\|_{C[0,\infty)} + \left\|P_n^{[\beta]}((e_1-x)^2;x)\right\|_{C[0,\infty)}\right)\|g\|_{C_B^2}$$
$$= \gamma_{n,x}\|g\|_{C_B^2}.$$

For each $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$,

$$\begin{aligned}\left\|P_n^{[\beta]}(f;x)-f(x)\right\|_{C_B^2} &\leq \left\|P_n^{[\beta]}(f;x)-P_n^{[\beta]}(g;x)\right\|_{C_B}+\left\|P_n^{[\beta]}(g;x)-g(x)\right\|_{C_B^2}+\|g-f\|_{C_B} \\ &\leq 2\|g-f\|_{C_B}+\left\|P_n^{[\beta]}(g;x)-g(x)\right\|_{C_B^2} \\ &\leq 2\|g-f\|_{C_B}+\gamma_{n,x}\|g\|_{C_B^2} \leq 2\left(\|g-f\|_{C_B}+\gamma_{n,x}\|g\|_{C_B^2}\right).\end{aligned}$$

Taking infimum over $g \in C_B^2[0,\infty)$, we get

$$\left\|P_n^{[\beta]}(f,x)-f(x)\right\|_{C_B^2} \leq \kappa(f;\gamma_{n,x}).$$

Since $st_A - \lim_n \beta_n = 0$, with $\beta_n \in [0,1)$,
we get $st_A - \lim_n \gamma_{n,x} = 0$, therefore $st_A - \lim_n \kappa(f;\gamma_{n,x}) = 0$. ■

Now following [26], we would like to find the order of A-statistical approximation for the sequence of operators $\left\{P_n^{[\beta]}\right\}$ given by (1.1).

We recall some definitions and notations. Let $A = (a_{nk})$ is a non-negative regular summability matrix and let (b_j) be positive non-increasing sequence. A sequence $x = (x_n)$ is called $A-$statistical convergent to the number L with the rate of $o(b_j)$ if for every $\varepsilon > 0$, $\lim_j \frac{1}{b_j} \sum_{n:\,|x_n-L|\geq\varepsilon} a_{jn} = 0$. In this case we write $x_n - L = st_A - o(b_n)$, (as $n \to \infty$). If for every $\varepsilon > 0$, $\sup_j \frac{1}{b_j} \sum_{n:|x_n|\geq\varepsilon} a_{jn} < \infty$, then x is $A-$statistically bounded with the rate of $O(b_n)$ and it is denoted by $x_n = st_A - O(b_n)$, as $n \to \infty$, [26].

In the above two definitions the "rate" is controlled more by the entries of the summability method than by the terms of the sequence $x = (x_n)$. For example if $A = (a_{jn}) = I$ the identity matrix and if $a_{nn} = o(b_n)$ then $x_n - L = st_A - o(b_n)$ for any convergent sequence $(x_n - L)$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation, considering the concept of convergence in measure from measure theory, the following two extra definitions are introduced in [26].

The sequence $x = (x_n)$ is $A-$statistically convergent to L with the rate of $o_\mu(b_n)$, denote by $x_n - L = st_A - o_\mu(b_n)$, (as $n \to \infty$), if for every $\varepsilon > 0$, $\lim_j \sum_{n:\,|x_n-L|\geq\varepsilon b_n} a_{jn} = 0$. Finally, the sequence $x = (x_n)$ is $A-$statistically bounded with the rate of $O\mu(b_n)$ provided that there is a positive number M such that $\lim_j \sum_{n:\,|x_n|\geq M\ b_n} a_{jn} = 0$. In this case we write $x_n = st_A - O\mu(b_n)$, as $n \to \infty$. In this case we write

$$x_k - L = st_A - o(a_n), \quad (\text{as } k \to \infty).$$

Theorem 4.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and $(b_n(x))$ non-increasing sequence. If the sequence of positive linear operators $\left\{P_n^{[\beta]}\right\}$ defined by (1.1) and*

$$\omega(f;\delta_{n,x}) = st_A - o(b_n(x)) \text{ with } \delta_{n,x} = \sqrt{P_n^{[\beta]}((e_1 - x)^2;x)} \tag{4.2}$$

then

$$P_n^{[\beta]}(f;x) - f(x) = st_A - o(b_n(x)),$$

where $\omega(f,\delta_n(x))$ *is the usual modulus of continuity of the function* f *for fixed* $x \geq 0$. *Similar results hold with little "o" replaced by big "O".*

Proof. Since $P_n^{[\beta]}(e_0;x) = 1 = e_0(x)$ using Cauchy-Schwarz inequality we can write

$$\left|P_n^{[\beta]}(f;x) - f(x)\right| \leq \left[P_n^{[\beta]}(e_0;x) + \frac{1}{\delta_{n,x}}\left(P_n^{[\beta]}((e_1-x)^2;x)\right)^{1/2}\right]\omega(f,\delta_{n,x}).$$

Hence with choosing $\delta_{n,x} = \sqrt{L_n((e_1-x)^2;x)}$ we obtain

$$\left|P_n^{[\beta]}(f;x) - f(x)\right| \leq 2\omega(f,\delta_{n,x}).$$

This implies that

$$\frac{1}{b_n(x)}\sum_{n:\left|P_n^{[\beta]}(f,x)-f(x)\right|\geq\varepsilon} a_{jn} \leq \frac{1}{b_n(x)}\sum_{n:\,2\omega(f;\delta_{n,x})\geq\varepsilon/2} a_{jn}.$$

The equality (4.2) gives the proof. ■

Replacing "o" by "o_μ" one can get the following result immediately.

Theorem 4.3. *Let* $A = (a_{jn})$, $(b_n(x)), \beta := \{\beta_n\}$ *and* $\left\{P_n^{[\beta]}\right\}$ *be the same as in Theorem 3.4. Then*

$$P_n^{[\beta]}(f;x) - f(x) = st_A - o_\mu(b_n(x))$$

where $\omega(f;\delta_{n,x}) = st_A - o_\mu(b_n(x))$ *with* $\delta_{n,x} = \sqrt{P_n^{[\beta]}((e_1-x)^2;x)}$ *for fixed* $x \in [0,\infty)$. *Similar conclusions hold when little "o_μ" is replaced by big "O_μ".*

Remark. Notice that when $A = (a_{jn})$ is replaced by the identity matrix in Theorem 3.4 or Theorem 3.5 by choosing $\beta' := \left\{\beta'_n\right\} = \left\{\frac{1}{n}\right\}$, we get Theorem 3.1 in [9]. It is clear that $st_A - \lim_n \beta'_n = 0$.

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript. The third author is extremely thankful to Science and Engineering Research Board (SERB), Govt. of India, for providing the financial support under Teachers Associateship for Research Excellence (TARE) Award (TAR/2018/000356).

References

[1] O. Szâsz, Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Nat. Bur. Standards 45 (1950) 239–245.

[2] W. Meyer-König, K. Zeller, Bernsteinsche Potenzreihen, Studia Math. 19 (1960) 89–94.

[3] A. Mei, A. Sharma, Approximation methods by polynomials and power series, Indg. Math. 29 (1967) 395–403.

[4] D.D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rev. Roumaine Math. Pures Appl. XIII (8) (1968) 1173–1194.

[5] K. Balázs, Approximation by Bernstein type rational functions, Acta Math. Acad. Sci. Hungar. 26 (1–2) (1975) 123–134.

[6] A. Kajla, T. Acar, Blending type approximation by generalized Bernstein-Durrmeyer type operators, Miskolc Math. Notes 19 (1) (2018) 319–336.

[7] N. Deo, N. Bhardwaj, Some approximation theorems for multivariate Bernstein operators, Southeast Asian Bull. Math. 34 (2010) 1023–1034.

[8] G.M. Mirakyan, Approximation of continuous functions with the aid of polynomials (Russian), Docl. Akad. Nauk. SSSR 31 (1941) 201–205.

[9] G.C. Jain, Approximation of functions by a new class of linear positive operators, J. Aust. Math. Soc. 13 (3) (1972) 271–276.

[10] N. Deo, N. Bhardwaj, S.P. Singh, Simultaneous approximation on generalized Bernstein-Durrmeyer operators, Afr. Mat. 24 (1) (2011) 77–82.

[11] N. Deo, M.A. Noor, M.A. Siddiqui, On approximation by a class of new Bernstein type operators, Appl. Math. Comput. 201 (2008) 604–612.

[12] V. Gupta, Some approximation properties on q-Durrmeyer operators, Appl. Math. Comput. 197 (2008) 172–178.

[13] G.C. Jain, S. Pethe, On the generalizations of Bernstein and Szâsz-Mirakyan operators, Nanta. Math. 10 (1977) 185–193.

[14] M. Bodur, Modified Lupas-Jain operators, Math. Slovaca. 70 (2) (2020) 431–440.

[15] V.N. Mishra, P. Patel, Some approximation properties of modified Jain-Beta operators, J. Calc. Var. 2013 (2013) Article ID 489249.

[16] S. Tarabie, On Jain-Beta linear operators, Appl. Math. Inf. Sci. 6 (2) (2012) 213–216.

[17] B. Tunca, G. Bodur, D. Soylemez, On Lupaş-Jain operators, Stud. Univ. Babes-Bolyai Math. 63 (4) (2018) 525–537.

[18] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.

[19] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[20] J.A. Fridy, H.L. Miller, A matrix characterization of statistical convergence, Analysis 11 (1991) 59–66.

[21] E. Kolk, Matrix summability of statistically convergent sequences, Analysis 13 (1993) 77–83.

[22] H.I. Miller, A measure theoretical subsequence characterization of statistical convergence, Trans. Amer. Math. Soc. 347 (1995) 1811–1819.

[23] E. Kolk, Statistically convergent sequences in normed spaces, Reports of convergence, Methods of algebra and analysis, Acta Comment. Univ. Tartu. Math. (1998) 63–66.

[24] E. Kolk, The statistical convergence in Banach spaces, Acta Comment. Univ. Tartu. Math. 928 (1991) 41–52.

[25] A.D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32 (1) (2002) 129–138.

[26] O. Duman, M.K. Khan, C. Orhan, A-Statistical convergence of approximating operators, Math. Ineq. Appl. 6 (4) (2003) 689–699.

[27] N. İspir, V. Gupta, A-Statistical approximation by the generalized Kantorovich-Bernstein type rational operators, Southeast Asian Bull. Math. 32 (2008) 87–97.

[28] O. Dogru, O. Duman, C. Orhan, Statistical approximation by generalized Meyer-Köning and Zeller type operators, Studia Sci. Math. Hung. 40 (2003) 359–371.

[29] M. Mursaleen, V. Karakaya, M. Ertürk, F. Gürsoy, Weighted statistical convergence and its application to Korovkin type approximation theorem, Appl. Math. Comput. 218 (18) (2012) 9132–9137.