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4-Hyperclique Decompositions of 3-Hyperdistance Multihypergraphs of Multipartite Graphs

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Abstract This paper investigates hyperclique decompositions of a hypergraph transformation, namely the hyperdistance multihypergraph defined as follows: Given an integer $k \geq 3$ and any hypergraph H, the *k-hyperdistance multihypergraph* of H, denoted by $D^{(k)}(H)$, is the k-uniform hypergraph which has the same vertex set as H and for any k-subset $\{v_1,v_2,\ldots v_k\}$ of $V(H)$, $D^{(k)}(H)$ has exactly $\sum_{i\neq j} d_H(v_i,v_j)$ copies of hyperedges $\{v_1,v_2,\ldots,v_k\}$ where $d_H(v_i,v_j)$ is the distance between vertices v_i and v_j in H.

We study 4-hyperclique decompositions of $D^{(3)}(H)$ where H is a complete multipartite graph. Our construction technique includes classic designs such as Latin cubes and factorizations of graphs.

MSC: 05C65
Keywords: hyperclique decomposition; 3-uniform hypergraph; hyperdistance multihypergraph

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1. Introduction

A *hypergraph* H is an ordered $(V(H),E(H))$ where $V(H)$ is the vertex set and $E(H)$ is the hyperedge set which is a multi-set of non-empty subsets of $V(H)$. That is, any set of hyperedges can have repeated elements. A *multihypergraph* is a hypergraph with repeated hyperedges. A hypergraph H is *k-uniform* if each hyperedge of H contains exactly k vertices, denoted $H^{(k)}$. A *complete k-uniform hypergraph* with n ($\geq k$) vertices on the vertex n-set V (or *n-hyperclique*), denoted $K_n^{(k)}(V)$ or $K_n^{(k)}$, is the k-uniform hypergraph on n vertices such that all k-subsets of V form the hyperedge set. Hence the vertex set determines the hyperclique, we then can simply refer to any hyperclique as its vertex set. A *k-uniform complete m-partite hypergraph* on m partite sets, V_1, V_2, …,V_m, is the k-uniform hypergraph on the vertex set $V=V_1\cup V_2\cup\ldots\cup V_m$ whose hyperedge set consists of all k-subsets of V except those of the form $\{x_1,x_2,\ldots,x_k\}\subseteq V_i$ for some i, and it is denoted by $K^{(k)}_{n_1,n_2,\ldots,n_m}(V_1,V_2,\ldots,V_m)$ if $|V_i|=n_i$ for all i. If all m partite sets have the same size n, $K^{(k)}_{n,n,\ldots,n}$ is denoted by $K^{(k)}_{m(n)}$. Since 2-uniform hypergraphs are *graphs*, we will use the same notation without specified $k=2$, for example, K_n, $K_{m(n)}$.

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To define our hypergraph transformation, we recall the notion in terms of graphs. The *distance* between vertices u and v in graph G, denoted by $d_G(u,v)$, is the number of edges in a shortest path connecting u and v.

Definition 1.1. The *distance multigraph of a graph* G, denoted $D(G)$, is a graph with the same vertex set as G such that 2-subset of $V(D(G))$, namely $\{u,v\}$, forms exactly $d_G(u,v)$ copies of edges $\{u,v\}$ in $E(D(G))$.

A distance multigraph is a graph model to solve some real world problems. There are wide range of its applications to the area of communication network, for example finding a shortest route to transmit messages in a computer network. See more details regarding the model of the problem in [1]. In particular, it is the problem of decomposition of the distance multigraph of graph into bicliques [2], [3], [4]. In 2008, Cavers *et al.* [5] studied the decomposition of the distance multigraphs of a graph G into cliques where G is a path, cycle or complete multipartite graph. Boonthong [1] obtained further results on clique decompositions of the distance multigraph of the Cartesian product of graphs.

Later on, Boonthong [6] extended the notion of the distance multigraph into hypergraphs which is called the k-hyperdistance multihypergraphs. In hypergraphs, a *path* connecting vertices v_1 and v_{n+1} is a vertex-hyperedge alternative sequence $v_1e_1v_2e_2\ldots e_nv_{n+1}$ of distinct vertices $v_1, v_2, v_3, \ldots, v_{n+1}$ and distinct hyperedges $e_1, e_2, e_3, \ldots, e_n$ with possibility $v_1 = v_{n+1}$ such that v_i and v_{i+1} are elements in e_i for all $i \in \{1,2,3,\ldots,n\}$. The *distance* between vertices u and v in a hypergraph H, denoted by $d_H(u,v)$, is the number of hyperedges in a shortest path connecting u and v.

Definition 1.2. For any hypergraph H, the *k-hyperdistance multihypergraph of* H, denoted by $D^{(k)}(H)$, is the k-uniform hypergraph with the same vertex set as H such that any k-subset of $V(D^{(k)}(H))$, namely $\{v_1, v_2, \ldots, v_k\}$, forms exactly $\sum_{i\neq j} d_H(v_i,v_j)$ copies of hyperedge $\{v_1, v_2, \ldots, v_k\}$ in $E(D^{(k)}(H))$.

A *hyperclique decomposition* of a hypergraph H is a collection of hypercliques of H such that each hyperedge of H belongs to exactly one hyperclique in the collection. If each hyperclique in the collection is isomorphic to a hyperclique K, then the decomposition is called a *K-decomposition*.

In this paper we study a problem of hyperclique decompositions of $D^{(k)}(H)$ when $k = 3$ and $H = K_{m(n)}$. Since a $K_3^{(3)}$- decomposition of $D^{(3)}(K_{m(n)})$ always exists (by its hyperedge set), ones investigate a decomposition of a given 3-uniform hypergraph into k-hypercliques when $k \geq 4$. A necessary condition for the existence of a $K_4^{(3)}$-decomposition of a hypergraph is that the number of its hyperedges must be divisible by 4, for example, $D^{(3)}(K_{5,5,5})$ has no $K_4^{(3)}$-decomposition. Thus finding a $K_4^{(3)}$- decomposition of $D^{(3)}(K_{m(n)})$ is a nontrivial interesting problem.

A hyperedge e of $D^{(3)}(K_{m(n)})$ is called a *hyperedge of Type i* if e contains vertices from different i partite sets of $K_{m(n)}$. Any hyperedge of Type 1, Type 2 and Type 3 of $D^{(3)}(K_{m(n)})$ has six, four and three copies, respectively, in $E(D^{(3)}(K_{m(n)}))$.

We would like to emphasize here that any set of hyperedges or 4-hypercliques are multi-sets, and α copies of a set S is denoted by αS.

Example 1.3. There exists a $K_4^{(3)}$-decomposition of $D^{(3)}(K_{3,3})$.

Let $U = \{u_1,u_2,u_3\}$ and $V = \{v_1,v_2,v_3\}$ be two partite sets of $K_{3,3}$. Then the set of hyperedges of $D^{(3)}(K_{3,3})$ is $\{6\{u_1,u_2,u_3\}, 6\{v_1,v_2,v_3\}, 4\{u_i,u_j,v_k\},\ 4\{v_i,v_j,u_k\} : 1 \leq$

$i < j \leq 3$ and $1 \leq k \leq 3\}$. Consider $\mathscr{P}$ and $\mathscr{P}'$ as follows:

$$\mathscr{P} = \{\{v_i, v_j, u_s, u_t\} : 1 \leq i < j \leq 3 \text{ and } 1 \leq s < t \leq 3\},$$

$$\mathscr{P}' = \{2\{v_1, v_2, v_3, u_i\}, 2\{u_1, u_2, u_3, v_i\} : 1 \leq i \leq 3\}.$$

It can be verified that each two copies of distinct hyperedges of Type 2 in $D^{(3)}(K_{3,3})$ form 4-hypercliques in $\mathscr{P}$ and all hyperedges of Type 1 together with the remaining of hyperedges of Type 2 form 4-hypercliques in $\mathscr{P}'$. Hence $\mathscr{P} \cup \mathscr{P}'$ is a $K_4^{(3)}$-decomposition of $D^{(3)}(K_{3,3})$. □

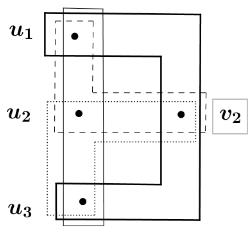


FIGURE 1. 4-hyperclique $\{u_1, u_2, u_3, v_2\}$ in $\mathscr{P}'$ containing 4 hyperedges.

Remark that $D^{(3)}(K_{m(n)})$ can be partitioned into three smaller subhypergraphs in two following ways:

(i) *The layer decomposition.* As a hyperedge of Type 1, Type 2 and Type 3 has six, four and three copies, respectively, each distinct hyperedge has at least three copies; use them to form a $3K_{mn}^{(3)}$. Now the remaining three copies of each distinct hyperedge of Type 1 in each partite set form a $3K_n^{(3)}$, and the remaining hyperedges of Type 2 form the union of complete bipartite hypergraphs on each pair of partite sets.

(ii) *The type decomposition.* The partition depends on the type of hyperedges. We partition $D^{(3)}(K_{m(n)})$ into $\bigcup_{1\leq i\leq 3} H_i$, where H_i consists of all hyperedges of Type i for $i \in \{1, 2, 3\}$.

The above two decompositions are concluded in Remark 1.4 for future references. Note first that λH stands for λ copies of the hypergraph H on the same vertex set.

Remark 1.4. Let $m \geq 2$ and n be positive integers.
$G = D^{(3)}(K_{m(n)}(V_1, V_2, \ldots, V_m))$ can be partitioned into three disjoint subhypergraphs in two ways as follows:

(i) *The layer decomposition.*

$$G = 3K_{mn}^{(3)}(\bigcup_{1\leq i\leq m} V_i) \cup \bigcup_{1\leq i\leq m} 3K_n^{(3)}(V_i) \cup \bigcup_{1\leq i<j\leq m} K_{n,n}^{(3)}(V_i, V_j)$$

(ii) *The type decomposition.*
Let H_i be the *Type i-subhypergraph* of G defined by $V(H_i) = V(G)$ and $E(H_i)$ consists of all hyperedges of Type i. Then

$$G = H_1 \cup H_2 \cup H_3.$$

In particular, $H_1 = \bigcup_{1\leq i\leq m} 6K_n^{(3)}(V_i)$ and $H_2 = \bigcup_{1\leq i\neq j\leq m} 4K_{n,n}^{(3)}(V_i, V_j)$.

2. $K_4^{(3)}$-Decompositions of $D^{(3)}(K_{m(n)})$

To decompose our distance multihypergraphs of $K_{m(n)}$, we mainly rely on some well known decompositions of graphs, namely 1-factorizations and 2-factorizations of complete graphs.

A 1-*factor* (2-*factor*) of a graph is a spanning subgraph that is regular of degree one (two, respectively). A 1-*factorization* (2-*factorization*) is a decomposition of a graph into 1-factors (2-factors, respectively). The existence of a 1-factorization and a 2-factorization of complete graphs are provided in the following theorems. (See more details in [7], [8]). We can refer to a 1-factor(or 2-factor) as its edge set.

Theorem 2.1. [7] *A complete graph of even vertices has a* 1*-factorization.*

Theorem 2.2. [8] *The complete graph of odd vertices has a* 2*-factorization.*

Our construction is divided into two parts depending on the parity of the size of each partite set.

2.1. $D^{(3)}(K_{m(n)})$ when n is Even

By the layer decomposition in Remark 1.4 (i), our distance multihypergraphs can be partitioned into subhypergraphs of the form $3K_s^{(3)}$ and $K_{n,n}^{(3)}$. Thus it suffices to decompose $3K_s^{(3)}$ and $K_{n,n}^{(3)}$ into 4-hypercliques. Lemmas 2.3 and 2.5 use 1-factorizations in the construction.

Lemma 2.3. *Let $s \geq 4$ be an even integer. There exists a $K_4^{(3)}$-decomposition of the hypergraph $3K_s^{(3)}$.*

Proof. Let V be a vertex set of a 3-uniform complete hypergraph $K_s^{(3)}$. Consider the *complete graph* K_s on the same vertex set V. Since s is even, by Theorem 2.1, there exists a 1-factorization of $K_s(V)$, say $\mathscr{M} = \{M_i : 1 \leq i \leq s-1\}$. Let

$$\mathscr{P} = \{\{a,b,c,d\} : \{a,b\} \neq \{c,d\} \in M_i, 1 \leq i \leq s-1\}.$$

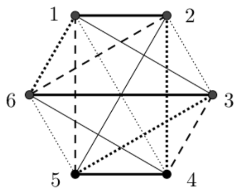
Then $\mathscr{P}$ contains $\binom{s/2}{2}(s-1)$ 4-hypercliques. Observe that any 3-subset e of V belongs to a hyperclique in $\mathscr{P}$ if e contains two vertices of an edge in some 1-factor of $\mathscr{M}$. Since $\mathscr{M}$ is a 1-factorization of $K_s(V)$ and e contains three distinct pairs of vertices, $\mathscr{P}$ covers exactly three copies of each distinct hyperedge in $E(K_s^{(3)})$. Therefore $\mathscr{P}$ is a $K_4^{(3)}$-decomposition of $3K_s^{(3)}(V)$. ∎

Example 2.4. An illustration of Lemma 2.3 when $s = 6$.

Let $V = \{1,2,3,4,5,6\}$ be the vertex set of $3K_6^{(3)}$, and let $\mathscr{M} = \{M_i : 1 \leq i \leq 5\}$ be a 1-factorization of $K_6(V)$ where

$$M_1 = \{\{1,2\},\{3,6\},\{4,5\}\}, \qquad M_4 = \{\{1,5\},\{2,6\},\{3,4\}\},$$
$$M_2 = \{\{1,3\},\{2,5\},\{4,6\}\}, \qquad M_5 = \{\{1,6\},\{2,4\},\{3,5\}\},$$
$$M_3 = \{\{1,4\},\{2,3\},\{5,6\}\},$$

as in Figure 2(a). Then we can construct 15 4-hypercliques in the collection $\mathscr{P}$ of $3K_6^{(3)}(V)$ from $\mathscr{M}$ as shown in Figure 2(b).



(a)

4-hypercliques in $\mathscr{P}$ which are associated with M_i														
M_1			M_2			M_3			M_4			M_5		
1	1	3	1	1	2	1	1	2	1	1	2	1	1	2
2	2	6	3	3	5	4	4	3	5	5	6	6	6	4
3	4	4	2	4	4	2	5	5	2	3	3	2	3	3
6	5	5	5	6	6	3	6	6	6	4	4	4	5	5

(b)

FIGURE 2. (a) 1-factorization $\mathscr{M}$ of K_6. (b) The collection $\mathscr{P}$ of 4-hypercliques.

Lemma 2.5. *Let n be an even integer. There exists a $K_4^{(3)}$-decomposition of the hypergraph $K_{n,n}^{(3)}$.*

Proof. Let X and Y be two partite sets of $K_{n,n}^{(3)}$. Since n is even, by Theorem 2.1, there exists a 1-factorization of $K_n(X)$ and $K_n(Y)$, say $\mathcal{E} = \{E_i : 1 \leq i \leq n-1\}$ and $\mathcal{F} = \{F_i : 1 \leq i \leq n-1\}$, respectively.

We will create each 4-hyperclique by choosing certain two vertices from each partite set. The four vertices are endpoint of an edge in 1-factor E_i and another edge from F_i for $1 \leq i \leq n-1$ as follows. Let $\mathscr{P}$ be the collection of 4-hypercliques in $K_{n,n}^{(3)}$ associated with $\mathcal{E}$ and $\mathcal{F}$ defined by

$$\mathscr{P} = \{\{a, b, c, d\} : \{a, b\} \in E_i, \{c, d\} \in F_i, 1 \leq i \leq n-1\}.$$

Hence $\mathscr{P}$ contains a total of $\left(\frac{n}{2}\right)^2(n-1)$ 4-hypercliques which together cover $4(\frac{n}{2})^2(n-1)$ hyperedges. It remains to show that $\mathscr{P}$ covers exactly one copy of each hyperedge of $K_{n,n}^{(3)}(X, Y)$.

Let $\{v, v', w\}$ be any hyperedge of $K_{n,n}^{(3)}(X, Y)$. Without loss of generality, say $v, v' \in X$ and $w \in Y$. Since $\{v, v'\}$ is an edge in E_i for some $1 \leq i \leq n-1$ and F_i is a spanning 1-regular subgraph, the hyperedge $\{v, v', w\}$ belongs to exactly one 4-hyperclique in $\mathscr{P}$. ■

Example 2.6. An illustration of Lemma 2.5 when $n = 4$.

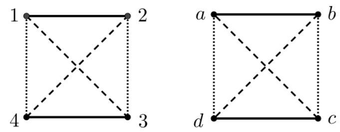
Let $X = \{1,2,3,4\}$ and $Y = \{a,b,c,d\}$ be two partite sets of $K_{4,4}^{(3)}$, and let $\mathcal{E} = \{E_1, E_2, E_3\}$ and $\mathcal{F} = \{F_1, F_2, F_3\}$ be 1-factorizations of $K_4(X)$ and $K_4(Y)$ respectively, where

$$E_1 = \{\{1, 2\}, \{3, 4\}\}, \qquad E_2 = \{\{1, 3\}, \{2, 4\}\}, \qquad E_3 = \{\{1, 4\}, \{2, 3\}\},$$
$$F_1 = \{\{a, b\}, \{c, d\}\}, \qquad F_2 = \{\{a, c\}, \{b, d\}\}, \qquad F_3 = \{\{a, d\}, \{b, c\}\},$$

as in Figure 3(a). Then we can construct 12 4-hypercliques in the collection $\mathscr{P}$ of $K_{n,n}^{(3)}(X, Y)$ from $\mathcal{E}$ and $\mathcal{F}$ as shown in Figure 3(b).

When n is even, it follows from Lemmas 2.3 and 2.5 that each subhypergraph in the layer decomposition of $D^{(3)}(K_{m(n)})$ has a $K_4^{(3)}$-decomposition. Hence the following first main theorem holds.

Theorem 2.7. *Let $m, n \geq 2$ be integers such that n is even. Then, there exists a $K_4^{(3)}$-decomposition of the hyperdistance multihypergraph $D^{(3)}(K_{m(n)})$.*



4-hyperclique in $\mathscr{P}$ which are associated with E_i and F_i											
E_1, F_1				E_2, F_2				E_3, F_3			
1	1	3	3	1	1	2	2	1	1	2	2
2	2	4	4	3	3	4	4	4	4	3	3
a	c	a	c	a	b	a	b	a	b	a	b
b	d	b	d	c	d	c	d	d	c	d	c

(a) (b)

FIGURE 3. (a) 1-factorizations $\mathcal{E}$ and $\mathcal{F}$ of $K_4(X)$ and $K_4(Y)$, respectively. (b) The collection $\mathscr{P}$ of 4-hypercliques.

2.2. $D^{(3)}(K_{m(n)})$ WHEN n IS ODD

When $n \geq 3$ is odd, the construction is getting more complicated. We present a technique to resolve some small cases.

First start with $m = 2$. To decompose the Type 2-subhypergraph of $D^{(3)}(K_{5,5})$ and $D^{(3)}(K_{9,9})$, we modify the construction of the even case in the previous section by using 2-factorizations instead of 1-factorizations due to the number of vertices of each partite set is odd.

Theorem 2.8. *There exist $K_4^{(3)}$-decompositions of $D^{(3)}(K_{5,5})$ and $D^{(3)}(K_{9,9})$.*

Proof. (i) Let X and Y be two partite sets of $K_{5,5}$. First, by the type decomposition in Remark 1.4 (ii), we have

$$D^{(3)}(K_{5,5}(X,Y)) = 6K_5^{(3)}(X) \,\cup\, 6K_5^{(3)}(Y) \,\cup\, 4K_{5,5}^{(3)}(X,Y).$$

Let $\mathscr{P}_X$ be the collection of all 4-subsets of X each of which form a 4-hyperclique. Since $|X| = 5$, each 3-subset of X must belong to exactly two 4-hypercliques in $\mathscr{P}_X$. Then $\mathscr{P}_X$ covers exactly two copies of each distinct hyperedge on partite set X. Hence, $3\mathscr{P}_X$ form a $K_4^{(3)}$-decomposition of $6K_5^{(3)}(X)$. Similarly, there exists a $K_4^{(3)}$-decomposition of $6K_5^{(3)}(Y)$.

Now, it remains to construct a $K_4^{(3)}$-decomposition of the Type 2-subhypergraph of $D^{(3)}(K_{5,5}(X,Y))$, say $2\mathscr{P}$. In other words, $\mathscr{P}$ will be a $K_4^{(3)}$-decomposition of $2K_{5,5}^{(3)}(X,Y)$. We create hypercliques in $\mathscr{P}$ using a 2-factorization of K_5 as follows. By Theorem 2.2, there exist 2-factorizations, say $\mathcal{E}_X = \{E_1, E_2\}$ and $\mathcal{F}_Y = \{F_1, F_2\}$ of $K_5(X)$ and $K_5(Y)$, respectively. Let

$$\mathscr{P} = \{\{a, b, c, d\} : \{a, b\} \in E_i, \{c, d\} \in F_i, 1 \leq i \leq 2\}.$$

Since both E_i and F_i have 5 edges, there are 25 4-hypercliques in the collection $\mathscr{P}$ which are associated with E_i and F_i.

Hence, $\mathscr{P}$ consists of 50 4-hypercliques. It follows that $\mathscr{P}$ covers at most 200 hyperedges counted repeatedly. Moreover, since $2K_{5,5}^{(3)}(X,Y)$ has a total of 200 hyperedges, it suffices to show that $\mathscr{P}$ covers at least two copies of each distinct hyperedge of Type 2 in $K_{5,5}^{(3)}(X,Y)$. Now let $\{v, v', w\}$ be a hyperedge where $v, v' \in X$ and $w \in Y$, then $\{v, v'\}$ is an edge in E_t for some $t \in \{1, 2\}$. Together with the fact that F_t is a spanning 2-regular subgraph of Y, so the hyperedge $\{v, v', w\}$ belongs to at least two 4-hypercliques in $\mathscr{P}$.

Hence, $\mathscr{P}$ covers $2K^{(3)}_{5,5}(X,Y)$, and thus $2\mathscr{P}$ decomposes $4K^{(3)}_{5,5}(X,Y)$ into 4-hypercliques.

(ii) Let A and B be two partite sets in $K_{9,9}$. Then

$$D^{(3)}(K_{9,9}(A,B)) = 6K^{(3)}_9(A) \ \cup \ 6K^{(3)}_9(B) \ \cup \ 4K^{(3)}_{9,9}(A,B).$$

Let $\mathscr{C}_A$ be the collection of all 4-subsets of A. Similar to (i) except now $|A| = 9$, each hyperedge in $K^{(3)}_9(A)$ belongs to exactly six 4-hypercliques in $\mathscr{C}_A$. Therefore, there exists a $K^{(3)}_4$-decomposition of $6K^{(3)}_9(A)$, and so does one of $6K^{(3)}_9(B)$. To decompose $4K^{(3)}_{9,9}(A,B)$ into 4-hypercliques, we use the similar argument as in (i). Therefore, $D^{(3)}(K_{9,9}(A,B))$ has a $K^{(3)}_4$-decomposition. ■

Corollary 2.9. *For an integer $n \geq 2$, there exists the $K^{(3)}_4$-decomposition of $2K^{(3)}_{n,n}$.*

Proof. If n is even, then the decomposition exists by Lemma 2.5. If n is odd, then we create a collection of 4-hypercliques by using a 2-factorization of K_n as in the proof of Theorem 2.8 which form a $K^{(3)}_4$-decomposition of $2K^{(3)}_{n,n}$. ■

It could be noted that each 4-hyperclique in the decomposition from the construction of Theorem 2.8 always contains hyperedges of the same type. However, when $n = 4k+3$, the number of hyperedges of Type 1 in $D^{(3)}(K_{m(n)})$ is not divisible by 4. This implies that some hyperedges of Type 1 must form a 4-hyperclique with some hyperedges of another type, for instance, Example 1.3 shows a $K^{(3)}_4$-decomposition of $D^{(3)}(K_{3,3})$ and a result in [9] reveals a $K^{(3)}_4$-decomposition of $D^{(3)}(K_{7,7})$. The construction for $D^{(3)}(K_{7,7})$ in [9] is a bit subtle which uses a 2-factorization of K_7 and a length function of edges in K_7. This result is concluded here for future reference in Theorem 2.10.

Theorem 2.10. [9] *There exists a $K^{(3)}_4$-decomposition of $D^{(3)}(K_{7,7})$.*

Now we move on to the complete multipartite hypergraphs with more than two partite sets. If the number of partite sets is an even integer at least four, we utilize *Latin cube* to handle hyperedges of Type 3 in our construction.

A *Latin cube* $L = \{L_{i,j,k}\}$ of order n on the set $X = \{1,2,\ldots,n\}$ is an $n \times n \times n$ cube (n rows, n columns and n layers) such that the entry $L_{i,j,k} \in X$, where $L_{i,j,k}$ denotes the entry in row i column j and layer k, and if two indices are fixed and the remaining index is allowed to vary from 1 to n, then these entries form a permutation of X.

1	2	3
2	3	1
3	1	2

$k = 1$

2	3	1
3	1	2
1	2	3

$k = 2$

3	1	2
1	2	3
2	3	1

$k = 3$

FIGURE 4. A Latin cube $L = \{L_{i,j,k}\}$ of order 3.

The existence of Latin cubes is shown in the following theorem. (See more details in [3])

Theorem 2.11. [10],[3] *Let $n \geq 2$ be an integer. There exists a Latin cube of order n.*

The next lemma is to decompose the Type 3-subhypergraph of $D^{(3)}(K_{m(n)})$ for an even m.

Lemma 2.12. *Let $m, n \geq 2$ be integers such that m is even. The Type 3-subhypergraph of $D^{(3)}(K_{m(n)})$ has a $K_4^{(3)}$-decomposition.*

Proof. Let H_3 be the Type 3-subhypergraph of $D^{(3)}(K_{m(n)})$. Let $V_1, V_2, \ldots, V_m$ be the m-partite sets of H_3. Then $|V_i| = n$ for all $i \in \{1, 2, \ldots, m\}$.

We will create 4-hypercliques in our decomposition using hyperedges of Type 3 from certain groups of four partite sets of H_3.

To specify groups of four partite sets, we consider the set of indices $I = \{1, 2, \ldots, m\}$ as the vertex set of complete graph K_m. Since m is even, by Theorem 2.1, $K_m(I)$ has a 1-factorization, say $\mathscr{M} = \{M_i : 1 \leq i \leq m-1\}$.

Now, define the collection of sets, called *groups*, of four partite sets of H_3 according to $\mathscr{M}$ as follows;

$$\mathscr{Q} = \{\{V_a, V_b, V_c, V_d\} : \{a, b\} \neq \{c, d\} \in M_i, 1 \leq i \leq m-1\}.$$

Then $\mathscr{Q}$ contains $\binom{m/2}{2}(m-1)$ groups of four partite sets.

Next, for each group in $\mathscr{Q}$, we will create the collection of 4-hypercliques of H_3 as follows. By Theorem 2.11, let L be a Latin cube of order n.

We define a collection of 4-hypercliques of H_3 by

$$\mathscr{P} = \{\{a_i, b_j, c_k, d_{L_{i,j,k}}\} : 1 \leq i, j, k \leq n, \{V_a, V_b, V_c, V_d\} \in \mathscr{Q}\}$$

where $V_a = \{a_1, a_2, \ldots, a_n\}$, $V_b = \{b_1, b_2, \ldots, b_n\}$, $V_c = \{c_1, c_2, \ldots, c_n\}$, and $V_d = \{d_1, d_2, \ldots, d_n\}$.

Then $\mathscr{P}$ contains $\binom{m/2}{2}(m-1)n^3$ 4-hypercliques. Observe that any hyperedge of Type 3 contains three vertices from different three partite sets.

Let e be a hyperedge of Type 3. Since $\mathscr{M}$ is a 1-factorization of $K_m(I)$, any three partite sets are together in the same group in $\mathscr{Q}$ exactly three different groups. Moreover, e belongs to a hyperclique in $\mathscr{P}$ once by the property of a Latin cube. It means that $\mathscr{P}$ covers exactly three copies of each distinct hyperedge in H_3. Consequently, $\mathscr{P}$ is a $K_4^{(3)}$-decomposition of H_3. ∎

Example 2.13. An illustration of Lemma 2.12 when $m = 4$ and $n = 3$.

Let $V_a = \{a_1, a_2, a_3\}$, $V_b = \{b_1, b_2, b_3\}$, $V_c = \{c_1, c_2, c_3\}$, and $V_d = \{d_1, d_2, d_3\}$ be four partite sets of $D^{(3)}(K_{4(3)})$, and let $\mathcal{F} = \{F_1, F_2, F_3\}$ be a 1-factorization of $K_4(\{a, b, c, d\})$ where

$$F_1 = \{\{a, b\}, \{c, d\}\}, \qquad F_2 = \{\{a, c\}, \{b, d\}\}, \qquad F_3 = \{\{a, d\}, \{b, c\}\}.$$

Let $L = \{L_{i,j,k}\}$ be the Latin cube of order 3 in Figure 4. Figure 5 lists all 4-hypercliques of $D^{(3)}(K_{4(3)}(V_a, V_b, V_c, V_d))$ constructed from $F_2 \in \mathcal{F}$ and $L = \{L_{i,j,k}\}$ for $k = 1, 2$.

4-hypercliques in $\mathscr{P}$ which are associated with F_2 and $L = \{L_{i,j,k}\}$ for $k = 1, 2$.																	
$k=1$									$k=2$								
a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
c_1	c_1	c_1	c_2	c_2	c_2	c_3	c_3	c_3	c_1	c_1	c_1	c_2	c_2	c_2	c_3	c_3	c_3
b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1
d_1	d_2	d_3	d_2	d_3	d_1	d_3	d_1	d_2	d_2	d_3	d_1	d_3	d_1	d_2	d_1	d_2	d_3

FIGURE 5. The 4-hypercliques in $\mathscr{P}$ associated with $F_2 \in \mathcal{F}$ and Latin cube $L = \{L_{i,j,k}\}$ for $k = 1, 2$.

The next main theorem shows that if $D^{(3)}(K_{2(n)})$ has a $K_4^{(3)}$-decomposition, so does $D^{(3)}(K_{m(n)})$ whenever m is even.

Theorem 2.14. *Let $n, m \geq 2$ be integers such that m is even. If there exists a $K_4^{(3)}$-decomposition of $D^{(3)}(K_{n,n})$, then there exists a $K_4^{(3)}$-decomposition of $D^{(3)}(K_{m(n)})$.*

Proof. Let $G = D^{(3)}(K_{m(n)}(V_1, V_2, \ldots, V_m))$. Then $|V_i| = n$ for all $i \in \{1, 2, \ldots, m\}$.

First, by the type decomposition, we have

$$G = H_1 \cup H_2 \cup H_3, \text{ where } H_i \text{ is a Type } i\text{-subhypergraph.}$$

Next we will partition H_2 into two smaller subhypergraphs using a 1-factor as follows: Consider the set of indices $I = \{1, 2, \ldots, m\}$ as the vertex set of the complete graph K_m. Since m is even, $K_m(I)$ has a 1-factor, say $M = \{\{2i-1, 2i\} : 1 \leq i \leq \frac{m}{2}\}$. Let $E = E(K_m(I))$. We have that

$$H_2 = \bigcup_{\{i,j\} \in M} 4K_{n,n}^{(3)}(V_i, V_j) \cup \bigcup_{\{i,j\} \in E \setminus M} 4K_{n,n}^{(3)}(V_i, V_j).$$

Let $K = H_1 \cup \bigcup_{\{i,j\} \in M} 4K_{n,n}^{(3)}(V_i, V_j)$. Then K can be viewed as the (hyperedge) disjoint union of $\frac{m}{2}$ copies of $D^{(3)}(K_{n,n})$, and hence it has a $K_4^{(3)}$-decomposition by the assumption. Furthermore, by Corollary 2.9 and Lemma 2.12, there exists a $K_4^{(3)}$-decomposition of $\bigcup_{\{i,j\} \in E \setminus M} 4K_{n,n}^{(3)}(V_i, V_j)$ and H_3, respectively. Therefore our desired decomposition exists. ∎

When $n = 3, 5, 7, 9$, the existence of $K_4^{(3)}$-decomposition of $D^{(3)}(K_{2(n)})$ is provided in Example 1.3 and, Theorems 2.8 and 2.10. Corollary 2.15 concludes the current result, and hence the problem remains open when m and n are odd.

Corollary 2.15. *Let m be an even integer and $n = 3, 5, 7, 9$. There exists a $K_4^{(3)}$-decomposition of $D^{(3)}(K_{m(n)})$.*

For our final remark, we would like to illustrate more application of our technique using 2-factorizations and Latin cubes which presented in Corollary 2.9 and Lemma 2.12, respectively. The technique yields a decomposition not only of a distance multihypergraph but also one of a complete 3-uniform multipartite hypergraph.

Corollary 2.16. *Let m, n and λ be positive integers such that m is even, n is odd and $\lambda \equiv 0 \pmod 6$. Then there exists a $K_4^{(3)}$-decomposition of $\lambda K_{m(n)}^{(3)}$.*

Proof. Let $G = K_{m(n)}^{(3)}(V_1, V_2, \ldots, V_m)$ and $\lambda = 6t$ for some positive integer t. Then $|V_i| = n$ for all $i \in \{1, 2, \ldots, m\}$. First note that

$$6tG = (6tH_3) \cup \bigcup_{1 \leq i < j \leq m} 6tK_{n,n}^{(3)}(V_i, V_j),$$

where H_3 is the Type 3-subhypergraph of $D^{(3)}(K_{m(n)})$.

Since $\bigcup_{1 \leq i < j \leq m} 2tK_{n,n}^{(3)}(V_i, V_j)$ and $3tH_3$ can be decomposed into a collection of 4-hypercliques by Corollary 2.9 and Lemma 2.12, respectively, $6tG$ has a $K_4^{(3)}$-decomposition as desired. ∎

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References

[1] V. Boonthong, C. Uiyyasathian, Clique decompositions of the distance multigraph of the Cartesian product of graphs, East-West J. Math 20 (2018) 9–18.

[2] R.J. Elzinga, D.A. Gregory, K.N. Vander Meulen, Addressing the Petersen graph, Discrete Math. 286 (2004) 241–244.

[3] R.L. Graham, H.O. Pollak, On the addressing problem for loop switching, Bell Syst. Tech. J. 50 (1971) 2495–2519.

[4] P. Winkler, Proof of the squashed cube conjecture, Combinatorica 3 (1983) 135–139.

[5] M.S. Cavers, R.J. Elzinga, D.A. Gregory, S.E. Vanderlinde, K.N. Vander Meulen, Clique partition of distance multigraph, Discrete Math. 308 (2008) 3230–3240.

[6] V. Boonthong, Hyperclique Decompositions of Hyperdistance Multihypergraphs, PhD Thesis, Chulalongkorn University, Bangkok, Thailand, 2014.

[7] W.D. Wallis, Introduction to Combinatorial Design, Chapman and Hall/CRC, New York, 2008.

[8] L. Lovsz, M.D. Plummer, Matching Theory, American Mathematical Society, Rhode Island, 2009.

[9] C. Saengchampa, Hyperclique Decompositions of 3-Hyperdistance Multihypergraphs of Complete Multipartite Graphs, Master Thesis, Chulalongkorn University, Bangkok, Thailand, 2016.

[10] J.T. Ethier, G.L. Mullen, Strong forms of orthogonality for sets of hypercubes, Discrete Math. 312 (2012) 2050–2061.