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On Asymptotically Wijsman (λ, σ)-Statistical Convergence of Set Sequences

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Abstract In this paper we presents three definitions which is a natural combination of the definition of asymptotic equivalence, statistical convergence, (λ, σ)-statistical convergence and Wijsman convergence. In addition, we also present asymptotically equivalent sequences of sets in sense of Wijsman and study some properties of this concept.

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1. Introduction

In [1], Marouf peresented definitions for asymptotically equivalent and asymptotic regular matrices. In [2], Pobyvancts introduced the concepts of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences. In [3], Patterson extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence. The concept of Wijsman statistical convergence which is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades in [4]. Similar to the concept, the concept of Wijsman lacunary statistical convergence presented by Ulusu and Nuray in [5]. In [6], Hazarika and Esi introduced the notion asymptotically Wijsman generalized statistical convergence of sequences of sets. For more works on convergence of sequences of sets, we refer to ([7–16]).

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The idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 ([17]). The concept was formally introduced by Steinhaus [18] and Fast [19] and later was introduced by Schoenberg [20], and also independently by Buck [21]. A lot of developments have been made in this area after the works of Šalát [22] and Fridy [23], Çanak et al. [24], Totur and Çanak [25, 26]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory.

In this paper we define asymptotically (λ, σ)-statistical equivalent sequences of sets in sense of Wijsman and establish some basic results regarding the notions asymptotically (λ, σ)-statistical equivalent sequences of sets in sense of Wijsman and asymptotically Wijsman statistical equivalent sequences of sets.

Now we recall the definitions of statistical convergence, λ-statistical convergence, σ-statistical convergence and Wijsman convergence.

Definition 1.1. A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0.$$

In this case, we write $S-\lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V,λ)-summable to number L [27] if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then $(V, \lambda)-$summability reduces to $(C, 1)$-summability.
Mursaleen [28] defined $\lambda-$statistically convergent sequence as follows:

Definition 1.2. A sequence $x = (x_k)$ is said to be $\lambda-$ statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in I_n : \ |x_k - L| \geq \varepsilon\}\right| = 0.$$

Let S_λ denotes the set of all $\lambda-$statistically convergent sequences. If $\lambda_n = n$, then S_λ is the same as S.

Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ is such that $x_k \geq 0$ for all k,
(2) $\phi(e) = 1$ where $e = (1, 1, 1, ...)$, and
(3) $\phi(x) = \phi(x_{\sigma(k)})$ for all $x \in \ell_\infty$.

Throughout this paper we shall consider the mapping σ has having on finite orbits, that is, $\sigma^m(k) \neq k$ for all nonnegative integers with $m \geq 1$, where $\sigma^m(k)$ is the m-th iterate of σ at k. We denote V_σ is the set of bounded sequences all of whose σ-mean are equal. If $\sigma(k) = k + 1$, then it is the set of almost convergent sequences in [29].

Definition 1.3. [30] A sequence $x = (x_k)$ is said to be σ-statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : \ \left|x_{\sigma^k(m)} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly on } m.$$

Let S_σ denotes the set of all σ-statistically convergent sequences.

We define the strongly (λ, σ)-convergence and (λ, σ)-statistical convergence as follows:

Definition 1.4. A sequence $x = (x_k)$ is said to be strongly (λ, σ)-convergent to the number L if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} \left|x_{\sigma^k(m)} - L\right| = 0, \text{ uniformly on } m.$$

Let $\mathcal{L}_\lambda$ denotes the set of all strongly (λ, σ)-convergent sequences.

Definition 1.5. A sequence $x = (x_k)$ is said to be (λ, σ)-statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : \ \left|x_{\sigma^k(m)} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly on } m.$$

Let $S_{\lambda,\sigma}$ denotes the set of all (λ, σ)-statistically convergent sequences. If $\lambda_n = n$, then $S_{\lambda,\sigma}$ is the same as S_σ.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y\in A} \rho(x, y).$$

Definition 1.6. [8] Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman convergent to A if $\lim_k d(x, A_k) = d(x, A)$ for each $x \in X$. In this case we write $W - \lim A_k = A$.

2. Definitions and Notations

Definition 2.1. [1] Two nonnegative sequences $x = (x_k)$ and $0 \neq y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1,$$

denoted by $x \sim y$.

Definition 2.2. [3] Two nonnegative sequences $x = (x_k)$ and $0 \neq y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| = 0,$$

denoted by $x \sim^{S^L} y$ and simply asymptotically statistical equivalent if $L = 1$.

In, [30], Savas and Nuray, defined the asymptotically σ-statistical equivalent and strongly asymptotically σ-statistical equivalent sequences as follows:

Definition 2.3. Two nonnegative sequences $x = (x_k)$ and $0 \neq y = (y_k)$ are said to be asymptotically σ-statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| = 0, \text{ uniformly on } m,$$

denoted by $x \sim^{S_\sigma^L} y$ and simply asymptotically σ-statistical equivalent if $L = 1$.

Definition 2.4. Two nonnegative sequences $x = (x_k)$ and $0 \neq y = (y_k)$ are said to be strongly asymptotically σ-statistical equivalent of multiple L provided that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| = 0, \text{ uniformly on } m,$$

denoted by $x \sim^{[V_\sigma]^L} y$ and simply strongly asymptotically σ-statistical equivalent if $L = 1$.

The concepts of Wijsman statistical convergence and boundedless for the sequence (A_k) were given by Nuray and Rhoades [4] as follows:

Definition 2.5. [4] Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman statistical convergent to A if the sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{n} \left| \{ k \leq n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| = 0.$$

In this case, we write $st - \lim_k A_k = A$ or $A_k \to A\,(WS)$. The sequence (A_k) is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets denoted by L_∞.

In [14], Ulusu and Nuray define asymptotically equivalent and asymptotically statistical equivalent sequences of sets as follows:

Definition 2.6. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically equivalent (Wijsman sense) if for each $x \in X$,

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1,$$

denoted by $(A_k) \sim (B_k)$.

Definition 2.7. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically statistical equivalent (Wijsman sense) if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0$$

denoted by $(A_k) \sim^{WS^L} (B_k)$ and simply asymptotically statistical equivalent (Wijsman sense) if $L = 1$.

In [12], Hazarika and Esi introduced the notion Wijsman λ-statistical convergence of sequences of sets as follows:

Definition 2.8. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman λ-statistical convergent to A if the sequence $(d(x, A_k))$ is λ-statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{\lambda_n} \left|\{k \in I_n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\}\right| = 0.$$

In this case, we write $S_\lambda^W - \lim_k A_k = A$ or $A_k \to A \left(S_\lambda^W\right)$.

In [6], Hazarika and Esi introduced the following definitions.

Definition 2.9. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically Wijsman λ-equivalent of multiple L if for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{d(x, A_k)}{d(x, B_k)} = L$$

denoted by $(A_k) \sim^{W(V,\lambda)^L} (B_k)$ and simply asymptotically Wijsman λ-equivalent if $L = 1$.

Definition 2.10. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are strongly asymptotically Wijsman λ-statistical equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| = 0$$

denoted by $(A_k) \sim^{W[V,\lambda]^L} (B_k)$ and simply strongly asymptotically Wijsman λ-equivalent if $L = 1$.

Example 2.11. We consider the following sequences:

$$A_k = \begin{cases} \left\{(x, y, z) \in \mathbb{R}^3 : \frac{(x-\sqrt{k})^2}{k} + \frac{(y-\sqrt{2k})^2}{2k} + \frac{z^2}{3k} = 1\right\}, & \text{if } n - [|\lambda_n|] + 1 \leq k \leq n; \\ \{(1,1,1)\}, & \text{otherwise} \end{cases}$$

and

$$B_k = \begin{cases} \left\{(x, y, z) \in \mathbb{R}^3 : \frac{(x+\sqrt{k})^2}{k} + \frac{(y+\sqrt{2k})^2}{2k} + \frac{z^2}{3k} = 1\right\}, & \text{if } n - [|\lambda_n|] + 1 \leq k \leq n; \\ \{(1,1,1)\}, & \text{otherwise} \end{cases}$$

Since

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left|\frac{d(x, A_k)}{d(x, B_k)} - 1\right| = 0,$$

therefore the sequences (A_k) and (B_k) are strongly asymptotically Wijsman λ-equivalent, i.e., $(A_k) \sim^{W[V,\lambda]^1} (B_k)$.

If we take $\lambda_n = n$, then we get the following definitions.

Definition 2.12. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically Wijsman Cesáro-equivalent of multiple L if for each $x \in X$,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \frac{d(x, A_k)}{d(x, B_k)} = L$$

denoted by $(A_k) \sim^{W(C,1)^L} (B_k)$ and simply asymptotically Wijsman Cesáro-equivalent if $L = 1$.

Definition 2.13. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are strongly asymptotically Wijsman Cesáro equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| = 0$$

denoted by $(A_k) \sim^{W[C,1]^L} (B_k)$ and simply strongly asymptotically Wijsman Cesáro-equivalent if $L = 1$.

Definition 2.14. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically Wijsman λ-statistical equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0$$

denoted by $(A_k) \sim^{WS_\lambda^L} (B_k)$ and simply asymptotically Wijsman λ-statistical equivalent if $L = 1$.

Example 2.15. We consider the following sequences:

$$A_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \frac{1}{k^2}\right\}, & \text{if } n - [|\lambda_n|] + 1 \leq k \leq n \text{ and } k = i^2, \\ & i = 1, 2, 3, ...; \\ \{(0, 0)\}, & \text{otherwise} \end{cases}$$

and

$$B_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : x^2 + (y+1)^2 = \frac{1}{k^2}\right\}, & \text{if } n - [|\lambda_n|] + 1 \leq k \leq n \text{ and } k = i^2, \\ & i = 1, 2, 3, ...; \\ \{(0, 0)\}, & \text{otherwise} \end{cases}$$

Since

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_k)}{d(x, B_k)} - 1 \right| \geq \varepsilon \right\} \right| = 0,$$

therefore the sequences (A_k) and (B_k) are asymptotically Wijsman λ-equivalent, i.e., $(A_k) \sim^{WS_\lambda^1} (B_k)$.

3. Main Results

In this section, we define asymptotically Wijsman (λ,σ)-statistical equivalent sequences of sets and proved some interesting results.

Definition 3.1. Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically Wijsman (λ,σ)-equivalent of multiple L if for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} = L$$

denoted by $(A_k) \sim^{W(\mathcal{L}_\lambda)^L} (B_k)$ and simply asymptotically Wijsman (λ,σ)-equivalent if $L = 1$.

Definition 3.2. Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are strongly asymptotically Wijsman (λ,σ)-statistical equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| = 0$$

denoted by $(A_k) \sim^{W[\mathcal{L}_\lambda]^L} (B_k)$ and simply strongly asymptotically Wijsman (λ,σ)-equivalent if $L = 1$.

Definition 3.3. Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically Wijsman (λ,σ)-statistical equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \geq \varepsilon \right\} \right| = 0$$

denoted by $(A_k) \sim^{WS^L_{\lambda,\sigma}} (B_k)$ and simply asymptotically Wijsman (λ,σ)-statistical equivalent if $L = 1$.

Remark 3.4. For $\sigma(m) = m + 1$, we obtain the results discuss in our paper [6].

Theorem 3.5. *Let (X,ρ) be a metric space and A_k, B_k be non-empty closed subsets X $(k \in \mathbb{N})$. Then*

(a) $(A_k) \sim^{W[\mathcal{L}_\lambda]^L} (B_k)$ *implies* $(A_k) \sim^{WS^L_{\lambda,\sigma}} (B_k)$
(b) $(A_k) \sim^{WS^L_{\lambda,\sigma}} (B_k)$ *implies* $(A_k) \nsim^{W[\mathcal{L}_\lambda]^L} (B_k)$
(c) $(A_k) \sim^{WS^L_{\lambda,\sigma}} (B_k)$ *and* $(A_k) \in L_\infty$ *implies* $(A_k) \sim^{W[\mathcal{L}_\lambda]^L} (B_k)$
(d) $(A_k) \sim^{WS^L_{\lambda,\sigma}} (B_k)$ *and* $(A_k) \in L_\infty$ *if and only if* $(A_k) \sim^{W[\mathcal{L}_\lambda]^L} (B_k)$.

Proof. (a) Let $\varepsilon > 0$ and $(A_k) \sim^{W[\mathcal{L}_\lambda]^L} (B_k)$. Then we can write

$$\sum_{k \in I_n} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \sum_{\substack{k \in I_n \\ \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \geq \varepsilon}} \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right|$$

$$\geq \varepsilon \left| \left\{ k \in I_n : \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \geq \varepsilon \right\} \right|$$

which gives the result.

(b) Suppose that $W[\mathcal{L}_\lambda]^L \subset WS^L_{\lambda,\sigma}$. Let (A_k) and (B_k) be two sequences defined as follows:

$$A_k = \begin{cases} \{k\}, & \text{if } n - [|\lambda_n|] + 1 \leq k \leq n, k = 1, 2, 3, ...; \\ \{0\}, & \text{otherwise} \end{cases}$$

and

$$B_k = \{0\} \text{ for all } k \in \mathbb{N}.$$

It is clear that $(A_k) \notin L_\infty$ and for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - 1 \right| \geq \varepsilon \right\} \right| = \lim_n \frac{1}{\lambda_n} [|\lambda_n|] = 0.$$

So $(A_k) \sim^{WS^1_{\lambda,\sigma}} (B_k)$, but

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \neq 0.$$

Therefore $(A_k) \not\sim^{W[\mathcal{L}_\lambda]^L} (B_k)$.

(c) Suppose that $(A_k) \sim^{WS^L_{\lambda,\sigma}} (B_k)$ and $(A_k) \in L_\infty$. We assume that $\left| \frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L \right| \leq M$ for each $x \in X$ and for all $k \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right|$$

$$= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ \left| \frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L \right| \geq \varepsilon}} \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ \left| \frac{d(x,A_{\sigma^k(m)})}{d(x,B_{\sigma^k(m)})} - L \right| < \varepsilon}} \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right|$$

$$\leq \frac{M}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_{\sigma^k(m)})}{d(x, B_{\sigma^k(m)})} - L \right| \geq \varepsilon \right\} \right| + \varepsilon,$$

from which the result follows.

(d) It follows from (a), (b) and (c). ∎

Theorem 3.6. *Let (X, ρ) be a metric space and A_k, B_k be non-empty closed subsets X $(k \in \mathbb{N})$. Then $(A_k) \sim^{WS^L_\sigma} (B_k) \Rightarrow (A_k) \sim^{WS^L_{\lambda,\sigma}} (B_k)$ if and only if* $\liminf \frac{\lambda_n}{n} > 0$.

Proof. Suppose that $\liminf \frac{\lambda_n}{n} > 0$. For given $\varepsilon > 0$, we have

$$\left\{ k \leq n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \right\} \supset \left\{ k \in I_n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \right\}.$$

Therefore

$$\frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \right\} \right| \geq \frac{1}{n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \right\} \right|$$

$$\geq \frac{\lambda_n}{n} \cdot \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \right\} \right|.$$

Taking the limit as $n \to \infty$ and using $\liminf \frac{\lambda_n}{n} > 0$, we get the desired result. Conversely, suppose that $\liminf_n \frac{\lambda_n}{n} = 0$. Then we can select a subsequence $(n(i))_{i=1}^{\infty}$ such that

$$\frac{\lambda_{n(i)}}{n(i)} < \frac{1}{i}.$$

We define sequences (A_k) and (B_k) as follows:

$$A_k = \begin{cases} \{1\}, & \text{if } n(i) - \left[\left|\lambda_{n(i)}\right|\right] + 1 \leq k \leq n(i), i = 1, 2, 3, \ldots; \\ \{0\}, & \text{otherwise.} \end{cases}$$

and

$$B_k = \{0\} \text{ for all } k \in \mathbb{N}.$$

Then $(A_k) \sim^{WS_\sigma^L} (B_k)$. But $(A_k) \nsim^{WS_{\lambda,\sigma}^L} (B_k)$ (A_k). This completes the proof. ∎

Theorem 3.7. *Let (X, ρ) be a metric space and A_k, B_k be non-empty closed subsets X $(k \in \mathbb{N})$. Then $(A_k) \sim^{WS_{\lambda,\sigma}^L} (B_k) \Rightarrow (A_k) \sim^{WS_\sigma^L} (B_k)$ if* $\liminf \frac{\lambda_n}{n} = 1$.

Proof. Since $\lim_n \frac{\lambda_n}{n} = 1$, then for $\varepsilon > 0$, we observe that

$$\frac{1}{n} \left| \{ k \leq n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \} \right|$$

$$\leq \frac{1}{n} \left| \{ k \leq n - \lambda_n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \} \right|$$

$$+ \frac{1}{n} \left| \{ k \in I_n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \} \right|$$

$$\leq \frac{n - \lambda_n}{n} + \frac{1}{n} \left| \{ k \in I_n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \} \right|$$

$$= \frac{n - \lambda_n}{n} + \frac{\lambda_n}{n} \frac{1}{\lambda_n} \left| \{ k \in I_n : \left| \frac{d(x, A_{\sigma^m(k)})}{d(x, B_{\sigma^m(k)})} - L \right| \geq \varepsilon \} \right|.$$

This implies that $(A_k) \sim^{WS_{\lambda,\sigma}^L} (B_k) \Rightarrow (A_k) \sim^{WS_\sigma^L} (B_k)$. ∎

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