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# Finite-Time Synchronization of Neural Networks with Interval and Distributed Time-Varying Delays via Feedback Control

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**Abstract** This research presents the problem of the finite-time synchronization of neural networks with interval and distributed time-varying delays. A state feedback control is planned for finite-time synchronization of neural networks. By constructing the Lyapunov-Krasovskii functional (LKF) is derived for finite-time stability criteria of neural network systems with interval and continuous differentiable time-varying delays. An extended reciprocally convex matrix inequality, a free-matrix-based integral inequality, Jensen's inequality and Wirtinger-based integral inequality are used to estimate the upper bound of the derivative of the LKF. The new sufficient finite-time stability conditions have been proposed in the form of linear matrix inequalities. Finally, a numerical example is presented to show the effectiveness of the proposed methods.

**MSC:** 34D06; 34D20; 93D15

**Keywords:** finite-time synchronization; neural networks; interval and distributed time-varying delays; feedback control

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## 1. Introduction

In the past decades, neural networks have generally been recognized as one of the simplest forms of neural processing in the human brain, which have an excellent ability to process various complicated engineering problems and improve the efficiency of dynamic systems. Neural networks have already been used in many majors, such as associate memories, robotics and control, optimization problems, pattern recognition, and other engineering areas [1–3]. In such applications, an important factor is the stability feature

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of equilibrium points of the designed neural networks. In addition, owing to the external interference and finite speed of data processing, time-delay unavoidably exists in neural networks [4]. As far as we well-know an important factor affecting the dynamic behavior of the system is time-delay, which may create complicated dynamical behavior such as chaos, instability, and oscillations [5–7]. Particularly, some primary stable systems may reach into an oscillating or chaotic state, which caused huge damage for engineering. Hence, the stability of the neural networks with time-delay has allured many researchers, and some stability criteria have been shown in [8–13]. The improved stability criteria for neural networks with time-delay can be divided into two types: delay-independent and delay-dependent. After comparing, the delay-dependent stability criteria, which contain the data of time delay, normally have less conservative, principally when applied to neural networks with time-delay with small delay, because the latter takes advantage of the further information of the time delays. Therefore, most researchers are interested in delay-dependent stability analysis and the main goal is to reduce the conservatism of the stability condition.

In many systems, consideration of long-time behavior of status variables is not enough because the state variable values during the transient period may be too large or unrealistic before reaching the equilibrium point. In a chemical process, for instance, the temperature inside a container must be maintained within certain criteria for a period of time for the chemicals to take effect. This situation is commonly known as finite-time stability (FTS) which was introduced by Dorato in 1961 [14]. As a result, a large number of researchers are more interested in studying the FTS of various systems. During the past decades, researchers presented criteria that guarantee FTS of various systems with finding the smallest upper bound of the norm square of state variables or finding the maximum time that guarantees values of the state variables to be within the given bounds for a certain time. Some examples of FTS of linear system with constant delay are studied in [15–22]. FTS of linear system with time-varying delays [23–27] and FTS on other systems [28–34].

However, in some practical situations, stabilization and synchronization should be executed in finite time. Thus, a study for finite-time synchronization is necessary. Some authors have investigated synchronization based on finite-time stability theory [35, 36]. In [35], the authors studied the finite-time synchronization of dynamical networks with complex-variable chaotic systems. The Finite-time synchronization control for uncertain Markov jump neural networks with input constraints was investigated [36]. In [37], the authors showed the finite-time synchronization of time-delayed neural networks with unknown parameters via adaptive control. The authors of [38] investigated the finite-time synchronization of Markovian jumping complex dynamical networks and hybrid couplings. As far as we know, there are few reports on the list of finite-time synchronization of delayed neural networks.

As mentioned above, FTS is one of the important topics that should have been further studied. Thus, in this article, we investigate the finite-time synchronization of neural networks with interval and distributed time-varying delays. This article is organized as follows. In Section 2, we introduce the considered systems and review important definitions and lemmas. Then, proof of the new integral inequality in the form of one free matrix is proposed. This inequality will be used for bounding the derivative of LKF which allows us to obtain delay-dependent FTS criteria in Section 3. A numerical example is given in Section 4 to show the effectiveness of the proposed criteria. The conclusion is drawn in Section 5.

## 2. Problem Statement and Preliminaries

The following notations will be used in this paper: $\mathbb{R}^n$ denotes the $n-$dimensional space; $\mathbb{R}^{n\times m}$ denotes real value matrix with dimension $n \times m$; $A^T$ denotes the transpose of matrix $A$; $A$ is symmetric if $A = A^T$; $\lambda(A)$ denotes all the eigenvalue of $A$; $\lambda_{\max}(A) = \max\{Re\ \lambda : \lambda \in \lambda(A)\}$; $\lambda_{\min}(A) = \min\{Re\ \lambda : \lambda \in \lambda(A)\}$; $A > 0$ or $A < 0$ denotes that the matrix $A$ is a symmetric and positive definite or negative definite matrix; If $A, B$ are symmetric matrices, $A > B$ means that $A - B$ is positive definite matrix; $I$ denotes the identity matrix with appropriate dimensions. The symmetric term in the matrix is denoted by $*$. The following norms will be used: $||\cdot||$ refer to the Euclidean vector norm and diag$\{\ldots\}$ denotes a block diagonal matrix; $sym\{A\} = A + A^T$ and $col\{a_1, a_2, ..., a_n\} = [a_1^T, a_2^T, ..., a_n^T]^T$.

In this paper, the master-slave neural networks with interval and distributed time-varying delays are described as follows:

$$\begin{cases} \dot{x}(t) = -Ax(t) + C\tilde{f}(x(t)) + D\tilde{f}(x(t-h(t))) + E\int_{t-d(t)}^{t}\tilde{f}(x(s))ds + J(t), \\ x(t) = \phi_1(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{y}(t) = -Ay(t) + C\tilde{f}(y(t)) + D\tilde{f}(y(t-h(t))) + E\int_{t-d(t)}^{t}\tilde{f}(y(s))ds + J(t) \\ \qquad\quad +Bu(t), \\ y(t) = \phi_2(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.2}$$

where $x(t) = [x_1(t), x_2(t), ..., x_n(t)] \in \mathbb{R}^n$ and $y(t) = [y_1(t), y_2(t), ..., y_n(t)] \in \mathbb{R}^n$ are the master systems state vector and the slave systems state vector of the neural networks, respectively. $\phi_1(t)$ and $\phi_2(t) \in \mathbb{C}[[-\tau, 0], \mathbb{R}^n]$ are the continuous initial function, $\tilde{f}(x(t)) = [\tilde{f}(x_1(t)), \tilde{f}(x_2(t)), ..., \tilde{f}(x_n(t))]^T$ is the neuron activation function, $A = diag\{a_1, ..., a_n\} > 0$ is a diagonal matrix, $B, C, D, E$ are the known real constant matrices with appropriate dimensions.

The synchronization error $e(t)$ is the form $e(t) = y(t) - x(t)$. Therefore, the neural networks with mixed time varying delays of synchronization error between the master-slave systems given in (2.1) and (2.2) can be described by

$$\begin{cases} \dot{e}(t) = -Ae(t) + Cf(e(t)) + Df(e(t-h(t))) + E\int_{t-d(t)}^{t} f(e(s))ds + Bu(t), \\ e(t) = \phi_2(t) - \phi_1(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.3}$$

where $f(e(t)) = \tilde{f}(e(t)+x(t)) - \tilde{f}(x(t))$. Now, we consider the state feedback control law:

$$u(t) = Ke(t), \tag{2.4}$$

where $K$ is a constant matrix control gain. Then, substituting (2.4) into (2.3), it is easy to get the following:

$$\begin{cases} \dot{e}(t) = -(A - BK)e(t) + Cf(e(t)) + Df(e(t-h(t))) + E\int_{t-d(t)}^{t} f(e(s))ds, \\ e(t) = \phi_2(t) - \phi_1(t) = \phi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{2.5}$$

The time-varying delay functions $h(t)$ and $d(t)$ satisfy the conditions

$$\begin{aligned} 0 \leq h_1 \leq h(t) \leq h_2, \qquad \dot{h}(t) \leq \mu, \\ 0 \leq d(t) \leq d, \end{aligned} \tag{2.6}$$

where $h_1$, $h_2$, $\mu$, $d$, $\tau = \max\{h_2, d\}$ are known real constant scalars and we denote $h_{12} = h_2 - h_1$, $h_{1t} = h(t) - h_1$, $h_{2t} = h_2 - h(t)$.

**Assumption 1.** The neuron activation functions $\tilde{f}(\cdot)$ is bounded, $\tilde{f}(0) = 0$ and there exits constant $l_i^-, l_i^+$ such that

$$l_i^- \leq \frac{\tilde{f}_i(y) - \tilde{f}_i(x)}{y - x} \leq l_i^+, i = 1, 2, ..., n, \tag{2.7}$$

where $y, x \in \mathbb{R}$ with $y \neq x$. Denote $L_1 = diag\{l_1^- l_1^+, l_2^- l_2^+, ..., l_n^- l_n^+\}$, $L_2 = diag\{\frac{l_1^- + l_1^+}{2}, ..., \frac{l_n^- + l_n^+}{2}\}$ and $L = \max\{|l_i^-|, |l_i^+|\}$.

**Definition 2.1** ([25]). Given three positive constants $c_1, c_2, T$ with $c_1 < c_2$, the time-delay system described by (2.5) and delay condition as in (2.6) is said to be finite-time stable with respect to $(c_1, c_2, T, \tau)$, if the state variables satisfy the relationship:

$$\sup_{-\tau \leq s \leq 0} \{e^T(s)e(s), \dot{e}^T(s)\dot{e}(s)\} \leq c_1 \Rightarrow e^T(t)e(t) < c_2, \forall t \in [0, T].$$

**Lemma 2.2** ([39]). *For a positive definite matrix $Z \in \mathbb{R}^{n\times n}$, and two scalars $0 \leq r_1 < r_2$ and vector function $x : [r_1, r_2] \to \mathbb{R}^n$ such that the following integrals are well defined, one has*

$$\Big(\int_{r_1}^{r_2} x(s)ds\Big)^T Z \Big(\int_{r_1}^{r_2} x(s)ds\Big) \quad \leq \quad (r_2 - r_1)\int_{r_1}^{r_2} x^T(s)Zx(s)ds.$$

**Lemma 2.3** ([40]). *Let $\omega$ be a differential function $\omega : [\alpha, \beta] \to \mathbb{R}^n$. For a positive definite symmetric matrix $R$, scalars $\beta > \alpha$ and any matrices $M_{1i} \in \mathbb{R}^{5n\times n}$, $M_{2i} \in \mathbb{R}^{4n\times n}, i = 1, 2, 3$, the following integral inequalities hold:*

$$-\int_{\alpha}^{\beta} \omega^T(s)R\omega(s)ds \leq \tilde{\zeta}^T(t)\Phi_1\tilde{\zeta}(t), \tag{2.8}$$

$$-\int_{\alpha}^{\beta} \dot{\omega}^T(s)R\dot{\omega}(s)ds \leq \tilde{\zeta}^T(t)\Phi_2\tilde{\zeta}(t), \tag{2.9}$$

*where*

$$
\begin{aligned}
\Phi_1 &= (\beta-\alpha)\Upsilon_1^T\left(M_{11}R^{-1}M_{11}^T+\frac{1}{3}M_{12}R^{-1}M_{12}^T+\frac{1}{5}M_{13}R^{-1}M_{13}^T\right)\Upsilon_1\\
&\quad +sym\{\Upsilon_1^TM_{11}\Xi_{11}+\Upsilon_1^TM_{12}\Xi_{12}+\Upsilon_1^TM_{13}\Xi_{13}\}\\
\Phi_2 &= (\beta-\alpha)\Upsilon_2^T\left(M_{21}R^{-1}M_{21}^T+\frac{1}{3}M_{22}R^{-1}M_{22}^T+\frac{1}{5}M_{23}R^{-1}M_{23}^T\right)\Upsilon_2\\
&\quad +sym\{\Upsilon_2^TM_{21}\Xi_{21}+\Upsilon_2^TM_{22}\Xi_{22}+\Upsilon_1^TM_{23}\Xi_{23}\}\\
\tilde{\zeta}(t) &= col\left\{\omega(\beta),\omega(\alpha),\frac{1}{\beta-\alpha}\int_\alpha^\beta\omega(s)ds,\frac{1}{(\beta-\alpha)^2}\int_\alpha^\beta\int_\theta^\beta\omega(s)dsd\theta,\right.\\
&\quad\left.\frac{1}{(\beta-\alpha)^3}\int_\alpha^\beta\int_\theta^\beta\int_r^\beta\omega(s)dsd\theta dr\right\}\\
\Upsilon_1 &= col\{e_1,e_2,e_3,e_4,e_5\},\quad \Upsilon_2=col\{e_1,e_2,e_3,e_4\}\\
\Xi_{11} &= (\beta-\alpha)e_3,\quad \Xi_{12}=(\beta-\alpha)(2e_4-e_3),\quad \Xi_{13}=(\beta-\alpha)(e_3-6e_4+12e_5),\\
\Xi_{21} &= e_1-e_2,\quad \Xi_{22}=e_1+e_2-2e_3,\quad \Xi_{23}=e_1-e_2+6e_3-12e_4,\\
e_i &= [0_{n\times(i-1)n}\quad I\quad 0_{n\times(5-i)n}], i=1,2,3,4,5.
\end{aligned}
$$

**Lemma 2.4** ([40]). *For a block symmetric matrix $R_1 = diag\{R, 3R, 5R\}$ with $R > 0$, and any matrices $S_1$, then the following single integral inequality hold:*

$$
\begin{aligned}
-\int_{t-h_2}^{t-h_1}\dot{\omega}^T(s)R\dot{\omega}(s)ds &= -\int_{t-h(t)}^{t-h_1}\dot{\omega}^T(s)R\dot{\omega}(s)ds-\int_{t-h_2}^{t-h(t)}\dot{\omega}^T(s)R\dot{\omega}(s)ds\\
&\le -\frac{1}{h_{12}}\eta^T(t)\begin{bmatrix}E_1\\E_2\end{bmatrix}^T\left\{\begin{bmatrix}R_1 & S_1\\ * & R_1\end{bmatrix}\right.\\
&\quad\left.+\begin{bmatrix}\frac{h_{2t}}{h_{12}}T_1 & 0\\ 0 & \frac{h_{1t}}{h_{12}}T_2\end{bmatrix}\right\}\begin{bmatrix}E_1\\E_2\end{bmatrix}\eta(t),
\end{aligned}\tag{2.10}
$$

*where*

$$
\begin{aligned}
T_1 &= R_1-S_1R_1^{-1}S_1^T,\quad T_2=R_1-S_1^TR_1^{-1}S_1,\\
\eta(t) &= col\left\{\omega(t-h_1),\omega(t-h(t)),\omega(t-h_2),\frac{1}{h_{1t}}\int_{t-h(t)}^{t-h_1}\omega(s)ds,\frac{1}{h_{2t}}\int_{t-h_2}^{t-h(t)}\omega(s)ds,\right.\\
&\quad\left.\frac{1}{h_{1t}^2}\int_{t-h(t)}^{t-h_1}\int_\theta^{t-h_1}\omega(s)dsd\theta,\frac{1}{h_{2t}^2}\int_{t-h_2}^{t-h(t)}\int_\theta^{t-h(t)}\omega(s)dsd\theta\right\},\\
E_1 &= col\{e_1-e_2,e_1+e_2-2e_4,e_1-e_2+6e_4-12e_6\},\\
E_2 &= col\{e_2-e_3,e_2+e_3-2e_5,e_2-e_3+6e_5-12e_7\},\\
e_i &= [0_{n\times(i-1)n}\quad I\quad 0_{n\times(7-i)n}], i=1,2,...,7.
\end{aligned}
$$

**Lemma 2.5** ([41] Wirtinger-Based Integral Inequality). *For a positive definite matrix $R > 0$, scalar $\alpha$ and $\beta$ with $\alpha < \beta$, and continuously differentiable function $\omega : [\alpha, \beta] \to$*

$\mathbb{R}^n$, *the following integral inequality holds:*

$$\int_{\alpha}^{\beta} \dot{\omega}^T(s)R\dot{\omega}(s)ds \geq \frac{1}{\beta-\alpha}\Omega_1^T R\Omega_1 + \frac{3}{\beta-\alpha}\Omega_2^T R\Omega_2, \tag{2.11}$$

*where*

$$\begin{aligned} \Omega_1 &= \omega(\beta) - \omega(\alpha), \\ \Omega_2 &= \omega(\beta) + \omega(\alpha) - \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} \omega(s)ds. \end{aligned}$$

**Lemma 2.6** ([42] Extended Reciprocally Convex Matrix Inequality)**.** *For any real scalars $\alpha_i > 0 (i = 1, 2, ..., m)$, $n \times n$ symmetric matrices $R_i > 0 (i = 1, 2, ..., m)$, and any $mn \times mn$ matrix $M$, the following matrix inequality holds:*

$$\begin{aligned} &\begin{bmatrix} \frac{1}{\alpha_1}R_1 & 0 & \dots & 0 \\ 0 & \frac{1}{\alpha_2}R_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\alpha_m}R_m \end{bmatrix} \\ \geq\ & -M - M^T - M^T \begin{bmatrix} \alpha_1 R_1^{-1} & 0 & \dots & 0 \\ 0 & \alpha_2 R_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_m R_m^{-1} \end{bmatrix} M. \end{aligned} \tag{2.12}$$

**Lemma 2.7** ([43])**.** *For a positive definite matrix $R > 0$, scalar $\alpha$ and $\beta$ with $\alpha < \beta$, and continuously differentiable function $\omega : [\alpha, \beta] \to \mathbb{R}^n$, the following integral inequality holds:*

$$\int_{\alpha}^{\beta}\int_{u}^{\beta} \dot{\omega}^T(s)R\dot{\omega}(s)dsdu \geq 2\Omega_3^T R\Omega_3 + 4\Omega_4^T R\Omega_4 + 6\Omega_5^T R\Omega_5, \tag{2.13}$$

*where*

$$\begin{aligned} \Omega_3 &= \omega(\beta) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \omega(s)ds, \\ \Omega_4 &= \omega(\beta) + \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} \omega(s)ds - \frac{6}{(\beta-\alpha)^2}\int_{\alpha}^{\beta}\int_{u}^{\beta} \omega(s)dsdu, \\ \Omega_5 &= \omega(\beta) - \frac{3}{\beta-\alpha}\int_{\alpha}^{\beta} \omega(s)ds + \frac{24}{(\beta-\alpha)^2}\int_{\alpha}^{\beta}\int_{u}^{\beta} \omega(s)dsdu \\ &\quad - \frac{60}{(\beta-\alpha)^3}\int_{\alpha}^{\beta}\int_{u}^{\beta}\int_{r}^{\beta} \omega(s)dsdrdu. \end{aligned}$$

## 3. Main Results

Before introducing the main result, following notations are defined for simplicity

$$
\begin{aligned}
e_i &= [0_{n\times(i-1)n} \quad I \quad 0_{n\times(23-i)n}], i=1,2,...,23,\\
\tilde{e}_i &= [0_{n\times(i-1)n} \quad I \quad 0_{n\times(6-i)n}], i=1,2,...,6,\\
\gamma_1 &= h_2L^2, \gamma_2=\frac{L^2d^3}{2}, \gamma_3=\frac{h_1^3}{2}, \gamma_4=\frac{h_{12}^3}{2}, \gamma_5=\frac{h_2^2}{2}, \gamma_6=\frac{h_2^3}{6}, \gamma_7=\frac{h_1^3}{6}, \gamma_8=\frac{h_{12}^3}{6},\\
\xi(t) &= col\Big\{e(t), e(t-h_1), e(t-h_2), e(t-h(t)), \dot{e}(t), \dot{e}(t-h_1), \dot{e}(t-h_2), f(e(t)),\\
&\quad f(e(t-h(t))), \int_{t-d(t)}^{t} f(e(s))ds, \frac{1}{h_1}\int_{t-h_1}^{t} e(s)ds, \frac{1}{h_1^2}\int_{t-h_1}^{t}\int_{u}^{t} e(s)dsdu,\\
&\quad \frac{1}{h_1^3}\int_{t-h_1}^{t}\int_{u}^{t}\int_{v}^{t} e(s)dsdvdu, \frac{1}{h_{12}}\int_{t-h_2}^{t-h_1} e(s)ds, \frac{1}{h_{12}^2}\int_{t-h_2}^{t-h_1}\int_{u}^{t-h_1} e(s)dsdu,\\
&\quad \frac{1}{h_{12}^3}\int_{t-h_2}^{t-h_1}\int_{u}^{t-h_1}\int_{v}^{t-h_1} e(s)dsdvdu, \frac{1}{h_{1t}}\int_{t-h(t)}^{t-h_1} e(s)ds,\\
&\quad \frac{1}{h_{1t}^2}\int_{t-h(t)}^{t-h_1}\int_{u}^{t-h_1} e(s)dsdu, \frac{1}{h_{2t}}\int_{t-h_2}^{t-h(t)} e(s)ds, \frac{1}{h_{2t}^2}\int_{t-h_2}^{t-h(t)}\int_{u}^{t-h(t)} e(s)dsdu,\\
&\quad \frac{1}{h_2}\int_{t-h_2}^{t} e(s)ds, \frac{1}{h_2^2}\int_{t-h_2}^{t}\int_{u}^{t} e(s)dsdu, \frac{1}{h_2^3}\int_{t-h_2}^{t}\int_{u}^{t}\int_{v}^{t} e(s)dsdvdu\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\eta_1(t) &= col\left\{e(t)-e(t-h_1), e(t)+e(t-h_1)-\frac{2}{h_1}\int_{t}^{t-h_1} e(s)ds\right\},\\
\eta_2(t) &= col\left\{e(t-h_1)-e(t-h(t)), e(t-h_1)+e(t-h(t))-\frac{2}{h_{1t}}\int_{t-h(t)}^{t-h_1} e(s)ds\right\},\\
\eta_2(t) &= col\left\{e(t-h(t))-e(t-h_2), e(t-h(t))+e(t-h_2)-\frac{2}{h_{2t}}\int_{t-h_2}^{t-h(t)} e(s)ds\right\}.
\end{aligned}
$$

Now, we provide a stability criterion for the error system (2.5) with time-varying delay $h(t)$ and $d(t)$ satisfy (2.6).

**Theorem 3.1.** *The error systems (2.5) with time-varying delay $h(t)$ and $d(t)$ satisfying (2.6) is finite-time stable with respect to $(c_1, c_2, T, \tau)$, $0 \le c_1 < c_2$, if there exist positive scalar $\alpha$, $\lambda_m$, $(m = 1, 2, ..., 17)$, symmetric positive definite matrices $P$, $W_1$, $W_2$, $Q_i$, $(i = 1, 2, 3, 4, 5)$, $R_j$, $Z_j$, $(j = 1, 2, 3, 4) \in \mathbb{R}^{n\times n}$, positive diagonal matrices $G_1$, $G_2 \in \mathbb{R}^{n\times n}$, and any matrices $N$, $X$, $Y \in \mathbb{R}^{n\times n}$, $S_1 \in \mathbb{R}^{3n\times 3n}$, $M \in \mathbb{R}^{6n\times 6n}$, $M_{1k}, M_{3k} \in \mathbb{R}^{5n\times n}$ and $M_{2k} \in \mathbb{R}^{4n\times n}$, $k = 1, 2, 3$ such that the following LMIs hold:*

$$
\Pi_1 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ * & \Sigma_{22} \end{bmatrix}_{h(t)=h_2} < 0, \tag{3.1}
$$

$$
\Pi_2 = \begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ * & \Sigma_{22} \end{bmatrix}_{h(t)=h_1} < 0, \tag{3.2}
$$

$$\lambda_1 I < P < \lambda_2 I, \quad Q_1 < \lambda_3 I, \quad Q_2 < \lambda_4 I, \quad Q_3 < \lambda_5 I, \quad Q_4 < \lambda_6 I, \quad Q_5 < \lambda_7 I,$$
$$W_1 < \lambda_8 I, \quad W_2 < \lambda_9 I, \quad Z_1 < \lambda_{10} I, \quad Z_2 < \lambda_{11} I, \quad Z_3 < \lambda_{12} I, \quad Z_4 < \lambda_{13} I,$$
$$R_1 < \lambda_{14} I, \quad R_2 < \lambda_{15} I, \quad R_3 < \lambda_{16} I, \quad R_4 < \lambda_{17} I, \tag{3.3}$$
$$e^{\alpha T}\Lambda c_1 - \lambda_1 c_2 < 0, \tag{3.4}$$

*where*

$$\begin{aligned}
\Lambda &= \lambda_2 + h_1(\lambda_3+\lambda_6) + h_{12}(\lambda_4+\lambda_5+\lambda_7) + \gamma_1\lambda_8 + \gamma_2\lambda_9 + \gamma_3(\lambda_{10}+\lambda_{11}) \\
&\quad +\gamma_4(\lambda_{12}+\lambda_{13}) + \gamma_5\lambda_{14} + \gamma_6\lambda_{15} + \gamma_7\lambda_{16} + \gamma_8\lambda_{17}, \\
\Sigma_{11} &= \psi + \psi_{51} + \psi_9, \\
\tilde{\Sigma}_{11} &= \psi + \psi_{52} + \psi_9, \\
\Sigma_{12} &= [h_1\Upsilon_1^T M_{11}, h_1\Upsilon_1^T M_{12}, h_1\Upsilon_1^T M_{13}, h_1\Upsilon_2^T M_{21}, h_1\Upsilon_2^T M_{22}, h_1\Upsilon_2^T M_{23}, \\
&\quad h_{12}\Upsilon_3^T M_{31}, h_{12}\Upsilon_3^T M_{32}, h_{12}\Upsilon_3^T M_{33}, E_2^T S_1^T, E_3^T M^T E_{31}^T, E_3^T M^T E_{32}^T], \\
\tilde{\Sigma}_{12} &= [h_1\Upsilon_1^T M_{11}, h_1\Upsilon_1^T M_{12}, h_1\Upsilon_1^T M_{13}, h_1\Upsilon_2^T M_{21}, h_1\Upsilon_2^T M_{22}, h_1\Upsilon_2^T M_{23}, \\
&\quad h_{12}\Upsilon_3^T M_{31}, h_{12}\Upsilon_3^T M_{32}, h_{12}\Upsilon_3^T M_{33}, E_1^T S_1, E_3^T M^T E_{31}^T, E_3^T M^T E_{33}^T], \\
\Sigma_{22} &= -diag\{Z_1, 3Z_1, 5Z_1, Z_2, 3Z_2, 5Z_2, Z_3, 3Z_3, 5Z_3, \tilde{Z}_4, \frac{1}{h_1}\tilde{R}_1, \frac{1}{h_{12}}\tilde{R}_1\}, \\
\tilde{Z}_4 &= diag\{Z_4, 3Z_4, 5Z_4\}, \quad \tilde{R}_1 = diag\{R_1, 3R_1\}, \\
\psi &= \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_6 + \psi_7 + \psi_8 - e_1^T \alpha P e_1,
\end{aligned}$$

$$\begin{aligned}
\psi_1 &= sym\{e_1^T P e_5\}, \\
\psi_2 &= e_1^T Q_1 e_1 - (1-\mu) e_4^T Q_3 e_4 + e_2^T(Q_2 - Q_1 + Q_3)e_2 - e_3^T Q_2 e_3 + e_5^T Q_4 e_5 \\
&\quad + e_6^T(Q_5 - Q_4)e_6 - e_7^T Q_5 e_7, \\
\psi_3 &= e_8^T W_1 e_8 - (1-\mu) e_9^T W_2 e_9 + e_8^T d^2 W_2 e_8 - e_{10}^T W_2 e_{10}, \\
\psi_4 &= e_1^T h_1^2 Z_1 e_1 + e_5^T h_1^2 Z_2 e_5 + h_1 sym\{\Upsilon_1^T M_{11}\Xi_{11} + \Upsilon_1^T M_{12}\Xi_{12} + \Upsilon_1^T M_{13}\Xi_{13}\} \\
&\quad + h_1 sym\{\Upsilon_2^T M_{21}\Xi_{21} + \Upsilon_2^T M_{22}\Xi_{22} + \Upsilon_2^T M_{23}\Xi_{23}\}, \\
\psi_5 &= e_2^T h_{12}^2 Z_3 e_2 + e_6^T h_{12}^2 Z_4 e_6 + h_{12} sym\{\Upsilon_3^T M_{31}\Xi_{31} + \Upsilon_3^T M_{32}\Xi_{32} + \Upsilon_3^T M_{33}\Xi_{33}\} \\
\tilde{\psi}_5 &= \psi_5 - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} \left(1+\frac{h_{2t}}{h_{12}}\right)\tilde{Z}_4 & S_1 \\ * & \left(1+\frac{h_{1t}}{h_{12}}\right)\tilde{Z}_4 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \\
\psi_{51} &= \psi_5 - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} \tilde{Z}_4 & S_1 \\ * & 2\tilde{Z}_4 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \\
\psi_{52} &= \psi_5 - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} 2\tilde{Z}_4 & S_1 \\ * & \tilde{Z}_4 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \\
\psi_6 &= e_5^T h_2 R_1 e_5 + sym\{E_3^T M E_3\}, \\
\psi_7 &= e_5^T \frac{h_2^2}{2} R_2 e_5 + e_5^T \frac{h_1^2}{2} R_3 e_5 + e_6^T \left(\frac{h_{12}^2 - h_1 h_{12}}{2}\right) R_4 e_6 - 2\pi_1^T R_2 \pi_1 - 4\pi_2^T R_2 \pi_2 \\
&\quad -6\pi_3^T R_2 \pi_3 - 2\pi_4^T R_3 \pi_4 - 4\pi_5^T R_3 \pi_5 - 6\pi_6^T R_3 \pi_6 - 2\pi_7^T R_4 \pi_7 - 4\pi_8^T R_4 \pi_8 \\
&\quad -6\pi_9^T R_4 \pi_9,
\end{aligned}$$

$$
\begin{aligned}
\psi_8 &= -e_1^T L_1 G_1 e_1 - e_8^T G_1 e_8 - e_4^T L_1 G_2 e_4 - e_9^T G_2 e_9 + sym\{e_1^T L_2 G_1 e_8 + e_4^T L_2 G_2 e_9\},\\
\psi_9 &= sym\{e_1^T (BY - NA) e_1 - e_1^T N e_5 + e_1^T N C e_8 + e_1^T N D e_9 + e_1^T N E e_{10}\\
&\quad + e_5^T (BY - NA) e_1 - e_5^T N e_5 + e_5^T N C e_8 + e_5^T N D e_9 + e_5^T N E e_{10}\},\\
\Upsilon_1 &= col\{e_1, e_2, e_{11}, e_{12}, e_{13}\}, \quad \Upsilon_2 = col\{e_1, e_2, e_{11}, e_{12}\},\\
\Upsilon_3 &= col\{e_2, e_3, e_{14}, e_{15}, e_{16}\},\\
\Xi_{11} &= h_1 e_{11}, \quad \Xi_{12} = h_1(2e_{12} - e_{11}), \quad \Xi_{13} = h_1(e_{11} - 6e_{12} + 12e_{13}),\\
\Xi_{21} &= e_1 - e_2, \quad \Xi_{22} = e_1 + e_2 - e_{11}, \quad \Xi_{23} = e_1 - e_2 + 6e_{11} - 12e_{12},\\
\Xi_{31} &= h_{12} e_{14}, \quad \Xi_{32} = h_{12}(2e_{15} - e_{14}), \quad \Xi_{33} = h_{12}(e_{14} - 6e_{15} + 12e_{16}),\\
E_1 &= col\{e_2 - e_4, e_2 + e_4 - 2e_{17}, e_2 - e_4 + 6e_{17} - 12e_{18}\},\\
E_2 &= col\{e_4 - e_3, e_4 + e_3 - 2e_{19}, e_4 - e_3 + 6e_{19} - 12e_{20}\},\\
E_3 &= col\{e_1 - e_2, e_1 + e_2 - 2e_{11}, e_2 - e_4, e_2 + e_4 - 2e_{17}, e_4 - e_3, e_4 + e_3 - 2e_{19}\},\\
E_{3i} &= col\{\tilde{e}_{2i-1}, \tilde{e}_{2i}\}(i = 1, 2, 3),\\
\pi_1 &= e_1 - e_{21}, \quad \pi_2 = e_1 + 2e_{21} - 6e_{22}, \quad \pi_3 = e_1 - 3e_{21} + 24e_{22} - 60e_{23},\\
\pi_4 &= e_1 - e_{11}, \quad \pi_5 = e_1 + 2e_{11} - 6e_{12}, \quad \pi_6 = e_1 - 3e_{11} + 24e_{12} - 60e_{13},\\
\pi_7 &= e_2 - e_{14}, \quad \pi_8 = e_2 + 2e_{14} - 6e_{15}, \quad \pi_9 = e_2 - 3e_{14} + 24e_{15} - 60e_{16},
\end{aligned}
$$

*Moreover, the desired controller is given as follows:*

$$K = YX^{-1}. \tag{3.5}$$

*Proof.* Consider the following Lypunov-Krasovskii functional:

$$V(e(t)) = \sum_{j=1}^{7} V_j(e(t)), \tag{3.6}$$

where

$$
\begin{aligned}
V_1(e(t)) &= e^T(t) P e(t),\\
V_2(e(t)) &= \int_{t-h_1}^{t} e^T(s) Q_1 e(s) ds + \int_{t-h_2}^{t-h_1} e^T(s) Q_2 e(s) ds + \int_{t-h(t)}^{t-h_1} e^T(s) Q_3 e(s) ds\\
&\quad + \int_{t-h_1}^{t} \dot{e}^T(s) Q_4 \dot{e}(s) ds + \int_{t-h_2}^{t-h_1} \dot{e}^T(s) Q_5 \dot{e}(s) ds,\\
V_3(e(t)) &= \int_{t-h(t)}^{t} f^T(e(s)) W_1 f(e(s)) ds + d \int_{-d}^{0} \int_{t+\theta}^{t} f^T(e(s)) W_2 f(e(s)) ds d\theta,\\
V_4(e(t)) &= h_1 \int_{t-h_1}^{t} \int_{u}^{t} e^T(s) Z_1 e(s) ds du + h_1 \int_{t-h_1}^{t} \int_{u}^{t} \dot{e}^T(s) Z_2 \dot{e}(s) ds du,\\
V_5(e(t)) &= h_{12} \int_{t-h_2}^{t-h_1} \int_{u}^{t-h_1} e^T(s) Z_3 e(s) ds du + h_{12} \int_{t-h_2}^{t-h_1} \int_{u}^{t-h_1} \dot{e}^T(s) Z_4 \dot{e}(s) ds du,
\end{aligned}
$$

$$\begin{aligned}
V_6(e(t)) &= \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)R_1\dot{e}(s)dsd\theta, \\
V_7(e(t)) &= \int_{t-h_2}^{t}\int_{\theta}^{t}\int_{r}^{t} \dot{e}^T(s)R_2\dot{e}(s)dsdrd\theta + \int_{t-h_1}^{t}\int_{\theta}^{t}\int_{r}^{t} \dot{e}^T(s)R_3\dot{e}(s)dsdrd\theta \\
&\quad + \int_{t-h_2}^{t-h_1}\int_{\theta}^{t-h_1}\int_{r}^{t-h_1} \dot{e}^T(s)R_4\dot{e}(s)dsdrd\theta.
\end{aligned}$$

The time derivative of $V(e(t))$ along the trajectory of system is given by:

$$\begin{aligned}
\dot{V}_1(e(t)) &= e^T(t)P\dot{e}(t) + \dot{e}^T(t)Pe(t) \\
&= \xi^T(t)\psi_1\xi(t), && (3.7) \\
\dot{V}_2(e(t)) &\leq e^T(t)Q_1e(t) - e^T(t-h_1)(Q_2 - Q_1 + Q_3) - (1-\mu)e^T(t-h(t))Q_3e(t-h(t)) \\
&\quad -e^T(t-h_2)Q_2e(t-h_2) + \dot{e}^T(t)Q_4\dot{e}(t) + \dot{e}^T(t-h_1)(Q_5 - Q_4)\dot{e}(t-h_1) \\
&\quad -\dot{e}^T(t-h_2)Q_5\dot{e}(t-h_2) \\
&= \xi^T(t)\psi_2\xi(t), && (3.8) \\
\dot{V}_3(e(t)) &\leq f^T(e(t))W_1f(e(t)) - (1-\mu)f^T(e(t-h(t)))W_1f(e(t-h(t))) \\
&\quad +d^2f^T(e(t))W_2f(e(t)) - d\int_{t-d}^{t} f^T(e(s))W_2f(e(s))ds.
\end{aligned}$$

Using Lemma 2.2 we can get

$$\begin{aligned}
\dot{V}_3(e(t)) &\leq f^T(e(t))W_1f(e(t)) - (1-\mu)f^T(e(t-h(t)))W_1f(e(t-h(t))) \\
&\quad +d^2f^T(e(t))W_2f(e(t)) - \left(\int_{t-d(t)}^{t} f(e(s))ds\right)^T W_2 \left(\int_{t-d(t)}^{t} f(e(s))ds\right) \\
&= \xi^T(t)\psi_3\xi(t). && (3.9)
\end{aligned}$$

By using integral inequalities (2.8)–(2.10) in Lemma 2.3 and Lemma 2.4, we can calculating the derivative of $V_4(x(t))$ and $V_5(x(t))$, respectively. We have

$$\begin{aligned}
\dot{V}_4(e(t)) &= h_1^2e^T(t)Z_1e(t) + h_1^2\dot{e}^T(t)Z_2\dot{e}(t) - h_1\int_{t-h_1}^{t} e^T(s)Z_1e(s)ds \\
&\quad -h_1\int_{t-h_1}^{t} \dot{e}^T(s)Z_2\dot{e}(s)ds \\
&\leq \xi^T(t)\Big\{\psi_4 + h_1^2\Upsilon_1^T(M_{11}Z_1^{-1}M_{11}^T + \frac{1}{3}M_{12}Z_1^{-1}M_{12}^T + \frac{1}{5}M_{13}Z_1^{-1}M_{13}^T)\Upsilon_1 \\
&\quad +h_1^2\Upsilon_2^T(M_{21}Z_2^{-1}M_{21}^T + \frac{1}{3}M_{22}Z_2^{-1}M_{22}^T + \frac{1}{5}M_{23}Z_2^{-1}M_{23}^T)\Upsilon_2\Big\}\xi(t), && (3.10)
\end{aligned}$$

$$
\begin{aligned}
\dot{V}_5(e(t)) &= h_{12}^2 e^T(t-h_1)Z_3e(t-h_1) + h_{12}^2\dot{e}^T(t-h_1)Z_4\dot{e}(t-h_1) \\
&\quad -h_{12}\int_{t-h_2}^{t-h_1} e^T(s)Z_3e(s)ds - h_{12}\int_{t-h_2}^{t-h_1}\dot{e}^T(s)Z_4\dot{e}(s)ds \\
&= h_{12}^2 e^T(t-h_1)Z_3e(t-h_1) + h_{12}^2\dot{e}^T(t-h_1)Z_4\dot{e}(t-h_1) \\
&\quad -h_{12}\int_{t-h_2}^{t-h_1} e^T(s)Z_3e(s)ds - h_{12}\int_{t-h_2}^{t-h(t)}\dot{e}^T(s)Z_4\dot{e}(s)ds \\
&\quad -h_{12}\int_{t-h(t)}^{t-h_1}\dot{e}^T(s)Z_4\dot{e}(s)ds \\
&\leq \xi^T(t)\Big\{\tilde{\psi}_5 + h_{12}^2\Upsilon_3^T(M_{31}Z_3^{-1}M_{31}^T + \frac{1}{3}M_{32}Z_3^{-1}M_{32}^T + \frac{1}{5}M_{33}Z_3^{-1}M_{33}^T)\Upsilon_3 \\
&\quad +E_1^T\frac{h_{2t}}{h_{12}}S_1\tilde{Z}_4^{-1}S_1^TE_1 + E_2^T\frac{h_{1t}}{h_{12}}S_1^T\tilde{Z}_4^{-1}S_1E_2\Big\}\xi(t), \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_6(e(t)) &= h_2\dot{e}^T(t)R_1\dot{e}(t) - \int_{t-h_2}^{t}\dot{e}^T(s)R_1\dot{e}(s)ds, \\
&= h_2\dot{e}^T(t)R_1\dot{e}(t) - \int_{t-h_1}^{t}\dot{e}^T(s)R_1\dot{e}(s)ds \\
&\quad -\int_{t-h(t)}^{t-h_1}\dot{e}^T(s)R_1\dot{e}(s)ds - \int_{t-h_2}^{t-h(t)}\dot{e}^T(s)R_1\dot{e}(s)ds, \qquad (3.12)
\end{aligned}
$$

by using Lemma 2.5 to estimate the upper bounds of the last three integral terms on the right hand side of equality (3.12), we get

$$
\begin{aligned}
\dot{V}_6(e(t)) &\leq h_2\dot{e}^T(t)R_1\dot{e}(t) - \frac{1}{h_1}\eta_1^T(t)\tilde{R}_1\eta_1(t) - \frac{1}{h_{1t}}\eta_2^T(t)\tilde{R}_1\eta_2(t) - \frac{1}{h_{2t}}\eta_3^T(t)\tilde{R}_1\eta_3(t) \\
&= h_2\dot{e}^T(t)R_1\dot{e}(t) + \begin{bmatrix}\eta_1(t)\\ \eta_2(t)\\ \eta_3(t)\end{bmatrix}^T\begin{bmatrix}-\frac{1}{h_1}\tilde{R}_1 & 0 & 0\\ 0 & -\frac{1}{h_{1t}}\tilde{R}_1 & 0\\ 0 & 0 & -\frac{1}{h_{2t}}\tilde{R}_1\end{bmatrix}\begin{bmatrix}\eta_1(t)\\ \eta_2(t)\\ \eta_3(t)\end{bmatrix},
\end{aligned}
\qquad (3.13)
$$

for matrix $M$, it follows Lemma 2.6 that

$$
\begin{aligned}
&\begin{bmatrix}-\frac{1}{h_1}\tilde{R}_1 & 0 & 0\\ 0 & -\frac{1}{h_{1t}}\tilde{R}_1 & 0\\ 0 & 0 & -\frac{1}{h_{2t}}\tilde{R}_1\end{bmatrix} \\
&\leq M + M^T + M^T\begin{bmatrix}h_1\tilde{R}_1^{-1} & 0 & 0\\ 0 & h_{1t}\tilde{R}_1^{-1} & 0\\ 0 & 0 & h_{2t}\tilde{R}_1^{-1}\end{bmatrix}M, \qquad (3.14)
\end{aligned}
$$

from (3.13) and (3.14), we obtain

$$
\begin{aligned}
\dot{V}_6(e(t)) &\leq \xi^T(t)\Big\{\psi_6 + h_1E_3^TM^TE_{31}^T\tilde{R}_1^{-1}E_{31}ME_3 + h_{1t}E_3^TM^TE_{32}^T\tilde{R}_1^{-1}E_{32}ME_3 \\
&\quad +h_{2t}E_3^TM^TE_{33}^T\tilde{R}_1^{-1}E_{33}ME_3\Big\}\xi(t). \qquad (3.15)
\end{aligned}
$$

By applying (2.13) in Lemma 2.7, we get

$$\begin{aligned}
\dot{V}_7(e(t)) &= \frac{h_2^2}{2}\dot{e}^T(t)R_2\dot{e}(t) - \int_{t-h_2}^{t}\int_{r}^{t}\dot{e}^T(s)R_2\dot{e}(s)dsdr + \frac{h_1^2}{2}\dot{e}^T(t)R_3\dot{e}(t) \\
&\quad - \int_{t-h_1}^{t}\int_{r}^{t}\dot{e}^T(s)R_3\dot{e}(s)dsdr + \frac{h_{12}^2 - h_1h_{12}}{2}\dot{e}^T(t-h_1)R_4\dot{e}(t-h_1) \\
&\quad - \int_{t-h_2}^{t-h_1}\int_{r}^{t-h_1}\dot{e}^T(s)R_4\dot{e}(s)dsdr \\
&\leq \xi^T(t)\psi_7\xi(t). \qquad (3.16)
\end{aligned}$$

From the Assumption 1, for any positive diagonal matrices $G_1$ and $G_2$, we have

$$\begin{bmatrix} e(t) \\ f(e(t)) \end{bmatrix}^T \begin{bmatrix} L_1G_1 & -L_2G_1 \\ * & G_1 \end{bmatrix} \begin{bmatrix} e(t) \\ f(e(t)) \end{bmatrix} \leq 0, \qquad (3.17)$$

$$\begin{bmatrix} e(t-h(t)) \\ f(e(t-h(t))) \end{bmatrix}^T \begin{bmatrix} L_1G_2 & -L_2G_2 \\ * & G_2 \end{bmatrix} \begin{bmatrix} e(t-h(t)) \\ f(e(t-h(t))) \end{bmatrix} \leq 0. \qquad (3.18)$$

For any matrix $N$ the following equality holds:

$$\begin{aligned}
&2[e^T(t)N][-\dot{e}(t) - (A-BK)e(t) + Cf(e(t)) + Df(e(t-h(t))) \\
&\qquad + E\int_{t-d(t)}^{t} f(e(s))ds] = 0, \qquad (3.19) \\
&2[\dot{e}^T(t)N][-\dot{e}(t) - (A-BK)e(t) + Cf(e(t)) + Df(e(t-h(t))) \\
&\qquad + E\int_{t-d(t)}^{t} f(e(s))ds] = 0. \qquad (3.20)
\end{aligned}$$

Combining (3.7)–(3.20), we can get

$$\begin{aligned}
\dot{V}(e(t)) &\leq \xi^T(t)\Theta\xi(t) + \alpha V_1(e(t)) \\
&< \xi^T(t)\Theta\xi(t) + \alpha V(e(t)), \qquad (3.21)
\end{aligned}$$

where

$$\begin{aligned}
\Theta &= \tilde{\psi} + \Delta + E_1^T\frac{h_{2t}}{h_{12}}S_1\tilde{Z}_4^{-1}S_1^TE_1 + E_2^T\frac{h_{1t}}{h_{12}}S_1^T\tilde{Z}_4^{-1}S_1E_2 + h_1E_3^TM^TE_{31}^T\tilde{R}_1^{-1}E_{31}ME_3 \\
&\quad + h_{1t}E_3^TM^TE_{32}^T\tilde{R}_1^{-1}E_{32}ME_3 + h_{2t}E_3^TM^TE_{33}^T\tilde{R}_1^{-1}E_{33}ME_3, \\
\tilde{\psi} &= \psi + \tilde{\psi}_5 + \tilde{\psi}_9, \\
\tilde{\psi}_9 &= sym\{e_1^T(-N(A-BK))e_1 - e_1^TNe_5 + e_1^TNCe_8 + e_1^TNDe_9 + e_1^TNEe_{10} \\
&\quad + e_5^T(-N(A-BK))e_1 - e_5^TNe_5 + e_5^TNCe_8 + e_5^TNDe_9 + e_5^TNEe_{10}\}, \\
\Delta &= h_1^2\Upsilon_1^T(M_{11}Z_1^{-1}M_{11}^T + \frac{1}{3}M_{12}Z_1^{-1}M_{12}^T + \frac{1}{5}M_{13}Z_1^{-1}M_{13}^T)\Upsilon_1 \\
&\quad + h_1^2\Upsilon_2^T(M_{21}Z_2^{-1}M_{21}^T + \frac{1}{3}M_{22}Z_2^{-1}M_{22}^T + \frac{1}{5}M_{23}Z_2^{-1}M_{23}^T)\Upsilon_2 \\
&\quad + h_{12}^2\Upsilon_3^T(M_{31}Z_3^{-1}M_{31}^T + \frac{1}{3}M_{32}Z_3^{-1}M_{32}^T + \frac{1}{5}M_{33}Z_3^{-1}M_{33}^T)\Upsilon_3,
\end{aligned}$$

pre-multiplying and post-multiplying $\Theta$ by $N^{-1}$ and $N^{-1}$ respectively, letting $X = N^{-1}$ and $K = YX^{-1}$, we obtain

$$\dot{V}(e(t)) \quad < \quad \xi^T(t)\tilde{\Pi}\xi(t) + \alpha V(e(t)), \tag{3.22}$$

where

$$\begin{aligned}
\tilde{\Pi} &= \hat{\psi} + \Delta + E_1^T \frac{h_{2t}}{h_{12}} S_1 \tilde{Z}_4^{-1} S_1^T E_1 + E_2^T \frac{h_{1t}}{h_{12}} S_1^T \tilde{Z}_4^{-1} S_1 E_2 + h_1 E_3^T M^T E_{31}^T \tilde{R}_1^{-1} E_{31} M E_3 \\
&\quad + h_{1t} E_3^T M^T E_{32}^T \tilde{R}_1^{-1} E_{32} M E_3 + h_{2t} E_3^T M^T E_{33}^T \tilde{R}_1^{-1} E_{33} M E_3, \\
\hat{\psi} &= \psi + \tilde{\psi}_5 + \psi_9.
\end{aligned}$$

Based on convex combination technique, $\tilde{\Pi} < 0$ holds if the following two inequalities hold:

$$\begin{aligned}
&\Sigma_{11} + \Delta + E_2^T S_1^T \tilde{Z}_4^{-1} S_1 E_2 + h_1 E_3^T M^T E_{31}^T \tilde{R}_1^{-1} E_{31} M E_3 \\
&\qquad + h_{12} E_3^T M^T E_{32}^T \tilde{R}_1^{-1} E_{32} M E_3 \quad < \quad 0, \qquad (3.23) \\
&\tilde{\Sigma}_{11} + \Delta + E_1^T S_1 \tilde{Z}_4^{-1} S_1^T E_1 + h_1 E_3^T M^T E_{31}^T \tilde{R}_1^{-1} E_{31} M E_3 \\
&\qquad + h_{12} E_3^T M^T E_{33}^T \tilde{R}_1^{-1} E_{33} M E_3 \quad < \quad 0. \qquad (3.24)
\end{aligned}$$

Applying Schur complement the inequality (3.23) and (3.24) are equivalent to $\Pi_1 < 0$ and $\Pi_2 < 0$ respectively. From (3.22) we get

$$\dot{V}(e(t)) \quad < \quad \alpha V(e(t)). \tag{3.25}$$

Multiplying the above inequality by $e^{-\alpha t}$ and integrating form 0 to $t$ with $t \in [0, T]$, we obtain

$$V(e(t)) < e^{\alpha T} V(e(0)), \tag{3.26}$$

with

$$\begin{aligned}
V(e(0)) &= e^T(0)Pe(0) + \int_{-h_1}^{0} e^T(s)Q_1 e(s)ds + \int_{-h_2}^{-h_1} e^T(s)Q_2 e(s)ds \\
&\quad + \int_{-h(0)}^{-h_1} e^T(s)Q_3 e(s)ds + \int_{-h_1}^{0} \dot{e}^T(s)Q_4 \dot{e}(s)ds + \int_{-h_2}^{-h_1} \dot{e}^T(s)Q_5 \dot{e}(s)ds \\
&\quad + \int_{-h(0)}^{0} f^T(e(s))W_1 f(e(s))ds + d\int_{-d}^{0}\int_{\theta}^{0} f^T(e(s))W_2 f(e(s))dsd\theta \\
&\quad + h_1 \int_{-h_1}^{0}\int_{u}^{0} e^T(s)Z_1 e(s)dsdu + h_1 \int_{-h_1}^{0}\int_{u}^{0} \dot{e}^T(s)Z_2 \dot{e}(s)dsdu \\
&\quad + h_{12} \int_{-h_2}^{-h_1}\int_{u}^{-h_1} e^T(s)Z_3 e(s)dsdu + h_{12} \int_{-h_2}^{-h_1}\int_{u}^{-h_1} \dot{e}^T(s)Z_4 \dot{e}(s)dsdu
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-h_2}^{0}\int_{\theta}^{0}\dot{e}^T(s)R_1\dot{e}(s)dsd\theta+\int_{-h_2}^{0}\int_{\theta}^{0}\int_{r}^{0}\dot{e}^T(s)R_2\dot{e}(s)dsdrd\theta \\
&+\int_{-h_1}^{0}\int_{\theta}^{0}\int_{r}^{0}\dot{e}^T(s)R_3\dot{e}(s)dsdrd\theta+\int_{-h_2}^{-h_1}\int_{\theta}^{-h_1}\int_{r}^{-h_1}\dot{e}^T(s)R_4\dot{e}(s)dsdrd\theta, \\
= \;& \lambda_2 e^T(0)e(0)+\lambda_3\int_{-h_1}^{0}e^T(s)e(s)ds+\lambda_4\int_{-h_2}^{-h_1}e^T(s)e(s)ds \\
&+\lambda_5\int_{-h(0)}^{-h_1}e^T(s)e(s)ds+\lambda_6\int_{-h_1}^{0}\dot{e}^T(s)\dot{e}(s)ds \\
&+\lambda_7\int_{-h_2}^{-h_1}\dot{e}^T(s)\dot{e}(s)ds+\lambda_8\int_{-h(0)}^{0}f^T(e(s))f(e(s))ds \\
&+d\lambda_9\int_{-d}^{0}\int_{\theta}^{0}f^T(e(s))f(e(s))dsd\theta+h_1\lambda_{10}\int_{-h_1}^{0}\int_{u}^{0}e^T(s)e(s)dsdu \\
&+h_1\lambda_{11}\int_{-h_1}^{0}\int_{u}^{0}\dot{e}^T(s)\dot{e}(s)dsdu+h_{12}\lambda_{12}\int_{-h_2}^{-h_1}\int_{u}^{-h_1}e^T(s)e(s)dsdu \\
&+h_{12}\lambda_{13}\int_{-h_2}^{-h_1}\int_{u}^{-h_1}\dot{e}^T(s)\dot{e}(s)dsdu+\lambda_{14}\int_{-h_2}^{0}\int_{\theta}^{0}\dot{e}^T(s)\dot{e}(s)dsd\theta \\
&+\lambda_{15}\int_{-h_2}^{0}\int_{\theta}^{0}\int_{r}^{0}\dot{e}^T(s)\dot{e}(s)dsdrd\theta+\lambda_{16}\int_{-h_1}^{0}\int_{\theta}^{0}\int_{r}^{0}\dot{e}^T(s)\dot{e}(s)dsdrd\theta \\
&+\lambda_{17}\int_{-h_2}^{-h_1}\int_{\theta}^{-h_1}\int_{r}^{-h_1}\dot{e}^T(s)\dot{e}(s)dsdrd\theta \\
\leq \;& \Big\{\lambda_2+h_1\lambda_3+h_{12}\lambda_4+h_{12}\lambda_5+h_1\lambda_6+h_{12}\lambda_7+h_2L^2\lambda_8+\frac{d^3}{2}L^2\lambda_9 \\
&+\frac{h_1^3}{2}\lambda_{10}+\frac{h_1^3}{2}\lambda_{11}+\frac{h_{12}^3}{2}\lambda_{12}+\frac{h_{12}^3}{2}\lambda_{13}+\frac{h_2^2}{2}\lambda_{14}+\frac{h_2^3}{6}\lambda_{15} \\
&+\frac{h_1^3}{6}\lambda_{16}+\frac{h_{12}^3}{6}\lambda_{17}\Big\}\sup_{-\tau\leq s\leq 0}\{e^T(s)e(s),\dot{e}^T(s)\dot{e}(s)\} \\
\leq \;& \Lambda c_1.
\end{aligned}
$$

Since $V(e(t)) \geq \lambda_1 e^T(t)e(t)$ and $V(e(0)) \leq \Lambda c_1$, thus for any $t \in [0,T]$, we obtain

$$
e^T(t)e(t) < \frac{e^{\alpha T}\Lambda c_1}{\lambda_1} < c_2. \tag{3.27}
$$

Hence, the condition (3.4) holds and the proof is complete. ■

## 4. Numerical Example

In this section, we now provide an example to show the effectiveness of the result in Theorem 3.1.

**Example 1.** Consider the master-slave neural networks (2.1) and (2.2) with the following parameter:

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & -0.2 \\ 0.1 & -0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.8 & 0.4 \\ -0.3 & 0.5 \end{bmatrix},$$
$$E = \begin{bmatrix} 1.1 & 0.3 \\ 0.2 & -1.8 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.95 \end{bmatrix},$$
$$\phi_1 = [-0.4\cos t, 0.5\cos t], \qquad \phi_2 = [\sin t, \sin t].$$

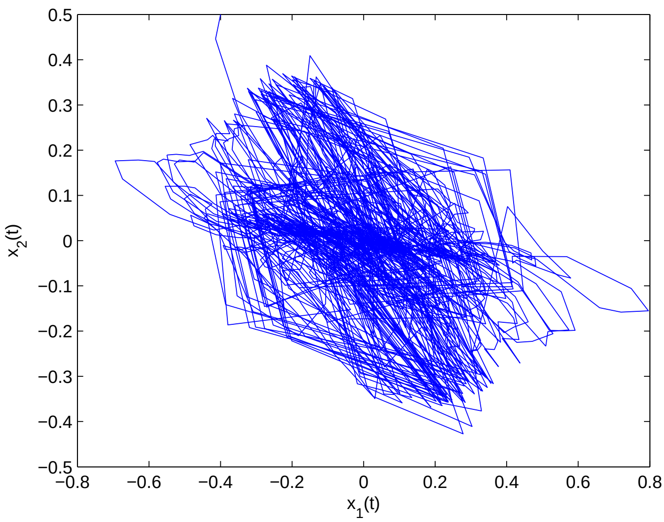


FIGURE 1. Chaotic behavior of master neural network (2.1)

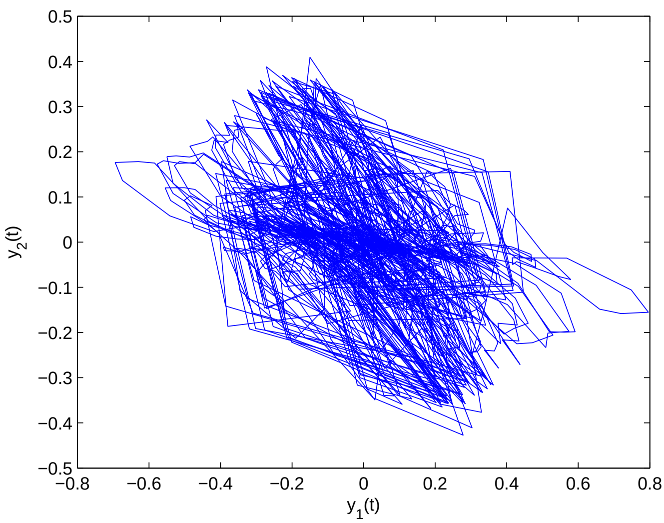


FIGURE 2. Chaotic behavior of slaver neural network (2.2)

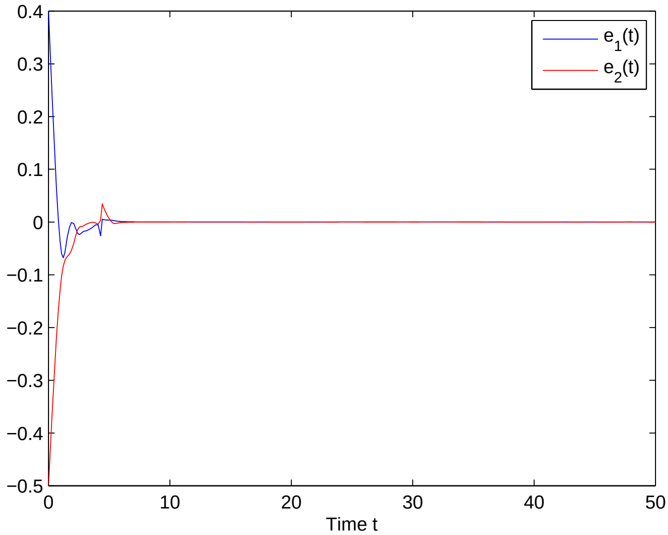


FIGURE 3. The state response of the resulting error system

*Solution:* From the condition (3.1)–(3.4) of Theorem 3.1, we let $\alpha = 0.00001$, $h_1 = 0.1$, $h_2 = 0.2$, $d = 0.15$ and $\mu = 0.3$. By using the LMI Toolbox in MATLAB, we obtain

$$
\begin{aligned}
P &= \begin{bmatrix} 6.1140 & -2\times 10^{-14} \\ -2\times 10^{-14} & 6.1140 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.0034 & 5.6259\times 10^{-5} \\ 5.6259\times 10^{-5} & 0.0033 \end{bmatrix},\\
R_2 &= 10^{-5}\begin{bmatrix} 2.9268 & 2\times 10^{-5} \\ 2\times 10^{-5} & 2.9268 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0.4571 & 0.0085 \\ 0.0085 & 0.4416 \end{bmatrix},\\
R_4 &= \begin{bmatrix} 0.0025 & 2\times 10^{-8} \\ 2\times 10^{-8} & 0.0025 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.2026 & -7\times 10^{-12} \\ -7\times 10^{-12} & 0.2026 \end{bmatrix},\\
Q_2 &= \begin{bmatrix} 0.3085 & -4\times 10^{-6} \\ -4\times 10^{-6} & 0.3082 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.3224 & 0.0027 \\ 0.0027 & 0.2656 \end{bmatrix},\\
Q_4 &= \begin{bmatrix} 0.0873 & 0.0133 \\ 0.0133 & 0.0632 \end{bmatrix}, \quad Q_5 = \begin{bmatrix} 0.0504 & 0.0074 \\ 0.0074 & 0.0370 \end{bmatrix},\\
W_1 &= \begin{bmatrix} 0.0521 & 0.0016 \\ 0.0016 & 0.0478 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1.0336 & -0.0086 \\ -0.0086 & 1.2004 \end{bmatrix},\\
G_1 &= \begin{bmatrix} 0.5321 & 0 \\ 0 & 0.5321 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.2678 & 0 \\ 0 & 0.2678 \end{bmatrix},\\
Z_1 &= \begin{bmatrix} 0.0307 & 1\times 10^{-5} \\ 1\times 10^{-5} & 0.0306 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.0311 & 0.0002 \\ 0.0002 & 0.0307 \end{bmatrix},\\
Z_3 &= \begin{bmatrix} 0.0307 & 1\times 10^{-5} \\ 1\times 10^{-5} & 0.0306 \end{bmatrix}, \quad Z_4 = \begin{bmatrix} 0.0005 & 3\times 10^{-8} \\ 3\times 10^{-8} & 0.0005 \end{bmatrix},\\
K &= \begin{bmatrix} -0.5889 & -0.0738 \\ -0.1860 & -0.9046 \end{bmatrix},
\end{aligned}
$$

and accordingly the feedback control is $u(t) = \begin{bmatrix} -1.1778 & -0.1477 \\ -0.1860 & -0.9046 \end{bmatrix} (y(t) - x(t))$.

We let $J(t) = 0$, $h(t) = 0.2 + 1.2\sin(t)$, $d(t) = |\sin t|$, $\phi_1 = [-0.4\cos t, 0.5\cos t]$, $\phi_2 = [\sin t, \sin t]$, for all $t \in [-0.2, 0]$, and the activation functions as follows:

$$f(s) = \frac{1}{2}(|s+1| - |s-1|).$$

Figure 1 shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of the master neural network system (2.1) with the initial condition $\phi_1(t) = [-0.4\cos t, 0.5\cos t]^T$. Figure 2 shows the trajectories of solutions $y_1(t)$ and $y_2(t)$ of the slaver neural network system (2.2) with the initial condition $\phi_2(t) = [\sin t, \sin t]^T$. Figure 3 shows the trajectories of solutions $e_1(t)$ and $e_2(t)$ of the error system with the activation function and mixed time-varying delays with feedback control $u(t) = \begin{bmatrix} -1.1778 & -0.1477 \\ -0.1860 & -0.9046 \end{bmatrix} (y(t) - x(t))$.

## 5. Conclusions

In this research, the problem of a finite-time synchronization of neural networks with interval and distributed time-varying delays via feedback control was investigated. Firstly, we considered a finite-time synchronization of neural networks with mixed time-varying delays and construct the LKF for synchronization neural network systems. Secondly, by using an extended reciprocally convex matrix inequality, a free-matrix-based integral inequality, Jensen's inequality and Wirtinger-based integral inequality are used to estimate the upper bound of the derivative of the LKF. Finally, a numerical simulation results have been given to illustrate the effectiveness of the proposed method.

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