Thai Journal of Mathematics
Volume 19 Number 3 (2021)
Pages 1134–1143

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Simple Poisson modules over a Poisson algebra S_q

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Abstract Let S be a $\mathbb{C}$-algebra generated by x, y, z, q and q^{-1} satisfies the relations

$$\begin{aligned} xy - qyx &= (q-1)(x+y+z), \\ yz - qzy &= (q-1)(x+y+z), \\ zx - qxz &= (q-1)(x+y+z) \quad \text{and} \end{aligned}$$

$$xq = qx, \quad yq = qy, \quad zq = qz, \quad qq^{-1} = 1 = q^{-1}q.$$

We focus on a Poisson algebra S_q, constructed from S, with the Poisson bracket $\{x, y\} = yx+x+y+z$, $\{y, z\} = zy + x + y + z$, and $\{z, x\} = xz + x + y + z$. There are only two Poisson maximal ideals of S_q . In this study, we characterize the simple Poisson modules which annihilated by each of the Poisson maximal ideals of S_q .

MSC: 16D25; 17B63
Keywords: Poisson algebra; Poisson modules; Simple Poisson modules

Submission date: 13.04.2021 / Acceptance date: 14.08.2021

1. Introduction

The Poisson algebra is an algebra with the brackets, called Poisson bracket, introduced in 1809 by Joseph-Louis Lagrange and his student, Siméon-Denis de Poisson, as an algorithm useful to produce solutions of motion. They defined the Poisson bracket by

$$\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right)$$

where $f, g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are smooth function and (q_k, p_k) are Lagrange's canonical coordinate. After that, in the thirties of the 19^{th} century, Carl G. J. Jacobi proved the basic properties of a Poisson bracket by using the Leibniz rule and the Leibniz identity of Poisson bracket. So, they have $\{fg, h\} = f\{g, h\} + \{f, h\} g$ for all mapping f, g, h. Then the vector field with this identity was called Hamiltonian vector fields and denoted by X_f.

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Published by The Mathematical Association of Thailand.

They extented the vector field X_f to $X_{\{f,g\}} = [X_f, X_g]$ where $[-,-]$ is a commutator, defined by $[x, y] = xy - yx$, and applied it to the mapping h, so they had the following identity, called Jacobi identity,

$$\{\{f, g\}, h\} = \{\{g, h\}, f\} - \{\{g, f\}, h\}.$$

The more systematic study of Poisson brackets was started by Marius Sophus Lie in the seventies of the 19^{th} century. After that, there are many researchers construct a new Poisson algebra and study the algebrais-structural properties on this algerbras.

In 2006, D. A. Jordan and N. Sasom [6–8] studied a Poisson algebra constructed from a $\mathbb{C}-$algebra, T, with Poisson bracket $\{x, y\} = yx + z, \{y, z\} = zy + x$, and $\{z, x\} = xz + y$. They saw that there are five Poisson maximal ideals for this Poisson algebra and classified the finite dimensional simple Poisson modules annihilated by the Poisson maximal ideals. In the same year, N. Sasom [8] used the method in [6] constructed a Poisson algebra from a quantized enveloping algebra $U_q(sl_2)$. She found that the Poisson bracket defined by $\{x, y\} = 2(1 - yx), \{y, z\} = 2(1 - zy)$ and $\{z, x\} = 2(1 - xz)$ and it had two Poisson maximal ideals. For each Poisson maximal ideal, there was the finite dimensional simple Poisson modules annihilated by it. In 2011, P. Chansuriya, N. Sasom and S. Seankarun [3] studied the Poisson algebra P_g with Poisson brackets $\{x, y\} = yx + az, \{y, z\} = zy + bx$, and $\{z, x\} = xz + cy$. It was a general form of a Poisson algebra constructed in [6]. There were five Poisson maximal ideals and, for each Poisson maximal ideal, there was the finite dimensional simple Poisson modules annihilated by it. In 2018, K. Changtong and N. Sasom [2] constructed a Poisson algebra S_q from a $\mathbb{C}$-algebra S generated by x, y, z, q and q^{-1} satisfying the relations

$$\begin{aligned} xy - qyx &= (q-1)(x+y+z), \\ yz - qtzy &= (q-1)(x+y+z), \\ zx - qxz &= (q-1)(x+y+z), \end{aligned}$$

and $xq = qx$, $yq = qy$, $zq = qz$, $qq^{-1} = 1 = q^{-1}q$. They was a Poisson bracket $\{x, y\} = yx + x + y + z, \{y, z\} = zy + x + y + z, \{z, x\} = xz + x + y + z$. They found that there were two Poisson maximal ideals as following

$$\begin{aligned} J_1 &= xS_q + yS_q + zS_q, \\ J_2 &= (x+3)S_q + (y+3)S_q + (z+3)S_q. \end{aligned}$$

Later, they used the method in [4] classified finite-dimensional simple Poisson modules over S_q annihilated by each Poisson maximal ideals. In this study, we focus on the Poisson algebra S_q and its Poisson maximal ideals. We use the method in [6] characterize the simple Poisson modules annihilated by Poisson maximal ideals J_1 and J_2.

2. Definitions and Notations

This section collect the basic definitions and notations of the Poisson algebra and Poisson module for using in this research.

Definition 2.1. Let $\mathfrak{g}$ be a vector space over a field $\mathbb{F}$. A **Lie algebra** is a vector space $\mathfrak{g}$ with a bilinear $[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies the following properties:

(1) $[x, x] = 0$ for all $x \in \mathfrak{g}$.

(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$ (Jacobi identity).

The bilinear map $[-,-]$ is called a **Lie beacket** on $\mathfrak{g}$.

Definition 2.2. A **Poisson algebra** A is a commutative algebra over a field $\mathbb{F}$ together with a bilinear map $\{-,-\} : A \times A \longrightarrow A$ such that $(A, \{-,-\})$ is a Lie algebra and satisfies a Leibniz identity $\{xy, z\} = x\{y, z\} + \{x, z\}y$, for all $x, y, z \in A$. We call $\{-,-\}$ a **Poisson bracket** on A.

A subalgebra B of A is a **Poisson subalgebra** if $\{b, c\} \in B$ for all $b, c \in B$.

Definition 2.3. Let A be a Poisson algebra. An ideal I of A is a **Poisson ideal** if $\{i, a\} \in I$ for all $i \in I, a \in A$.

Moreover, the factor A/I is also a Poisson algebra with the Poisson bracket $\{a + I, b + I\} = \{a, b\} + I$ for all $a, b \in A$.

Definition 2.4. Let A be a Poisson algebra. A Poisson ideal I of A, which $I \neq A$, is said to be a **Poisson maximal ideal** if it is also a maximal ideal of A.

Definition 2.5. Let A be a commutative Poisson algebra with the Poisson bracket $\{-,-\}$ and M be a module over A. An A-module M is called a **Poisson Module** if there is a bilinear form $\{-,-\}_M : A \times M \to M$ such that

(1) $\{a, bm\}_M = \{a, b\}m + b\{a, m\}_M$,
(2) $\{ab, m\}_M = a\{b, m\}_M + b\{a, m\}_M$,
(3) $\{\{a, b\}, m\}_M = \{a, \{b, m\}_M\}_M - \{b, \{a, m\}_M\}_M$,

for all $a, b \in A$ and $m \in M$.

A submodule N of a Poisson module M is called a **Poisson submodule** if $\{a, n\}_M \in N$ for all $a \in A$ and $n \in N$.

Definition 2.6. Let M be a Poisson A-module and $J \subseteq M$. The **annihilator** of J in A is defined by:

$$Ann_A(J) = \{a \in A : \{a, m\}_M = 0 \text{ for all } m \in J\}.$$

Lemma 2.7. *Let A be a finitely generated Poisson algebra and M be a Poisson A-module. Let $J = Ann_A(M)$.*

(1) J is a Poisson ideal of A.
(2) If M is finite-dimensional and simple module then J is a maximal ideal of A.

Proof. See [5], Lemma 1. ∎

3. Simple Poisson modules over a Poisson algebra S_q

In this section, we study on a Poisson algebra S_q with the Poisson maximal ideals $J_1 = xS_q + yS_q + zS_q$ and $J_2 = (x+3)S_q + (y+3)S_q + (z+3)S_q$. We characterize the simple Poisson module annihilated by each Poisson maximal ideal.

3.1. Simple Poisson modules annihilated by J_1

Lemma 3.1. *Let M be a Poisson S_q-module annihilated by $J_1 = xS_q + yS_q + zS_q$ and $m \in M$. Then we obtain :*

(1) $xm = ym = zm = 0$.
(2) $\{xy, m\}_M = \{yz, m\}_M = \{zx, m\}_M = 0$.
(3) *(a)* $\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M = \{x, m\}_M + \{y, m\}_M + \{z, m\}_M$,
(b) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = \{x, m\}_M + \{y, m\}_M + \{z, m\}_M$,

(c) $\{z, \{x, m\}_M\}_M - \{x, \{z, m\}_M\}_M = \{x, m\}_M + \{y, m\}_M + \{z, m\}_M.$

Proof. (1) It is easy to see that $xm = ym = zm = 0$.

(2) By (1) and Definition 2.5(2), we have

(a) $\{xy, m\}_M = x\{y, m\}_M + y\{x, m\}_M = 0,$

(b) $\{yz, m\}_M = y\{z, m\}_M + z\{y, m\}_M = 0,$

(c) $\{zx, m\}_M = z\{x, m\}_M + x\{z, m\}_M = 0.$

(3) (a) By (2) and Definition 2.5(3), we have

$$\begin{aligned}
\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M &= \{\{x, y\}, m\}_M \\
&= \{yx + x + y + z, m\}_M \\
&= \{yx, m\}_M + \{x, m\}_M + \{y, m\}_M + \{z, m\}_M \\
&= \{x, m\}_M + \{y, m\}_M + \{z, m\}_M.
\end{aligned}$$

Similarly, (b) and (c) are proved. ■

Lemma 3.2. *Let M be a Poisson S_q−module annihilated by J_1 and $m \in M$ be an eigenvector for $\{x, -\}_M$ with eigenvalue $\lambda \in \mathbb{C}$. Then $\{x, m\}_M = \lambda m$ and we also have*

(1) $\{x, \{y, m\}_M\}_M = (\lambda + 1)\{y, m\}_M a + \{z, m\}_M + \lambda m;$

(2) $\{x, \{z, m\}_M\}_M = (\lambda - 1)\{z, m\}_M - \{y, m\}_M - \lambda m;$

(3) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = \{y, m\}_M + \{z, m\}_M + \lambda m$

Proof. Let $\lambda \in \mathbb{C}$ such that $\{x, m\}_M = \lambda m$ for some $0 \neq m \in M$. Then, by Lemma 3.1(3), we have

(1)

$$\begin{aligned}
\{x, \{y, m\}_M\}_M &= \{y, \{x, m\}_M\}_M + \{x, m\}_M + \{y, m\}_M + \{z, m\}_M \\
&= \{y, \lambda m\}_M + \lambda m + \{y, m\}_M + \{z, m\}_M \\
&= \lambda\{y, m\}_M + \lambda m + \{y, m\}_M + \{z, m\}_M \\
&= (\lambda + 1)\{y, m\}_M + \{z, m\}_M + \lambda m.
\end{aligned}$$

Likewise, we can calculate (2) and (3). ■

To classify the finite-dimensional simple Poisson modules annihilated by J_1, we change the generators by setting $u := \frac{1}{2}(y - iz)$ and $v := \frac{1}{2}(z - iy)$. Here we have the new generators of a Poisson algebra S_q which are u, v and x.

Lemma 3.3. *Let $S_q = \mathbb{C}[x, y, z]$ be a Poisson algebra with the Poisson brackets:*

$$\{x, y\} = yx + x + y + z, \quad \{y, z\} = zy + x + y + z, \quad \{z, x\} = xz + x + y + z.$$

If $u = \frac{1}{2}(y - iz)$ and $v = \frac{1}{2}(z - iy)$, then S_q is generated by u, v, x and the Poisson brackets is given by

$$\begin{aligned}
\{x, v\} &= -(x + 1)ui - vi - \frac{1}{2}(1 + i)x; \\
\{x, u\} &= (x + 1)vi + ui + \frac{1}{2}(1 + i)x; \\
\{u, v\} &= \frac{1}{2}\left((x + u + v) + (u^2 + v^2 + u + v)i\right).
\end{aligned}$$

Proof. If $u = \frac{1}{2}(y - iz)$ and $v = \frac{1}{2}(z - iy)$, then we obtain $ui = \frac{1}{2}(yi + z)$ and $vi = \frac{1}{2}(zi + y)$. So $y = u + iv$ and $z = v + ui$. Thus S_q is generated by u, v, x.

$$\begin{aligned}
\{x, v\} &= \{x, \frac{1}{2}(z - iy)\} \\
&= \frac{1}{2}\{x, z\} - \frac{1}{2}i\{x, y\} \\
&= \frac{1}{2}(-xz - x - y - z) - \frac{1}{2}i(yx + x + y + z) \\
&= -\frac{1}{2}(z + iy)x - \frac{1}{2}(y + iz) - \frac{1}{2}(z + iy) - \frac{1}{2}(1 + i)x \\
&= -xui - vi - ui - \frac{1}{2}(1 + i)x \\
&= -(x + 1)ui - vi - \frac{1}{2}(1 + i)x.
\end{aligned}$$

$$\begin{aligned}
\{x, u\} &= \{x, \frac{1}{2}(y - iz)\} \\
&= \frac{1}{2}\{x, y\} - \frac{1}{2}i\{x, z\} \\
&= \frac{1}{2}(yx + x + y + z) - \frac{1}{2}i(-xz - x - y - z) \\
&= \frac{1}{2}(y + iz)x + + \frac{1}{2}(y + iz) + \frac{1}{2}(z + iy) + \frac{1}{2}(1 + i)x \\
&= xvi + vi + ui + \frac{1}{2}(1 + i)x \\
&= (x + 1)vi + ui + \frac{1}{2}(1 + i)x.
\end{aligned}$$

$$\begin{aligned}
\{u, v\} &= \left\{\frac{1}{2}(y - iz), \frac{1}{2}(z - iy)\right\} \\
&= \frac{1}{2}\left\{y - iz, \frac{1}{2}z\right\} - \frac{1}{2}\left\{y - iz, \frac{1}{2}iy\right\} \\
&= \frac{1}{4}\{y, z\} - \frac{1}{4}\{z, y\} \\
&= \frac{1}{4}(zy + x + y + z - (-zy - x - y - z)) \\
&= \frac{1}{2}(zy + x + y + z) \\
&= \frac{1}{2}\left[((v + iu)(u + iv) + x + u + iv + v + iu)\right] \\
&= \frac{1}{2}\left[(x + u + v) + (u^2 + v^2 + u + v)i\right].
\end{aligned}$$

Therefore the lemma is proved. ■

Lemma 3.4. *Let M be a Poisson S_q-module annihilated by J_1 and let $m \in M$. Then we have:*

(1) $xm = vm = um = 0$.

(2) $\{xu, m\}_M = \{xv, m\}_M = \{uv, m\}_M = \{u^2, m\}_M = \{v^2, m\}_M = 0.$

(3) *(a)*

$$\{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M = i\{u, m\}_M + i\{v, m\}_M + \frac{1}{2}(1+i)\{x, m\}_M;$$

(b)

$$\{x, \{v, m\}_M\}_M - \{v, \{x, m\}_M\}_M = -i\{u, m\}_M - i\{v, m\}_M - \frac{1}{2}(1+i)\{x, m\}_M;$$

(c)

$$\{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \frac{1}{2}\{x, m\}_M + \frac{1}{2}(1+i)\{u, m\}_M + \frac{1}{2}(1+i)\{v, m\}_M.$$

Proof. (1) Since $x \in J_1$ and $m \in M$, it is easy to see that $xm = 0$. In the case $vm = 0$ and $um = 0$, we can prove similarly.

(2) It is a routine calculation , by (1) and Definition 2.5(2), we have $\{xu, m\}_M = \{xv, m\}_M = \{uv, m\}_M = \{u^2, m\}_M = \{v^2, m\}_M = 0$.

(3) (a) By (2) and Definition 2.5(3), we have

$$\begin{aligned}
\{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M &= \{\{x, u\}, m\}_M \\
&= \left\{(x+1)vi + ui + \frac{1}{2}(1+i)x, m\right\}_M \\
&= i\{xv, m\}_M + \{vi, m\}_M + \{ui, m\}_M \\
&\quad + \frac{1}{2}(1+i)\{x, m\}_M \\
&= i\{v, m\}_M + i\{u, m\}_M + \frac{1}{2}(1+i)\{x, m\}_M.
\end{aligned}$$

Similarly, we can proved (b) and (c).

■

Lemma 3.5. *Let M be a Poisson S_q-module annihilated by J_1 and $\lambda \in \mathbb{C}$ such that $\{x, m\}_M = \lambda m$ for some $0 \neq m \in M$. If $u = \frac{1}{2}(y - iz)$ and $v = \frac{1}{2}(z - iy)$, then we have*

(1) $\{x, \{v, m\}_M\}_M = (\lambda - i)\{v, m\}_M - i\{u, m\}_M - \frac{1}{2}(1+i)\lambda m;$

(2) $\{x, \{u, m\}_M\}_M = (\lambda + i)\{u, m\}_M + i\{v, m\}_M + \frac{1}{2}(1+i)\lambda m;$

(3) $\{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \frac{1}{2}\lambda m + \frac{1}{2}(1+i)\{u, m\}_M + \frac{1}{2}(1+i)\{v, m\}_M.$

Proof. Let $\lambda \in \mathbb{C}$ such that $\{x, m\}_M = \lambda m$ for some $0 \neq m \in M$. It is a routine calculation by using Lemma 3.4(3). ■

3.2. Simple Poisson modules annihilated by J_2

Lemma 3.6. *Let M be a Poisson S_q-module annihilated by $J_2=(x+3)S_q+(y+3)S_q+(z+3)S_q$, and $m\in M$. Then we have:*

(1) $xm=ym=zm=-3m$.

(2) *(a)* $\{xy,m\}_M=-3\{y,m\}_M-3\{x,m\}_M$;

(b) $\{yz,m\}_M=-3\{z,m\}_M-3\{y,m\}_M$;

(c) $\{xz,m\}_M=-3\{z,m\}_M-3\{x,m\}_M$.

(3) *(a)* $\{x,\{y,m\}_M\}_M-\{y,\{x,m\}_M\}_M=\{z,m\}_M-2\{x,m\}_M-2\{y,m\}_M$;

(b) $\{y,\{z,m\}_M\}_M-\{z,\{y,m\}_M\}_M=\{x,m\}_M-2\{y,m\}_M-2\{z,m\}_M$;

(c) $\{z,\{x,m\}_M\}_M-\{x,\{z,m\}_M\}_M=\{y,m\}_M-2\{z,m\}_M-2\{x,m\}_M$.

Proof. (1) Since $x+3\in J_2$ and $m\in M$, $(x+3)m=0$.
Consider

$$\begin{aligned}(x+3)m &= 0\\ xm+3m &= 0\\ xm &= -3m.\end{aligned}$$

Similarly, $ym=zm=-3m$.

(2) By (1) and Definition 2.5(2), we have

(a) $\{xy,m\}_M=x\{y,m\}_M+y\{x,m\}_M=-3\{y,m\}_M-3\{x,m\}_M$.

(b) $\{yz,m\}_M=y\{z,m\}_M+z\{y,m\}_M=-3\{z,m\}_M-3\{y,m\}_M$.

(c) $\{zx,m\}_M=z\{x,m\}_M+x\{z,m\}_M=-3\{z,m\}_M-3\{x,m\}_M$.

(3) (a) By (2) and Definition 2.5(3), we have that

$$\begin{aligned}\{x,\{y,m\}_M\}_M-\{y,\{x,m\}_M\}_M &= \{\{x,y\},m\}_M\\ &= \{yx+x+y+z,m\}_M\\ &= \{yx,m\}_M+\{x,m\}_M+\{y,m\}_M+\{z,m\}_M\\ &= -3\{y,m\}_M-3\{x,m\}_M+\{x,m\}_M+\{y,m\}_M\\ &\quad +\{z,m\}_M\\ &= \{z,m\}_M-2\{x,m\}_M-2\{y,m\}_M.\end{aligned}$$

(b) and (c) are proved similarly. ∎

Lemma 3.7. *Let M be a Poisson S_q-module annihilated by J_2 and $m\in M$ be an eigenvector for $\{x,-\}_M$ with eigenvalue $\lambda\in\mathbb{C}$. Then $\{x,m\}_M=\lambda m$ and we have*

(1) $\{x,\{y,m\}_M\}_M=(\lambda-2)\{y,m\}_M+\{z,m\}_M-2\lambda m$,

(2) $\{x,\{z,m\}_M\}_M=(\lambda+2)\{z,m\}_M-\{y,m\}_M+2\lambda m$,

(3) $\{y,\{z,m\}_M\}_M-\{z,\{y,m\}_M\}_M=\lambda m-2\{z,m\}_M-2\{y,m\}_M$.

Proof. Let $\lambda\in\mathbb{C}$ such that $\{x,m\}_M=\lambda m$ for some $0\neq m\in M$. Then, by Lemma 3.6(3), we have

(1)

$$\begin{aligned}\{x,\{y,m\}_M\}_M &= \{y,\{x,m\}_M\}_M+\{z,m\}_M-2\{x,m\}_M-2\{y,m\}_M\\ &= \{y,\lambda m\}_M+\{z,m\}_M-2\lambda m-2\{y,m\}_M\\ &= \lambda\{y,m\}_M+\{z,m\}_M-2\lambda m-2\{y,m\}_M\\ &= (\lambda-2)\{y,m\}_M+\{z,m\}_M-2\lambda m.\end{aligned}$$

(2) and (3) are proved similarly. ■

We shall replace z by $u := z - 3x - 3y$ to simplify these results.

Lemma 3.8. *Let $S_q = \mathbb{C}[x, y, s]$ be a Poisson algebra with the Poisson brackets as the following: $\{x, y\} = yx + x + y + z$, $\{y, z\} = zy + x + y + z$, $\{z, x\} = xz + x + y + z$. If $u = z - 3x - 3y$ then S_q is generated by x, y, u and the Poisson brackets is given by*

(1) $\{x, y\} = (x + 4)y + 4x + u$,

(2) $\{x, u\} = -(x + 4)u - (3x + 16)x - (6x - 16)y$,

(3) $\{y, u\} = (y + 4)u + (6y + 16)x + (3y + 16)y.$,

Proof. Since $u = z - 3x - 3y$, then we obtain$z = u + 3x + 3y$. Thus S_q is generated by x, y, u and we have

(1)

$$\begin{aligned}\{x, y\} &= yx + x + y + z\\ &= yx + x + y + (u + 3x + 3y)\\ &= (x + 4)y + 4x + u.\end{aligned}$$

(2)

$$\begin{aligned}\{x, u\} &= \{x, z - 3x - 3y\}\\ &= \{x, z\} - 3\{x, y\}\\ &= (-xz - x - y - z) - 3(yx + x + y + z)\\ &= -xz - 3yx - 4x - 4y - 4z\\ &= -(x + 4)z - 3yx - 4x - 4y\\ &= -(x + 4)(u + 3x + 3y) - 3yx - 4x - 4y\\ &= -(x + 4)u - (3x + 16)x - (6x + 16)y.\end{aligned}$$

(3)

$$\begin{aligned}\{y, u\} &= \{y, z - 3x - 3y\}\\ &= \{y, z\} - 3\{y, x\}\\ &= (zy + x + y + z) - 3(-yx - x - y - z)\\ &= zy + 3yx4x + 4y + 4z\\ &= (y + 4)z + 3yx + 4x + 4y\\ &= (y + 4)(u + 3x + 3y) + 3yx + 4x + 4y\\ &= (y + 4)u + (3x + 16)y - (6x + 16)x.\end{aligned}$$

■

Lemma 3.9. *Let M be a Poisson S_q−module annihilated by $J_2 = (x+3)S_q + (y+3)S_q + (u - 3)S_q$. Then we have:*

(1) $xm = ym = -3m, um = 3m$.

(2) *(a)* $\{xy, m\}_M = -3\{y, m\}_M - 3\{x, m\}_M$;

(b) $\{xu, m\}_M = -3\{u, m\}_M + 3\{x, m\}_M$;

(c) $\{yu, m\}_M = -3\{u, m\}_M + 3\{y, m\}_M$.

(3) *(a)* $\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M = \{x, m\}_M + \{y, m\}_M + \{u, m\}_M$;

(b) $\{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M = 17\{x, m\}_M + 2\{y, m\}_M - \{u, m\}_M$;
(c) $\{y, \{u, m\}_M\}_M - \{u, \{y, m\}_M\}_M = \{u, m\}_M - 17\{y, m\}_M - 2\{x, m\}_M$.

Proof. (1) Since $x + 3 \in J_2$ and $m \in M$, $(x + 3)m = 0$.
Consider

$$\begin{aligned} (x+3)m &= 0 \\ xm + 3m &= 0 \\ xm &= -3m. \end{aligned}$$

Similarly, $ym = -3m$.
Since $u - 3 \in J_2$ and $m \in M$, $(u - 3)m = 0$.
Consider

$$\begin{aligned} (u-3)m &= 0 \\ um - 3m &= 0 \\ um &= 3m. \end{aligned}$$

(2) By (1) and Definition 2.5(2), we have
(a) $\{xy, m\}_M = x\{y, m\}_M + y\{x, m\}_M = -3\{y, m\}_M - 3\{x, m\}_M$.
(b) $\{xu, m\}_M = x\{u, m\}_M + u\{x, m\}_M = -3\{u, m\}_M + 3\{y, m\}_M$.
(c) $\{yu, m\}_M = y\{x, m\}_M + u\{y, m\}_M = -3\{u, m\}_M + 3\{x, m\}_M$.

(3) (a) By (2) and Definition 2.5(3), we have that

$$\begin{aligned} \{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M &= \{\{x, y\}, m\}_M \\ &= \{xy + 4y + 4x + u, m\}_M \\ &= \{xy, m\}_M + 4\{y, m\}_M + 4\{x, m\}_M + \{u, m\}_M \\ &= -3\{y, m\}_M - 3\{x, m\}_M + 4\{y, m\}_M + 4\{x, m\}_M \\ &\quad +\{u, m\}_M \\ &= \{x, m\}_M + \{y, m\}_M + 2\{u, m\}_M. \end{aligned}$$

(b) and (c) are proved similarly. ∎

Lemma 3.10. *Let $u = z - 3x - 3y$. Let M be a Poisson S_q-module annihilated by J_2. Let $\lambda \in \mathbb{C}$ such that $\{u, m\}_M = \lambda m$ for some $0 \neq m \in M$. Then*

(1) $\{u, \{x, m\}_M\}_M = (\lambda - 17)\{x, m\}_M - 2\{y, m\}_M + \lambda m$;
(2) $\{u, \{y, m\}_M\}_M = (\lambda + 17)\{x, m\}_M + 2\{y, m\}_M - \lambda m$;
(3) $\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M = \{x, m\}_M + \{y, m\}_M + \lambda m$.

Proof. Let $\lambda \in \mathbb{C}$ such that $\{s, m\}_M = \lambda m$ for some $0 \neq m \in M$. It is a routine calculation by using Lemma 3.9(3). ∎

Conclusion and Discussion

We have been followed the same method of [6], but the problem is the relations of our Poisson brackets. It was very hard to change variables to get the nice Poisson brackets as in [6]. Comparing the Poisson brackets in [6] which are

$$\{x, y\} = yx + z, \{y, z\} = zy + x, \{z, x\} = xz + y \tag{3.1}$$

and our Poisson brackets:

$$\{x, y\} = yx + x + y + z, \{y, z\} = zy + x + y + z, \{z, x\} = xz + x + y + z. \quad (3.2)$$

We see that the polynomials in (3.1) has the different term of degree one, but in (3.2) all terms of degree one are the same. Therefore it was very hard to change the suitable variable as in [6]. However, we still work hard to find a nice condition for characterize the nice result and show that the results are finite dimensional.

Acknowledgements

This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-61-DRIVE-018. We would like to thank the referees for their comments and suggestions on the manuscript.

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