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An Inertial Projection and Contraction Scheme for Monotone Variational Inequality Problems

Abor Isa Garba[1*], **Jamilu Abubakar**[1,2,3*] **and Shehu Abubakar Sidi**[4]

[1] *Department of Mathematics, Faculty of Science, Usmanu Danfodiyo University Sokoto, PMB 2346, Nigeria.*
[2] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod,Thung Khru, Bangkok 10140, Thailand.*
[3] *KMUTTFixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, Faculty of Science, King Mongkuts University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand.*
[4] *College of Science and Technology, Umaru Ali Shankafi Polytechnic Sokoto, Nigeria.*
Email addresses: garba.isa@udusok.edu.ng (A.I. Garba), abubakar.jamilu@udusok.edu.ng (J. Abubakar), lynkcd@gmail.com (S.A. Sidi)

Abstract In this paper, we propose a one step-type inertial projection and contraction algorithm for solving monotone variational inequality problems. We establish a strong convergence theorem under mild assumptions imposed on the underlying operator. Furthermore, we give numerical experiments to illustrate the inertial - effect and the computational performance of our proposed algorithm and comparisons with some existing algorithms in the literature.

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1. Introduction

Variational inequality problem (VIP) is an approach of finding a point $\xi^* \in E$ such that

$$\langle \mathcal{A}(\xi^*), \xi - \xi^* \rangle \geq 0, \qquad \forall\, \xi \in E, \tag{1.1}$$

where E is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ is a monotone operator and $\langle \cdot, \cdot \rangle$ denotes an inner product space. It is a fundamental problem in optimization theory which is applied in many areas of study such as transportation problems, economics, engineering and so on (see [7, 8, 20–23, 28, 29]).

There are two approaches for solving the VIP problem, namely, regularization and the projection method. Based on these, many studies had carried out to study (1.1) with several algorithms being considered and proposed (see for example [1–3, 13, 14, 17, 27]).

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In this study we are interested in the projection method. The basic idea of the projection method comes from extending the *gradient projection method* for minimizing a function $f(\xi)$ subject to $\xi \in E$ which is given by:

$$\xi_{n+1} = P_E\left(\xi_n - \lambda_n \nabla f(\xi_n)\right), \qquad \forall\, n \geq 1, \tag{1.2}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers satisfying a particular condition and P_E is a metric projection onto E. Replacing the gradient operator $\nabla f(\xi_n)$ with an operator $\mathcal{A}$ gives an immediate extension of the method to the VIP, where a sequence $\{\xi_n\}$ is generated by the following iterative scheme:

$$\xi_{n+1} = P_E\left(\xi_n - \lambda_n \mathcal{A}\xi_n\right), \qquad \forall\, n \geq 1. \tag{1.3}$$

However, a slightly strong assumption of strong monotonicity or strong inverse monotonicity needs to be placed on the operator $\mathcal{A}$ to guarantee the convergence of the sequence generated by this scheme (see [40]). To solve this problem, Korpelevich proposed the extragradient method for solving saddle point problem in [24], which was further extented to solve (1.1). The method requires only the operator $\mathcal{A}$ to be monotone and L - Lipschitz continuous for the convergence of the generated sequence $\{\xi_n\}$ with $\lambda_n \in (0, 1/L)$.

The extragradient method needs to compute two projections onto the set E in each iteration, which is going to be difficult in a situation where E is not a simple set to project onto. Censor *et al.* introduced the *subgradient extragradient method* in [11, 12] to overcome this drawback, where they replaced the second projection with a projection onto a constructible half - space which has an explicit formula to compute. In the same vein, Tseng [38] proposed an algorithm that requires only one projection onto the feasible set.

Another effort towards solving the problem of double projections is the introduction of *projection and contraction method* by some researchers (see [10, 14, 33, 36]). This method just like the subgradient extragradient method has only one projection in each iteration with advantages in computation and implementation over the subgradient extra-gradient method [15]. It has the following form:

$$\begin{cases} z_n = P_E\left(\xi_n - \lambda \mathcal{A}\xi_n\right), \\ d(\xi_n, z_n) = \xi_n - z_n - \lambda(\mathcal{A}\xi_n - \mathcal{A}z_n), \\ \xi_{n+1} = \xi_n - \gamma\beta_n d(\xi_n, z_n), \qquad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $\gamma \in (0, 2)$ and $\beta_n = \frac{\langle \xi_n - z_n, d(\xi_n, z_n)\rangle}{\|d(\xi_n, z_n)\|^2}$. The sequences generated by the above mentioned algorithms converge weakly to the respective solutions.

Yekini *et al.* [33] introduced a projection contraction type algorithm which converges strongly to the solution of the problem. Their scheme only needs a single projection in each iteration with a pseudomonotone and L -Lipschitzs operator as the underlying operator. The algorithm is given by:

$$\begin{cases} z_n = P_E\left(\xi_n - \lambda_n \mathcal{A}\xi_n\right), \\ d(\xi_n, z_n) = \xi_n - z_n - \lambda_n(\mathcal{A}\xi_n - \mathcal{A}z_n), \\ \xi_{n+1} = \alpha_n \xi_1 + (1 - \alpha_n)(\xi_n - \gamma\tau_n d(\xi_n, z_n)), \qquad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\gamma \in (0, 2)$, $\alpha_n \subset (0, 1)$ and

$$\tau_n = \begin{cases} \frac{\langle \xi_n - z_n, d(\xi_n, z_n)\rangle}{\|d(\xi_n, z_n)\|^2}, & \text{if} \quad d(\xi_n, z_n) \neq 0, \\ 0, & \text{if} \quad d(\xi_n, z_n) = 0. \end{cases}$$

In virtue of the applicability of these methods, several works have been carried out in this area by numerous researchers to extend, modify, and come up with algorithms that are easier to implement and faster to converge to the solution of a given problem. Towards this aim, inertial-type algorithms that are based upon a discrete version of a second-order dissipative dynamical system [4] were proposed. The inertial step is considered to be a strategy for accelerating the convergence of the iterates. Alvarez and Attouch [5] used the procedure in solving the problem of finding zero of a maximal monotone operator.

Recently, inertial schemes have received increasing interests (see for instance [6, 14, 17, 30–32, 37, 39]). In particular, Dong *et al.* [14] incorporated projection-contraction scheme with inertial extrapolation step as follows:

$$\begin{cases} w_n = \xi_n + \alpha_n(\xi_n - \xi_{n-1}) \\ z_n = P_E\left(w_n - \lambda \mathcal{A} w_n\right), \\ d(w_n, z_n) = w_n - z_n - \lambda(\mathcal{A}w_n - \mathcal{A}z_n), \\ \xi_{n+1} = w_n - \gamma\beta_n d(w_n, z_n), \qquad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\gamma \in (0, 2)$,

$$\beta_n = \begin{cases} \frac{\phi(w_n, z_n)}{\|d(w_n, z_n)\|^2}, & \text{if} \quad d(w_n, z_n) \neq 0, \\ 0, & \text{if} \quad d(w_n, z_n) = 0 \end{cases}$$

and

$$\phi(w_n, z_n) = \left\langle w_n - z_n, d(w_n, z_n)\right\rangle.$$

They established that the sequence generated by (1.6) algorithm converges weakly to the solution of (1.1) similar to most inertial type algorithms. However, the paper [26] for fixed point problems and recently in [34] introduced inertial type algorithms with strong convergence.

In this paper, motivated and inspired by the above works, we propose an inertial one-step projection-contraction algorithm by incorporating the inertial extrapolation step. This modification aims to obtain an algorithm with faster and strong convergence properties which performs better under mild conditions imposed on the parameters. Furthermore, we present several numerical examples to illustrate the performance and the effect of the inertial step when compared to the existing algorithms in the literature.

This paper is organized as follows: In Section 2, we give some definitions and lemmas which we will use in our convergence analysis. In Section 3, we present the convergence analysis of our proposed algorithm. Lastly, in Section 4, we illustrate the inertial-effect and the computational performance of our algorithms by giving some examples.

2. Preliminaries

This section, recalls some known facts and necessary tools that we need for the convergence analysis of our method.

Throughout this article $\mathcal{H}$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, E is a nonempty, closed and convex subset of $\mathcal{H}$. The notations $\xi_n \rightharpoonup \xi$ and $\xi_n \longrightarrow \xi$ are used to indicate that, the sequence $\{\xi_n\}$ converges weakly and strongly to ξ respectively. The following are known to hold in a real Hilbert space:

$$\|\xi \pm \mathfrak{y}\|^2 = \|\xi\|^2 + \|\mathfrak{y}\|^2 \pm 2\left\langle \xi, \mathfrak{y}\right\rangle \tag{2.1}$$

and

$$\|\alpha\xi + (1-\alpha)\mathfrak{y}\|^2 = \alpha\|\xi\|^2 + (1-\alpha)\|\mathfrak{y}\|^2 - \alpha(1-\alpha)\|\xi - \mathfrak{y}\|^2 \tag{2.2}$$

for all $\xi, \mathfrak{y} \in \mathcal{H}$ and $\alpha \in \mathbb{R}$ [9].

Definition 2.1. Let $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ be a mapping defined on a real Hilbert space $\mathcal{H}$. $\mathcal{A}$ is said to be:

(1) *monotone* if

$$\langle \mathcal{A}(\xi) - \mathcal{A}(\mathfrak{y}), \xi - \mathfrak{y} \rangle \geq 0, \quad \forall \xi, \mathfrak{y} \in \mathcal{H}.$$

(2) L *-Lipschitz continuous* on $\mathcal{H}$ if there exists a constant $L > 0$ such that

$$\|\mathcal{A}\xi - \mathcal{A}\mathfrak{y}\| \leq L\,\|\xi - \mathfrak{y}\|\,, \quad \forall \xi, \mathfrak{y} \in \mathcal{H}.$$

One of the nice property of a solution set of the VIP defined by a continuous monotone operator is, it is closed and convex.

Lemma 2.2. [34] *Let E be a nonempty closed and convex subset of $\mathcal{H}$, $\xi \in \mathcal{H}$ be arbitrarily given, $\mathfrak{y} = P_E\xi$ and $\Gamma := \{\omega \in \mathcal{H} : \langle \omega - \xi, \omega - \mathfrak{y} \rangle \leq 0\}$. Then $\Gamma \cap E = \{\mathfrak{y}\}$.*

Lemma 2.3. [9] *Let E be a nonempty, closed and convex subset of $\mathcal{H}$ and P_E be the metric projection from $\mathcal{H}$ onto E (i.e., for $\xi \in \mathcal{H}$, $\|\xi - P_E\xi\| = \inf\{\|\xi - \mathfrak{y}\| : \mathfrak{y} \in E\}$. Then, for any $\xi \in \mathcal{H}$, $\mathfrak{y} = P_E\xi$ if and only if there holds the relation:*

$$\langle \xi - \mathfrak{y}, y - \mathfrak{y} \rangle \leq 0, \quad \forall\, y \in E.$$

Lemma 2.4. [34] *Let E be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\mathcal{A} : E \longrightarrow \mathcal{H}$ be a continuous, monotone mapping and $\xi^* \in E$. Then*

$$\xi^* \in \Omega, \quad \Longleftrightarrow \langle \mathcal{A}\xi, \xi - \xi^* \rangle\,, \quad \forall \xi \in E, \tag{2.3}$$

where Ω is the solution set of the VIP.

Lemma 2.5. [34] *Let E be a nonempty, closed and convex subset of $\mathcal{H}$, $\mathfrak{y} = P_E\xi$ and $\xi^* \in E$. Then*

$$\|\mathfrak{y} - \xi^*\|^2 \leq \|\xi - \xi^*\|^2 - \|\xi - \mathfrak{y}\|^2 \tag{2.4}$$

3. Inertial One Step Projection and Contraction Algorithm

Here we present our proposed projection-contraction type algorithm and convergence analysis of the sequence generated by it. In the sequel, we denote the solution set of (1.1) as Ω and we assume the following to hold:

Assumption 3.1. The following conditions are assumed for the convergence of our method:

(A1) The feasible set E is a nonempty closed and convex subset of the real Hilbert space $\mathcal{H}$.
(A3) The solution set Ω of the *VIP* (1.1) is nonempty.
(A2) $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ is a monotone, L - *Lipschitz* and uniformly continuous on bounded subsets of $\mathcal{H}$.
(A4) The real sequence $\{\alpha_n\}$ is a non-decreasing with $\{\alpha_n\} \in [0, \alpha)$ for all $n \in \mathbb{N}$ for some $\alpha \in [0, 1/3)$ and $\{\beta_n\} \subset [\frac{1}{2}, 1)$ is a non-increasing sequence with $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm 1 Inertial one-step projection and contraction algorithm for variational inequality

Initialization: Choose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that, the conditions from assumption 3.1 hold. Let $\lambda_n \subset (0,1)$ and $\gamma \subset (0,2)$. Let $\xi_0, \xi_1 \in H$ be given starting points:

Steps 1: Set

$$w_n = \beta_n \xi_0 + (1-\beta_n)\xi_n + \alpha_n(\xi_n - \xi_{n-1}),$$

and compute

$$z_n = P_E\left(w_n - \lambda_n \mathcal{A} w_n\right). \tag{3.1}$$

If $z_n = w_n$ STOP.

Step 2: Compute

$$\xi_{n+1} = \beta_n w_n + (1-\beta_n)\, t_n,$$

where

$$t_n = w_n - \gamma \tau_n d(w_n, z_n)$$

with

$$d(w_n, z_n) = w_n - z_n - \lambda_n\left(\mathcal{A} w_n - \mathcal{A} z_n\right), \tag{3.2}$$

and

$$\tau_n = \begin{cases} \frac{\langle w_n - z_n, d(w_n, z_n)\rangle}{\|d(w_n, z_n)\|^2}, & \text{if} \quad d(w_n, z_n) \neq 0, \\ 0, & \text{if} \quad d(w_n, z_n) = 0, \end{cases}$$

Step 3. Set $n = n+1$ and go back to **Step 1.**

Definition 3.2. Suppose k_1, k_2 are given positve constants. For given ξ_n, let z_n and $d(\xi_n, z_n)$ be defined by (3.1) and (3.2) respectively. We say that the parameter $\lambda_n \in (0, \infty)$ in Algorithm 1 satisfies the step-size conditions, if its related direction $d(\xi_n, z_n)$ satisfies

$$\langle w_n - z_n, d(w_n, z_n)\rangle \geq k_1 \|w_n - z_n\|^2 \tag{3.3}$$

and

$$\frac{\langle w_n - z_n, d(w_n, z_n)\rangle}{\|d(w_n, z_n)\|^2} \geq k_2. \tag{3.4}$$

Lemma 3.3. *Let $\xi^* \in \Omega$, then, from Algorithm 1, we have*

$$\|t_n - \xi^*\|^2 \leq \|w_n - \xi^*\|^2 - \frac{2-\gamma}{\gamma}\|t_n - w_n\|^2 \tag{3.5}$$

and

$$\|w_n - t_n\|^2 \leq \frac{1+\lambda_n^2 L^2}{\gamma^2(1-\lambda_n L)^2}\|z_n - w_n\|^2. \tag{3.6}$$

Proof. Observe that $t_n = w_n - \gamma \tau_n d(w_n, z_n)$ and

$$\begin{aligned} \|t_n - \xi^*\|^2 &= \|(w_n - \xi^*) - \gamma \tau_n d(w_n, z_n)\|^2 \\ &= \|w_n - \xi^*\|^2 - 2\gamma\tau_n \langle w_n - \xi^*, d(w_n, z_n)\rangle + \gamma^2 \tau_n^2 \|d(w_n, z_n)\|^2. \end{aligned} \tag{3.7}$$

Observe that

$$\langle w_n - \xi^*, d(w_n, z_n)\rangle = \langle w_n - z_n, d(w_n, z_n)\rangle + \langle z_n - \xi^*, d(w_n, z_n)\rangle . \tag{3.8}$$

Since $\xi^* \in \Omega$, it follows from (3.1) and Lemma 2.3 that

$$\langle z_n - \xi^*, w_n - z_n - \lambda_n \mathcal{A} w_n\rangle \geq 0. \tag{3.9}$$

Since $\langle \mathcal{A}\xi^*, z_n - \xi^*\rangle \geq 0$ and the fact that $\xi^* \in \Omega$, from the monotonicity of $\mathcal{A}$, we have

$$\langle \mathcal{A} z_n, z_n - \xi^*\rangle \geq 0. \tag{3.10}$$

Thus, we have

$$\langle \lambda_n \mathcal{A} z_n, z_n - \xi^*\rangle \geq 0. \tag{3.11}$$

Adding (3.9)and (3.11), we get

$$\langle z_n - \xi^*, w_n - z_n - \lambda_n \mathcal{A} w_n + \lambda_n \mathcal{A} z_n\rangle \geq 0.$$

Therefore, we have

$$\langle z_n - \xi^*, d(w_n, z_n)\rangle \geq 0. \tag{3.12}$$

From (3.8) and (3.12), it follows that

$$\langle w_n - \xi^*, d(w_n, z_n)\rangle \geq \langle w_n - z_n, d(w_n, z_n)\rangle . \tag{3.13}$$

Substituting (3.13) in to (3.7), we get

$$\begin{aligned} \|t_n - \xi^*\|^2 &\leq \|w_n - \xi^*\|^2 - 2\gamma\tau_n \langle w_n - z_n, d(w_n, z_n)\rangle + \gamma^2\tau_n^2\|d(w_n, z_n)\|^2 \\ &= \|w_n - \xi^*\|^2 - 2\gamma\tau_n \langle w_n - z_n, d(w_n, z_n)\rangle + \gamma^2\tau_n \langle w_n - z_n, d(w_n, z_n)\rangle \\ &= \|w_n - \xi^*\|^2 - \gamma(2-\gamma)\tau_n \langle w_n - z_n, d(w_n, z_n)\rangle . \end{aligned} \tag{3.14}$$

By the definition of t_n, we obtain

$$\tau_n \langle w_n - z_n, d(w_n, z_n)\rangle = \|\tau_n d(w_n, z_n)\|^2 = \frac{1}{\gamma^2}\|z_n - w_n\|^2. \tag{3.15}$$

Substituting (3.15) in (3.14), implies

$$\|t_n - \xi^*\|^2 \leq \|w_n - \xi^*\|^2 - \frac{2-\gamma}{\gamma}\|z_n - w_n\|^2. \tag{3.16}$$

Now, observe that, from the Cauchy - Schwarz inequality and the Lipchitz property of $\mathcal{A}$,

$$\begin{aligned} \langle w_n - z_n, d(w_n, z_n)\rangle &= \langle w_n - z_n, w_n - z_n - \lambda_n(\mathcal{A} w_n - \mathcal{A} z_n)\rangle \\ &= \|w_n - z_n\|^2 - \lambda_n \langle w_n - z_n, \mathcal{A} w_n - \mathcal{A} z_n\rangle \\ &\geq \|w_n - z_n\|^2 - \lambda_n\|w_n - z_n\|\|\mathcal{A} w_n - \mathcal{A} z_n\| \\ &\geq (1 - \lambda_n L)\|w_n - z_n\|^2. \end{aligned} \tag{3.17}$$

Using equation (2.1) and the monotonicity property of $\mathcal{A}$, we get

$$\begin{aligned} \|d(w_n, z_n)\|^2 &= \|w_n - z_n - \lambda_n(\mathcal{A} w_n - \mathcal{A} z_n)\|^2 \\ &= \|w_n - z_n\|^2 + \lambda_n^2\|\mathcal{A} w_n - \mathcal{A} z_n\|^2 - 2\lambda_n \langle w_n - z_n, \mathcal{A} w_n - \mathcal{A} z_n\rangle \\ &\leq \|w_n - z_n\|^2 + \lambda_n^2\|\mathcal{A} w_n - \mathcal{A} z_n\|^2 \\ &\leq (1 + \lambda_n^2 L^2)\|w_n - z_n\|^2. \end{aligned}$$

(3.18)

By (3.17) and (3.18), we have

$$\tau_n = \frac{\langle w_n - z_n, d(w_n, z_n)\rangle}{\|d(w_n, z_n)\|^2} \geq \frac{1-\lambda_n L}{1+\lambda_n^2 L^2}. \tag{3.19}$$

From (3.15), (3.17) and and (3.19), we obtain

$$\begin{aligned} \|t_n - w_n\|^2 &\geq \gamma^2 \tau_n (1-\lambda_n L)\|w_n - z_n\|^2 \\ &\geq \gamma^2 \frac{(1-\lambda_n L)^2}{1+\lambda_n^2 L^2}\|w_n - z_n\|^2, \end{aligned} \tag{3.20}$$

which implies that

$$\|w_n - t_n\|^2 \leq \frac{1+\lambda_n^2 L^2}{\gamma^2 (1-\lambda_n L)^2}\|z_n - w_n\|^2. \tag{3.21}$$

This completes the proof. ∎

Lemma 3.4. *Suppose that, Assumption 3.1 holds. Then for each $n \in \mathbb{N}$, we have*

$$\begin{aligned} -2\beta_n \langle \xi_n - \xi^*, \xi_n - \xi_0\rangle \geq\ & \|\xi_{n+1} - \xi^*\|^2 - \|\xi_n - \xi^*\|^2 + 2\alpha_{n+1}\|\xi_{n+1} - \xi_n\|^2 \\ & - 2\alpha_n \|\xi_n - \xi_{n-1}\|^2 + \beta_{n+1}\|\xi_0 - \xi_{n+1}\|^2 - \beta_n\|\xi_n - \xi_0\|^2 \\ & - \alpha_n\|\xi_n - \xi^*\|^2 + \alpha_{n-1}\|\xi_{n-1} - \xi^*\|^2 \\ & + (1 - 3\alpha_{n+1} - \beta_n)\|\xi_n - \xi_{n+1}\|^2 \end{aligned} \tag{3.22}$$

Proof. Now, it follows from the definition of ξ_{n+1} and equation (2.2) that

$$\begin{aligned} \|\xi_{n+1} - \xi^*\|^2 &= \|\beta_n(w_n - \xi^*) + (1-\beta_n)(t_n - \xi^*)\|^2 \\ &= \beta_n\|w_n - \xi^*\|^2 + (1-\beta_n)\|t_n - \xi^*\|^2 - \beta_n(1-\beta_n)\|t_n - w_n\|^2 \\ &= \beta_n\|w_n - \xi^*\|^2 + (1-\beta_n)\|t_n - \xi^*\|^2 - \frac{\beta_n}{(1-\beta_n)}\|\xi_{n+1} - w_n\|^2. \end{aligned} \tag{3.23}$$

From Lemma 3.3, we have,

$$\begin{aligned} \|\xi_{n+1} - \xi^*\|^2 &\leq \beta_n\|w_n - \xi^*\|^2 + (1-\beta_n)\left(\|w_n - \xi^*\|^2 - \frac{2-\gamma}{\gamma}\|\xi_{n+1} - w_n\|^2\right) \\ &\quad - \frac{\beta_n}{(1-\beta_n)}\|\xi_{n+1} - w_n\|^2 \\ &= \|w_n - \xi^*\|^2 - \frac{(1-\beta_n)(2-\gamma)}{\gamma}\|\xi_{n+1} - w_n\|^2 - \frac{\beta_n}{(1-\beta_n)}\|\xi_{n+1} - w_n\|^2 \\ &\leq \|w_n - \xi^*\|^2 - \frac{\beta_n}{(1-\beta_n)}\|\xi_{n+1} - w_n\|^2. \end{aligned} \tag{3.24}$$

Since $\beta_n \in [\frac{1}{2}, 1)$, this implies that $\frac{\beta_n}{(1-\beta_n)} \geq 1$. Therefore, we have

$$\|\xi_{n+1} - \xi^*\|^2 \leq \|w_n - \xi^*\|^2 - \|\xi_{n+1} - w_n\|^2. \tag{3.25}$$

Using (2.1) and the definition of w_n, we get

$$\begin{aligned}\|w_n-\xi^*\|^2 &= \|(\xi_n-\xi^*)+\alpha_n(\xi_n-\xi_{n-1})-\beta_n(\xi_n-\xi_0)\|^2\\ &= \|\xi_n-\xi^*\|^2+\|\alpha_n(\xi_n-\xi_{n-1})-\beta_n(\xi_n-\xi_0)\|^2\\ &\quad -2\left\langle \xi_n-\xi^*,\alpha_n(\xi_n-\xi_{n-1})-\beta_n(\xi_n-\xi_0)\right\rangle\\ &= \|\xi_n-\xi^*\|^2+2\alpha_n\left\langle \xi_n-\xi^*,\xi_n-\xi_{n-1}\right\rangle-2\beta_n\left\langle \xi_n-\xi^*,\xi_n-\xi_0\right\rangle\\ &\quad +\|\alpha_n(\xi_n-\xi_{n-1})-\beta_n(\xi_n-\xi_0)\|^2.\end{aligned} \tag{3.26}$$

Similarly, by replacing ξ^* with ξ_{n+1} in the previous equations, we have

$$\begin{aligned}&\|w_n-\xi_{n+1}\|^2\\ &= \|\xi_n-\xi_{n+1}\|^2+2\alpha_n\left\langle \xi_n-\xi_{n+1},\xi_n-\xi_{n-1}\right\rangle-2\beta_n\left\langle \xi_n-\xi_{n+1},\xi_n-\xi_0\right\rangle\\ &\quad +\|\alpha_n(\xi_n-\xi_{n-1})-\beta_n(\xi_n-\xi_0)\|^2.\end{aligned} \tag{3.27}$$

Now, substituting (3.26) and (3.27) in (3.25) and using (2.2), we obtain

$$\begin{aligned}&\|\xi_{n+1}-\xi^*\|^2\\ &\le \|\xi_n-\xi^*\|^2+2\alpha_n\left\langle \xi_n-\xi^*,\xi_n-\xi_{n-1}\right\rangle-2\beta_n\left\langle \xi_n-\xi^*,\xi_n-\xi_0\right\rangle\\ &\quad -\|\xi_n-\xi_{n+1}\|^2-2\alpha_n\left\langle \xi_n-\xi_{n+1},\xi_n-\xi_{n-1}\right\rangle\\ &\quad +2\beta_n\left\langle \xi_n-\xi_{n+1},\xi_n-\xi_0\right\rangle\\ &= \|\xi_n-\xi^*\|^2+2\alpha_n\left\langle \xi_n-\xi^*,\xi_n-\xi_{n-1}\right\rangle-2\beta_n\left\langle \xi_n-\xi^*,\xi_n-\xi_0\right\rangle\\ &\quad -\|\xi_n-\xi_{n+1}\|^2+\alpha_n\|\xi_n-\xi_{n+1}\|^2+\alpha_n\|\xi_n-\xi_{n-1}\|^2\\ &\quad -\alpha_n\|\xi_n-\xi_{n+1}+(\xi_n-\xi_{n-1})\|^2+2\beta_n\left\langle \xi_n-\xi_{n+1},\xi_n-\xi_0\right\rangle.\end{aligned} \tag{3.28}$$

Therefore, we have

$$\begin{aligned}&\|\xi_{n+1}-\xi^*\|^2-\|\xi_n-\xi^*\|^2-\alpha_n\|\xi_n-\xi_{n-1}\|^2+(1-\alpha_n)\|\xi_n-\xi_{n+1}\|^2\\ &\le -2\beta_n\left\langle \xi_n-\xi^*,\xi_n-\xi_0\right\rangle+2\alpha_n\left\langle \xi_n-\xi^*,\xi_n-\xi_{n-1}\right\rangle+2\beta_n\left\langle \xi_n-\xi_{n+1},\xi_n-\xi_0\right\rangle\\ &= -2\beta_n\left\langle \xi_n-\xi^*,\xi_n-\xi_0\right\rangle-\alpha_n\|\xi_{n-1}-\xi^*\|^2+\alpha_n\|\xi_n-\xi^*\|^2+\alpha_n\|\xi_n-\xi_{n-1}\|^2\\ &\quad -\|\xi_0-\xi_{n+1}\|^2+\beta\|\xi_{n+1}-\xi_n\|^2+\beta\|\xi_n-\xi_0\|^2,\end{aligned} \tag{3.29}$$

where we applied (2.2) in the last equation. Hence, we have

$$\begin{aligned}&-2\beta_n\left\langle \xi_n-\xi^*,\xi_n-\xi_0\right\rangle\\ &\ge \|\xi_{n+1}-\xi^*\|^2-\|\xi_n-\xi^*\|^2-2\alpha_{n+1}\|\xi_{n+1}-\xi_n\|^2-2\alpha_n\|\xi_n-\xi_{n-1}\|^2\\ &\quad +\alpha_n\left(\|\xi_{n-1}-\xi^*\|^2-\|\xi_n-\xi^*\|^2\right)+\beta_n\left(\|\xi_0-\xi_{n+1}\|^2-\|\xi_n-\xi_0\|^2\right)\\ &\quad +(1-\alpha_n-2\alpha_{n+1}-\beta_n)\|\xi_{n+1}-\xi_n\|^2.\end{aligned} \tag{3.30}$$

Therefore, from the fact that, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are non-decreasing and non-increasing sequences respectively, then (3.22) follows. Hence, the proof. ∎

Theorem 3.5. *Suppose that Assumption 3.1 holds. Then the sequence $\{\xi_n\}$ generated by Algorithm 1 is bounded.*

Proof. By the simple rearrangement of (3.22), we have

$$\begin{aligned}
&\|\xi_{n+1}-\xi^*\|^2-\|\xi_n-\xi^*\|^2\\
&\le \alpha_n\|\xi_n-\xi^*\|^2-\alpha_{n-1}\|\xi_{n-1}-\xi^*\|^2-(1-3\alpha_{n+1}-\beta_n)\|\xi_{n+1}-\xi_n\|^2\\
&-2\alpha_{n+1}\|\xi_{n+1}-\xi_n\|^2+2\alpha_n\|\xi_n-\xi_{n-1}\|^2-\beta_{n+1}\|\xi_0-\xi_{n+1}\|^2\\
&\quad+\beta_n\|\xi_n-\xi_0\|^2-2\beta_n\langle\xi_n-\xi_0,\xi_n-\xi^*\rangle\\
&=\alpha_n\|\xi_n-\xi^*\|^2-\alpha_{n-1}\|\xi_{n-1}-\xi^*\|^2-(1-3\alpha_{n+1}-\beta_n)\|\xi_{n+1}-\xi_n\|^2\\
&-2\alpha_{n+1}\|\xi_{n+1}-\xi_n\|^2+2\alpha_n\|\xi_n-\xi_{n-1}\|^2-\beta_{n+1}\|\xi_0-\xi_{n+1}\|^2\\
&\quad+\beta_n\|\xi_n-\xi_0\|^2+\beta_n\|\xi_0-\xi^*\|^2-\beta_n\|\xi_n-\xi_0\|^2-\beta_n\|\xi_n-\xi^*\|^2,
\end{aligned}\tag{3.31}$$

where we used (2.2) in the last equation. Hence we have

$$\begin{aligned}
&\|\xi_{n+1}-\xi^*\|^2-\|\xi_n-\xi^*\|^2+\beta_n\|\xi_n-\xi^*\|^2\\
&\le \alpha_n\|\xi_n-\xi^*\|^2-\alpha_{n-1}\|\xi_{n-1}-\xi^*\|^2-(1-3\alpha_{n+1}-\beta_n)\|\xi_{n+1}-\xi_n\|^2\\
&\quad-2\alpha_{n+1}\|\xi_{n+1}-\xi_n\|^2+2\alpha_n\|\xi_n-\xi_{n-1}\|^2+\beta_n\|\xi_0-\xi^*\|^2.
\end{aligned}\tag{3.32}$$

Let $\rho_k=\exp^{\sum_{i=1}^k\beta_i}$, $k\ge 1$. Notice that, since $\exp^x\ge 1+x$ for some $x\in\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{1}{\rho_{n+1}}\left(\rho_{n+1}\|\xi_{n+1}-\xi^*\|^2-\rho_n\|\xi_n-\xi^*\|^2\right)\\
&=\|\xi_{n+1}-\xi^*\|^2-\|\xi_n-\xi^*\|^2+\frac{1}{\rho_{n+1}}(\rho_{n+1}-\rho_n)\|\xi_n-\xi^*\|^2\\
&\le\|\xi_{n+1}-\xi^*\|^2-\|\xi_n-\xi^*\|^2+\beta_{n+1}\|\xi_n-\xi^*\|^2.
\end{aligned}\tag{3.33}$$

Using the fact the sequence $\{\beta_n\}$ is non-increasing in $(0,1]$, we have

$$\begin{aligned}
&\frac{1}{\rho_{n+1}}\left(\rho_{n+1}\|\xi_{n+1}-\xi^*\|^2-\rho_n\|\xi_n-\xi^*\|^2\right)\\
&\le\|\xi_{n+1}-\xi^*\|^2-\|\xi_n-\xi^*\|^2+\beta_n\|\xi_n-\xi^*\|^2.
\end{aligned}\tag{3.34}$$

Substituting (3.34) in (3.32), we have

$$\begin{aligned}
&\frac{1}{\rho_{n+1}}\left(\rho_{n+1}\|\xi_{n+1}-\xi^*\|^2-\rho_n\|\xi_n-\xi^*\|^2\right)\\
&\le \alpha_n\|\xi_n-\xi^*\|^2-\alpha_{n-1}\|\xi_{n-1}-\xi^*\|^2-(1-3\alpha_{n+1}-\beta_n)\|\xi_{n+1}-\xi_n\|^2\\
&\quad-2\alpha_{n+1}\|\xi_{n+1}-\xi_n\|^2+2\alpha_n\|\xi_n-\xi_{n-1}\|^2+\beta_n\|\xi_0-\xi^*\|^2.
\end{aligned}$$

From the fact that $\rho_n\le\rho_{n+1}=\rho_n\exp^{\beta n+1}$ and $\{\beta_n\}$ is non-increasing, we have

$$\begin{aligned}
&\rho_{n+1}\|\xi_{n+1}-\xi^*\|^2-\rho_n\|\xi_n-\xi^*\|^2\\
&\le\rho_{n+1}\alpha_n\|\xi_n-\xi^*\|^2-\rho_n\alpha_{n-1}\|\xi_{n-1}-\xi^*\|^2-\rho_{n+1}(1-3\alpha_{n+1}-\beta_n)\|\xi_{n+1}-\xi_n\|^2\\
&\quad-2\rho_{n+1}\alpha_{n+1}\|\xi_{n+1}-\xi_n\|^2+2\rho_n\alpha_n\exp^{\beta n+1}\|\xi_n-\xi_{n-1}\|^2+\rho_{n+1}\beta_n\|\xi_0-\xi^*\|^2.
\end{aligned}$$

Since $\beta_{n+1}\le\beta_n$, this can be written as

$$\begin{aligned}
&\rho_{n+1}\|\xi_{n+1}-\xi^*\|^2-\rho_n\|\xi_n-\xi^*\|^2\\
&\le\rho_{n+1}\alpha_n\|\xi_n-\xi^*\|^2-\rho_n\alpha_{n-1}\|\xi_{n-1}-\xi^*\|^2\\
&\quad-\rho_{n+1}\left[1-\alpha_{n+1}(3+2(\exp^{\beta n+1}-1))-\beta_n\right]\|\xi_{n+1}-\xi_n\|^2\\
&\quad-2\rho_{n+1}\alpha_{n+1}\|\xi_{n+1}-\xi_n\|^2+2\rho_n\alpha_n\exp^{\beta n+1}\|\xi_n-\xi_{n-1}\|^2+\rho_{n+1}\beta_n\|\xi_0-\xi^*\|^2.
\end{aligned}$$

By Assumption 3.1 and $\{\alpha_n\} \subset [0, \alpha_n]$, we have

$$1 - \alpha_{n+1}(3 + 2(\exp^{\beta n+1} - 1)) - \beta_n \geq 1 - \alpha(3 + 2(\exp^{\beta n+1} - 1)) - \beta_n.$$

Using the fact that $\lim_{n\to\infty} \beta_n = 0$ and $\alpha_n \in [0, 1/3)$, it follows that, for all $N \in \mathbb{N}$ large enough, there is a $\delta > 0$ such that

$$1 - \alpha(3 + 2(\exp^{\beta n+1} - 1)) - \beta_n \geq \delta$$

for all $n \geq N$. Therefore, we have

$$\begin{aligned} &\rho_{n+1}\|\xi_{n+1} - \xi^*\|^2 - \rho_n\|\xi_n - \xi^*\|^2 \\ &\leq \rho_{n+1}\alpha_n\|\xi_n - \xi^*\|^2 - \rho_n\alpha_{n-1}\|\xi_{n-1} - \xi^*\|^2 - \delta\rho_{n+1}\|\xi_{n+1} - \xi_n\|^2 \\ &\quad - 2\rho_{n+1}\alpha_{n+1}\|\xi_{n+1} - \xi_n\|^2 + 2\rho_n\alpha_n \exp^{\beta_{n+1}} \|\xi_n - \xi_{n-1}\|^2 \\ &\quad + \rho_{n+1}\beta_n\|\xi_0 - \xi^*\|^2. \end{aligned}$$

Now, for all $n \geq N$, we have

$$\begin{aligned} &\|\xi_0 - \xi^*\|^2 \sum_{j=N+1}^{n} \rho_{j+1}\beta_j \\ &\geq \rho_{n+1}\|\xi_{n+1} - \xi^*\|^2 + 2\rho_{n+1}\alpha_{n+1} \exp^{\beta_{n+1}} \|\xi_{n+1} - \xi_n\|^2 - \rho_{n+1}\alpha_n\|\xi_n - \xi^*\|^2 \\ &\quad - \rho_{N+1}\|\xi_{N+1} - \xi^*\|^2 - 2\rho_{N+1}\alpha_{N+1} \exp^{\beta_{N+1}} \|\xi_{N+1} - \xi_N\|^2 + \rho_{N+1}\alpha_N\|\xi_N - \xi^*\|^2. \end{aligned} \tag{3.35}$$

Dividing (3.35) by ρ_{N+1} and dropping the non-positive term $-2\rho_{n+1}\alpha_{n+1} \exp^{\beta_{n+1}} \|\xi_{n+1} - \xi_n\|^2$, we have

$$\begin{aligned} &\|\xi_{n+1} - \xi^*\|^2 - \alpha_n\|\xi_n - \xi^*\|^2 \\ &\leq \exp^{-t_{n+1}} \big[\rho_{N+1}\|\xi_{N+1} - \xi^*\|^2 + 2\rho_{N+1}\alpha_{N+1} \exp^{\beta_{N+1}} \|\xi_{N+1} - \xi_N\|^2 \\ &-\rho_{N+1}\alpha_N\|\xi_N - \xi^*\|^2\big] + \|\xi_0 - \xi^*\|^2 \exp^{-t_{n+1}} \sum_{j=N+1}^{n} \beta_j \exp^{t_{j+1}}, \end{aligned} \tag{3.36}$$

where $t_n = \sum_{j=1}^{n} \beta_j$. Notice that $\beta_j \exp^{t_{j+1}} \leq \exp^2(\exp^{t_j} - \exp^{t_{j-1}})$ for all $j \geq 2$. Therefore, we have

$$\sum_{j=N+1}^{n} \rho_{j+1}\beta_j = \sum_{j=N+1}^{n} \beta_j \exp^{t_{j+1}} \leq \exp^2(\exp^{t_n} - \exp^{t_N}) \leq \exp^2 \exp^{t_n}. \tag{3.37}$$

Putting (3.37) in (3.36) with $\exp^{-t_{n+1}} \leq 1$, we get

$$\begin{aligned} &\|\xi_{n+1} - \xi^*\|^2 \\ &\leq -\alpha_n\|\xi_n - \xi^*\|^2 + \rho_{N+1}\|\xi_{N+1} - \xi^*\|^2 + 2\rho_{N+1}\alpha_{N+1} \exp^{\beta_{N+1}} \|\xi_{N+1} - \xi_N\|^2 \\ &\quad - \rho_{N+1}\alpha_N\|\xi_N - \xi^*\|^2 + \exp^2 \exp^{t_n} \|\xi_0 - \xi^*\|^2 \\ &\leq -\alpha\|\xi_n - \xi^*\|^2 + \rho_{N+1}\|\xi_{N+1} - \xi^*\|^2 + 2\rho_{N+1}\alpha_{N+1} \exp^{\beta_{N+1}} \|\xi_{N+1} - \xi_N\|^2 \\ &\quad + \exp^2 \|\xi_0 - \xi^*\|^2 \end{aligned} \tag{3.38}$$

By the simple calculation and using the convergence of a geometric series, we obtain

$$\begin{aligned}\|\xi_{n+1}-\xi^*\|^2 &\le -\alpha^{n-N}\|\xi_{N+1}-\xi^*\|^2 \\ &\quad + \frac{1}{1-\alpha}\left(\rho_{N+1}\|\xi_{N+1}-\xi^*\|^2 + 2\rho_{N+1}\alpha_{N+1}\exp^{\beta_{N+1}}\|\xi_{N+1}-\xi_N\|^2 \right. \\ &\quad \left. + \exp^2\|\xi_0-\xi^*\|^2\right).\end{aligned} \tag{3.39}$$

Since the $\alpha \le 1$, It now follows that the sequence $\{\xi_n\}$ is bounded. ∎

Lemma 3.6. *Let Assumption 3.1 holds and let $\{\xi_n\}$ be the sequence generated by Algorithm (1). Suppose that*

$$\lim_{n\to\infty}\|\xi_{n+1}-\xi_n\| = 0$$

and

$$\lim_{n\to\infty}\left(\|\xi_{n+1}-\xi^*\|^2 - \alpha_n\|\xi_n-\xi^*\|^2\right) = 0.$$

Then, the sequence $\{\xi_n\}$ strongly converges to the solution $\xi^* \in \Omega$.

Proof. By the hypothesis, we have

$$\begin{aligned}0 &= \lim_{n\to\infty}\left(\|\xi_{n+1}-\xi^*\|^2 - \alpha_n\|\xi_n-\xi^*\|^2\right) \\ &= \lim_{n\to\infty}\left[\left(\|\xi_{n+1}-\xi^*\| + \sqrt{\alpha_n}\|\xi_n-\xi^*\|\right)\left(\|\xi_{n+1}-\xi^*\| - \sqrt{\alpha_n}\|\xi_n-\xi^*\|\right)\right].\end{aligned} \tag{3.40}$$

Now, we claim that

$$\lim_{n\to\infty}\left(\|\xi_{n+1}-\xi^*\| + \sqrt{\alpha_n}\|\xi_n-\xi^*\|\right) = 0.$$

To see this, assume that the above equality does not hold. Then there exist $\nu > 0$ and a subset $N \subset \mathbb{N}$ such that, for all $n \ge N$,

$$\|\xi_{n+1}-\xi^*\| + \sqrt{\alpha_n}\|\xi_n-\xi^*\| \ge \nu. \tag{3.41}$$

Using the fact that, $\alpha_n \le \alpha \le 1$ and (3.40), we have

$$\begin{aligned}0 &= \lim_{n\in N}\left(\|\xi_{n+1}-\xi^*\| - \alpha_n\|\xi_n-\xi^*\|\right) \\ &= \limsup_{n\in N}\left(\|\xi_{n+1}-\xi_n+\xi_n-\xi^*\| + \sqrt{\alpha_n}\|\xi_n-\xi^*\|\right) \\ &\ge \limsup_{n\in N}\left(\|\xi_n-\xi^*\| - \|\xi_{n+1}-\xi_n\| - \sqrt{\alpha_n}\|\xi_n-\xi^*\|\right) \\ &\ge \limsup_{n\in N}\left(1-\sqrt{\alpha_n}\right)\|\xi_n-\xi^*\| - \|\xi_{n+1}-\xi_n\| \\ &= \left(1-\sqrt{\alpha_n}\right)\limsup_{n\in N}\|\xi_n-\xi^*\| - \lim_{n\in N}\|\xi_{n+1}-\xi_n\| \\ &= \left(1-\sqrt{\alpha_n}\right)\limsup_{n\in N}\|\xi_n-\xi^*\|.\end{aligned} \tag{3.42}$$

This implies that $\limsup_{n\in N}\|\xi_n-\xi^*\| \le 0$. Clearly, $\liminf_{n\in N}\|\xi_n-\xi^*\| \ge 0$. Therefore, it follows that $\lim_{n\in N}\|\xi_n-\xi^*\| = 0$. By (3.41), this implies

$$\begin{aligned}\|\xi_{n+1}-\xi_n\| &\ge \|\xi_{n+1}-\xi^*\| - \|\xi_n-\xi^*\| \\ &= \|\xi_{n+1}-\xi^*\| + \sqrt{\alpha_n}\|\xi_n-\xi^*\| - \left(1+\sqrt{\alpha_n}\right)\|\xi_n-\xi^*\| \\ &\ge \frac{\nu}{2}\end{aligned}$$

for all $n \in N$ large enough, which is a contradiction. Hence the proof follows. ∎

In the next theorem, we prove the convergence of the sequence $\{\xi_n\}$ generated by Algorithm 1.

Theorem 3.7. *Suppose that Assumption 3.1 holds. Then, the sequence $\{\xi_n\}$ generated by Algorithm 1 strongly converges to a solution $\xi^* \in \Omega$.*

Proof. Let $\{T_n\}$ be a sequence defined by

$$T_n = \|\xi_n - \xi^*\|^2 - \alpha_{n-1}\|\xi_{n-1} - \xi^*\|^2 + 2\alpha_n\|\xi_n - \xi_{n-1}\|^2 + \beta_n\|\xi_n - \xi_0\|^2.$$

Now, we claim that the sequence $\{T_n\}$ is non-negative. To see this, using the assumptions on the sequence $\{\alpha_n\}$ in Assumption 3.1 and (2.1), we have

$$\begin{aligned} T_n &= \|\xi_n - \xi^*\|^2 - \alpha_{n-1}\Big(\|\xi_{n-1} - \xi_n\|^2 + \|\xi_n - \xi^*\|^2 + 2\left\langle \xi_{n+1} - \xi_n, \xi_n - \xi^*\right\rangle\Big) \\ &\quad + 2\alpha_n\|\xi_n - \xi_{n-1}\|^2 + \beta_n\|\xi_n - \xi_0\|^2 \\ &= \|\xi_n - \xi^*\|^2 - \alpha_{n-1}\Big(2\|\xi_{n-1} - \xi_n\|^2 + 2\|\xi_n - \xi^*\|^2 - \|\xi_{n+1} - 2\xi_n - \xi^*\|^2\Big) \\ &\quad + 2\alpha_n\|\xi_n - \xi_{n-1}\|^2 + \beta_n\|\xi_n - \xi_0\|^2 \\ &\geq \|\xi_n - \xi^*\|^2 - 2\alpha_{n-1}\|\xi_{n-1} - \xi_n\|^2 + \frac{2}{3}\|\xi_n - \xi^*\|^2 - \alpha_{n-1}\|\xi_{n+1} - 2\xi_n - \xi^*\|^2 \\ &\quad + 2\alpha_n\|\xi_n - \xi_{n-1}\|^2 + \beta_n\|\xi_n - \xi_0\|^2 \\ &\geq \frac{1}{3}\|\xi_n - \xi^*\|^2 + \beta_n\|\xi_n - \xi_0\|^2 \\ &\geq 0. \end{aligned} \tag{3.43}$$

Applying $\{T_n\}$ in Lemma 3.4, we have

$$\begin{aligned} &T_{n+1} - T_n + (1 - 3\alpha_{n+1} - \beta_n)\|\xi_n - \xi_{n+1}\|^2 \\ &\leq -2\beta_n\left\langle \xi_n - \xi^*, \xi_n - \xi_0\right\rangle. \end{aligned} \tag{3.44}$$

We now consider two cases

Case 1: Suppose for some $N \in \mathbb{N}$ sufficiently large enough $T_{n+1} \leq T_n$, that is, the sequence $\{T_n\}$ is monotonically non-decreasing. Then obviously $\{T_n\}$ is convergent. It follows from Assumption 3.1 and 3.5 that, there exists $\delta > 0$ and $M > 0$ such that $1 - 3\alpha_{n+1} - \beta_n \geq \delta$ and $2\left|\langle \xi_n - \xi^*, \xi_n - \xi_0\rangle\right| \leq M$ respectively. Therefore, from (3.44), we obtain

$$\delta\|\xi_{n+1} - \xi_n\|^2 \leq \beta_n M + T_n - T_{n+1} \to 0 \quad \text{as } n \to \infty. \tag{3.45}$$

Therefore, we have

$$\lim_{n\to\infty} \|\xi_{n+1} - \xi_n\|^2 \to 0.$$

Using the fact that $\beta_n \to 0$, the boundedness of the sequence $\{\xi_n\}$ and the convergence of $\{T_n\}$, the limit

$$\mu := \lim_{n\to\infty} \left(\|\xi_{n+1} - \xi^*\|^2 - \alpha_n\|\xi_n - \xi^*\|^2\right) \tag{3.46}$$

exists and is equal to $\lim_{n\to\infty} T_{n+1}$. In particular, (3.43) implies that $\mu \geq 0$. Now, we show that $\mu = 0$. Suppose that $\mu > 0$. Since the sequence $\{\xi_n\}$ is bounded, we can select a subsequence $\{\xi_{n_k}\}$ which converges weakly to $p \in \mathcal{H}$ such that

$$\liminf_{n\to\infty} \langle \xi_n - \xi^*, \xi^* - \xi_0\rangle = \lim_{k\to\infty} \langle \xi_{n_k} - \xi^*, \xi^* - \xi_0\rangle = \langle p - \xi^*, \xi^* - \xi_0\rangle .$$

Next, we show that $p \in \Omega$. Observe that

$$\begin{aligned} \delta\|w_n - \xi_n\| &\leq \|\beta_n(\xi_0 - \xi_n) + \alpha_n(\xi_n - \xi_{n-1})\| \\ &\leq \beta_n\|\xi_0 - \xi_n\| + \alpha_n\|\xi_n - \xi_{n-1}\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.47}$$

This implies that

$$\|\xi_{n+1} - w_n\| \leq \|\xi_n - w_n\| + \|\xi_{n+1} - \xi_n\| \to 0 \text{ as } n \to \infty. \tag{3.48}$$

Now, it follows from (3.25) and the assumption on the sequence $\{\beta_n\}$ in Assumption 3.1 that

$$\begin{aligned} \|\xi_{n+1} - \xi^*\|^2 &\leq \|w_n - \xi^*\|^2 - \|\xi_{n+1} - w_n\|^2 \\ &\leq \|w_n - \xi^*\|^2 - \beta_n^2\|w_n - z_n\|^2 \\ &\leq \|w_n - \xi^*\|^2 - \|w_n - z_n\|^2 \end{aligned} \tag{3.49}$$

Since $\{\xi_n\}$ is bounded and hence $\{w_n\}$ are bounded, it follows from (3.49) that

$$\begin{aligned} \|w_n - z_n\|^2 &\leq \|w_n - \xi^*\|^2 - \|\xi_{n+1} - \xi^*\|^2 \\ &= (\|w_n - \xi^*\| - \|\xi_{n+1} - \xi^*\|)\left(\|w_n - \xi^*\|^2 + \|\xi_{n+1} - \xi^*\|\right) \\ &\leq (\|w_n - \xi^*\| - \|\xi_{n+1} - \xi^*\|)\, M \\ &\leq \|w_n - \xi_{n+1}\|M, \end{aligned} \tag{3.50}$$

where $M := \sup_{n\geq 1}\{\|w_n - \xi^*\|, \|\xi_{n+1} - \xi^*\|\}$. It now follows from (3.48) that

$$\lim_{n\to\infty} \|w_n - z_n\| = 0.$$

It then follows from (3.6) that

$$\lim_{n\to\infty} \|w_n - t_n\| = 0.$$

This implies that $\xi_n - z_n \to 0$. Therefore, $z_{n_k} \rightharpoonup p$ and since $z_n \in E$, we have $p \in E$. Similarly, $w_{n_k} \rightharpoonup p$. Now, for all $\xi \in E$, using Lemma 2.3 and the monotonicity of $\mathcal{A}$, we have

$$\begin{aligned} 0 &\leq \langle z_{n_k} - w_{n_k} + \lambda_n \mathcal{A}w_{n_k}, \xi - z_{n_k}\rangle \\ &= \langle z_{n_k} - w_{n_k}, \xi - z_{n_k}\rangle + \lambda_n \langle \mathcal{A}w_{n_k}, w_{n_k} - z_{n_k}\rangle + \lambda_n \langle \mathcal{A}w_{n_k}, \xi - w_{n_k}\rangle \\ &\leq \langle z_{n_k} - w_{n_k}, \xi - w_{n_k}\rangle + \lambda_n \langle \mathcal{A}w_{n_k}, w_{n_k} - z_{n_k}\rangle + \lambda_n \langle \mathcal{A}\xi, \xi - w_{n_k}\rangle \end{aligned} \tag{3.51}$$

Passing through the limit, we have

$$\langle \mathcal{A}\xi, \xi - p\rangle \geq 0, \quad \forall\, \xi \in E.$$

Then Lemma (2.4), implies $p \in \Omega$ so that

$$\liminf_{n\to\infty} \langle \xi_n - \xi^*, \xi^* - \xi_0\rangle = \langle p - \xi^*, \xi^* - \xi_0\rangle \geq 0, \tag{3.52}$$

where the last inequality follows from Lemma 2.3. Now, from (3.46), we have

$$\liminf_{n\to\infty} \|\xi_{n+1} - \xi^*\|^2 \geq \lim_{n\to\infty} \left(\|\xi_{n+1} - \xi^*\|^2 - \alpha_n\|\xi_n - \xi^*\|^2\right) = \mu.$$

Since $\mu > 0$, for some $N_1 \in \mathbb{N}$ large enough, we have

$$\|\xi_{n+1} - \xi^*\|^2 \geq \frac{1}{2}\mu, \quad \forall n \geq N_1.$$

Observe that

$$\langle \xi_n - \xi^*, \xi_n - \xi_0 \rangle = \|\xi_n - \xi^*\|^2 + \langle \xi_n - \xi^*, \xi^* - \xi_0 \rangle . \tag{3.53}$$

Using (3.53), the equation (3.52) will be

$$\begin{aligned}\liminf_{n\to\infty} \langle \xi_n - \xi^*, \xi_n - \xi_0 \rangle &= \liminf_{n\to\infty} \left(\|\xi_n - \xi^*\|^2 + \langle \xi_n - \xi^*, \xi^* - \xi_0 \rangle \right) \\ &\geq \liminf_{n\to\infty} \left(\frac{1}{2}\mu + \langle \xi_n - \xi^*, \xi^* - \xi_0 \rangle \right) \\ &= \frac{1}{2}\mu + \liminf_{n\to\infty} \left(\langle \xi_n - \xi^*, \xi^* - \xi_0 \rangle \right) \\ &\geq \frac{1}{2}\mu.\end{aligned}$$

It implies once again that, for some $N_2 \in \mathbb{N}$ large enough,

$$\langle \xi_n - \xi^*, \xi_n - \xi_0 \rangle \geq \frac{1}{4}\mu \quad \forall n \geq N_2$$

Therefore, it follows from (3.44) that

$$T_{n+1} - T_n \leq -\frac{1}{2}\beta_n \mu, \quad \forall n \geq N_2,$$

which implies that

$$\frac{1}{2}\mu \sum_{k=N_2}^{n} \beta_k \leq T_{N_2} - T_n \leq T_{N_2}, \quad \forall n \geq N_2.$$

This contradicts our assumption on the sequence $\{\beta_n\}$ in Assumption 3.1. Therefore we must have $\mu = 0$. Hence the sequence $\{\xi_n\}$ converges strongly to a solution ξ^*.

Case 2: Suppose the sequence $\{T_n\}$ is not monotically decreasing. For some $N \in \mathbb{N}$ large enough, define a map $\phi : \mathbb{N} \to \mathbb{N}$ for all $n \geq N$ by

$$\phi(n) := \max\{k \in \mathbb{N} : k \leq n, T_n \leq T_{n+1}\}. \tag{3.54}$$

Clearly, $\phi(n)$ is a non-decreasing sequence such that $\phi(n) \to \infty$ as $n \to \infty$ and $T_{\phi(n)} \leq T_{\phi(n)+1}$ for all $n \geq N$. Therefore, in a similar manner as the proof of Case 1, it follows from (3.44) that

$$\delta \|\xi_{\phi(n)+1} - \xi_{\phi(n)}\|^2 \leq \beta_{\phi(n)} M \to 0 \quad \text{as } n \to \infty, \tag{3.55}$$

where $M > 0$ is a constant. Thus

$$\|\xi_{\phi(n)+1} - \xi_{\phi(n)}\| \to 0 \quad \text{as } n \to \infty. \tag{3.56}$$

Similarly as in Case 1 above, it follows that

$$\|\xi_{\phi(n)+1} - w_{\phi(n)}\| \to 0 \quad \text{as } n \to \infty,$$

$$\|w_{\phi(n)} - \xi_{\phi(n)}\| \to 0 \quad \text{as } n \to \infty, \tag{3.57}$$

and

$$\|\xi_{\phi(n)} - z_{\phi(n)}\| \to 0 \quad \text{as } n \to \infty. \tag{3.58}$$

Again, notice that, for $k \geq 0$, by (3.44) , $T_{k+1} < T_k$ when $\xi_k \notin \Gamma = \{\xi \in \mathcal{H} : \langle \xi - \xi_0, \xi - \xi^* \rangle \leq 0\}$. Therefore, since $T_{\phi(n)} \leq T_{\phi(n)+1}$, we have $\xi_{\phi(n)} \in \Gamma$ for all $n \geq N$. Now, since $\{\xi_{\phi(n)}\}$ is bounded, we can select a subsequence (for simplicity, we denote it by $\{\xi_{\phi(n)}\}$) which weakly converges to $\hat{\xi} \in H$. Since Γ is a closed convex and hence weakly closed, it then implies that $\hat{\xi} \in \Gamma$. Similarly as in Case 1, using (3.58), we have $z_{\phi(n)} \rightharpoonup \hat{\xi}$ and $\hat{\xi} \in \Omega$. Consequently, we have $\hat{\xi} \in \Gamma \cap \Omega$, and so, from Lemma 2.2, $\hat{\xi} = \xi^*$. Moreover, since $\xi_{\phi(n)} \in \Gamma$, we have

$$\begin{aligned}\|\xi_{\phi(n)} - \xi^*\|^2 &= \left\langle \xi_{\phi(n)} - \xi_0, \xi_{\phi(n)} - \xi^* \right\rangle - \left\langle \xi^* - \xi_0, \xi_{\phi(n)} - \xi^* \right\rangle \\ &\leq - \left\langle \xi^* - \xi_0, \xi_{\phi(n)} - \xi^* \right\rangle.\end{aligned}$$

Taking the limsup through the last inequality gives

$$\limsup_{n\to\infty} \|\xi_{\phi(n)} - \xi^*\| \leq 0.$$

Thus we have

$$\|\xi_{\phi(n)} - \xi^*\| \to 0 \quad \text{as } n \to \infty.$$

Now, we claim that $\lim_{n\to\infty} T_{\phi(n)+1} = 0$. It is easy to see by simple calculation and using (3.56) and (3.58) that, $\lim_{n\to\infty} T_{\phi(n)+1}$ which is given by

$$\begin{aligned}\lim_{n\to\infty} T_{\phi(n)+1} = \lim_{n\to\infty} \Big(&\|\xi_{\phi(n)+1} - \xi^*\|^2 - \alpha_{\phi(n)}\|\xi_{\phi(n)} - \xi^*\|^2 \\ &+ 2\alpha_{\phi(n)+1}\|\xi_{\phi(n)+1} - \xi_{\phi(n)}\|^2 + \beta_{\phi(n)+1}\|\xi_{\phi(n)+1} - \xi_0\|^2 \Big)\end{aligned}$$

goes to zero. We next show that, the limit of T_n is actually zero, that is, $\lim_{n\to\infty} T_n = 0$. Notice that, if $n \neq \phi(n)$, $T_n \leq T_{\phi(n)+1}$ for all $n \geq N$ since $T_k > T_{k+1}$ for $\phi(n) + 1 \leq k \leq n - 1$. It follows that, for all $n \geq N$,

$$T_n \leq \max\{T_{\phi(n)}, T_{\phi(n)+1}\} = T_{\phi(n)+1} \to 0,$$

and so $\limsup_{n\to\infty} T_n \leq 0$. On the other hand, (3.43) implies that $\liminf_{n\to\infty} T_n \geq 0$. Therefore, $\lim_{n\to\infty} T_n = 0$. Consequently, by the boundedness of the sequence $\{\xi_n\}$, (3.44) and Assumption 3.1, we have

$$\lim_{n\to\infty} \|\xi_n - \xi_{n+1}\| \to 0 \text{ as } n \to \infty.$$

Thus, the definition of $\{T_n\}$ implies

$$\lim_{n\to\infty} \left(\|\xi_{n+1} - \xi^*\|^2 - \alpha_n\|\xi_n - \xi^*\|^2\right) = 0.$$

It is now easy to see that, using Assumption 3.1, the last equation gives the strong convergence of $\{\xi_n\}$ to ξ^*. ∎

4. Numerical Illustrations

Here, we analyze the computational performance and the inertial-effect of our proposed algorithm by giving some computational experiments and comparisons with some existing algorithms in the literature. For reference, our codes were written using the Matlab program (Matlab R2016b) and were run on a PC with Intel(R) Core(TM)i5, CPU @ 1.4GHz, and RAM 4.00 GB.

Example 4.1. [14] Let consider the VIP (1.1) with an operator $\mathcal{A}$ defined as

$$\mathcal{A}(\xi_1, \xi_2) = (2\xi_1 + 2\xi_2 + \sin(\xi_1), -2\xi_1 + 2\xi_2 + \sin(\xi_2))$$

for all $\xi_1, \xi_2 \in \mathcal{H}$ with $E := \{\xi \in \mathbb{R}^2 : -10 \leq \xi_n \leq 10,\ n = 1, 2\}$. The unique solution of this problem is $\xi^* = (0, 0)^T$. It can be shown that $\mathcal{A}$ is a strongly monotone and L - *Lipschitz* with Lipschitz constant $L = \sqrt{26}$.

We give the numerical analysis of Example 4.1 in Table 1, where ε, ξ_0, "iter" and CPU denote respectively the tolerance, initial point, number of iterations and CPU time.

Table 1. Results of Example 4.1 for Algorithm 1 with different initial values

ε	10^{-3}	10^{-4}	10^{-5}	10^{-7}	10^{-3}	10^{-4}	10^{-5}	10^{-7}
ξ_0	Iter.	Iter.	Iter.	Iter.	cpu.	cpu.	cpu.	cpu.
$(1, 10)^T$	10	43	329	32334	0.2598	0.3939	3.2108	354.5379
$(10, 10)^T$	12	56	458	44957	0.2799	0.4388	4.2567	510.9625
$(-5, 20)^T$	15	77	662	66477	0.2898	0.6129	6.8006	627.5416

Example 4.2. [19] Let us consider the fractional programming problem defined as follows:

$$\min T(\xi) = \frac{\xi^T P\xi + c^T\xi + c_0}{d^T\xi + d_0}$$

$$\text{subject to } \xi \in \mathcal{X} := \{\xi \in \mathbb{R}^4 : d^T\xi + d_0 > 0\}$$

with

$$P = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad d = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad c_0 = 2, \quad d_0 = 4.$$

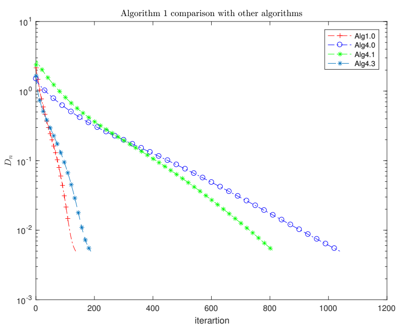
It can be seen that P is symmetric and positive definite in $\mathbb{R}^4$, therefore, T is pseudo-convex on X. Minimizing T over $E = \left\{\xi \in \mathbb{R}^4 : 1 \leq \xi_n \leq 10, n = 1, ..., 4\right\} \subset \mathcal{X}$ using Algorithm 1 with $\mathcal{A}(\xi) := \left((d^T\xi + d_0)(2P\xi + c) - d(\xi^T P\xi + c^T\xi + a_0)\right) / (d^T\xi + d_0)^2$. It is clear that $\xi^* = (1, 1, 1, 1)^T$ in E is the unique solution of this problem.

In Table 2 we present the numerical analysis of Example 4.2 with different initial values and stopping criteria.

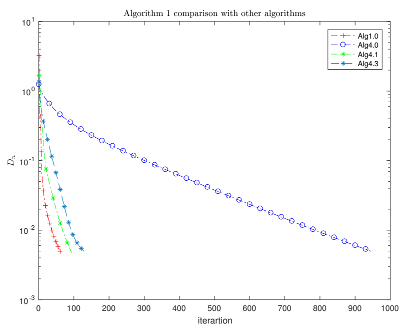
Table 2. Results of Example 4.1 for Algorithm 1 with different initial values

ε ξ_0	10^{-3} Iter.	10^{-4} Iter.	10^{-5} Iter.	10^{-6} Iter.	10^{-3} cpu.	10^{-4} cpu.	10^{-5} cpu.	10^{-6} cpu.
$(10,10,10,10)^T$	75	236	749	2369	0.5752	1.8721	7.4670	23.3459
$(5,10,15,20)^T$	89	283	898	44957	0.6249	2.2638	9.3607	29.9212
$(10,20,30,40)^T$	128	408	1292	4087	0.9729	3.4837	13.1561	43.9974

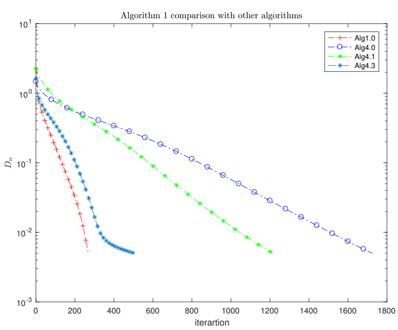
Example 4.3. [16] Let the operator $\mathcal{A}(\xi) := M\xi + q, \xi \in \mathbb{R}^n$. This is a prominent Example considered in the literature by many authors, see for example [33, 35], where $M = BB^T + C + D$, with B, C and D as $n \times n$ square matrices such that C is a skew-symmetric, and D is a diagonal matrix with nonnegative diagonal entries. Therefore, M is a positive semi-definite and q is a vector in $\mathbb{R}^n$. We defined the feasible set $E := \{\xi \in \mathbb{R}^n : Q\xi \leq b\}$ as a closed and convex subset $E \subset \mathbb{R}^n$, where Q is an $l \times n$ matrix and b is a nonnegative vector. It can be seen that $\mathcal{A}$ is monotone and M-Lipschitz-continuous. When $q = 0$, the solution set is $\Omega := \{0\}$.



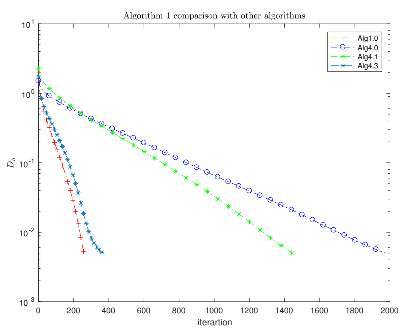
(A) $m = 20, l = 20$



(B) $m = 20, l = 50$



(C) $m = 30, l = 70$



(D) $m = 20, l = 100$

Figure 1. Comparison of error estimate with respect to the number of iterations of Alg.1, Alg.4.0, Alg.4.1 and Alg.4.3 with different sizes of m and l (see Example 4.3).

In this example we use $\mathcal{H} := \mathbb{R}^n$, and $L := \|M\|$. Similar to the example in [16], we choose randomly the starting points $\xi_0 \in [0,1]^m$ and $\xi_1 \in [0,1]^m$. We compare our proposed Algorithm with the Inertial Projection and Contraction Algorithm [14] given by (1.6), the Single Projection Method Algorithm introduced in [33] and the Halpern type Subgradient Extragradient Algorithm studied in [25]. For convenience we denote these algorithms respectively as Alg. 4.0, Alg. 4.3 and Alg. 4.1. We use $\|\xi_n - \xi^*\| \leq \epsilon = 0.005$ as the stopping criterion. For this experiment we take $\lambda_n = 0.9/\|M\|$ for Alg. 1, Alg. 4.3 and Alg. 4.0, $\lambda = 1/2\|M\|$ for Alg. 4.1. We randomly generate the matrices A, b, B, C, D and, we take $\beta_n = 1/(n+2)$, $\alpha_n = 0.3$ and $\gamma = 1.99$ for Alg. 1, $\alpha_n = 1/(13n+2)$ and $\gamma = 1.99$ for Alg.4.3, $\alpha_n = 0.4$ for Alg. 4.0 and $\alpha_n = 1/(100n+2)$ for Alg. 4.1.

We recall that, the Halpern-type subgradient extragradient algorithm studied in ([25]) is given by:

$$\begin{cases} \xi_0 \in \mathcal{H}, \\ z_n = P_E\left(\xi_n - \tau f(\xi_n)\right), \\ T_n = \{w \in \mathcal{H} : \langle \xi_n - \tau f(\xi_n) - z_n, w - z_n \rangle \leq 0\}, \\ \xi_{n+1} = \alpha_n \xi_0 + (1-\alpha_n) P_{T_n}\left(\xi_n - \tau f(z_n)\right) \end{cases} \tag{4.1}$$

where $f : \mathcal{H} \longrightarrow \mathcal{H}$ and $E \subset \mathcal{H}$ is nonempty, closed and convex. We also recall the *Single Projection Method Algorithm* introduced in [33] given as follows:

Initialization: Choose sequence $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Ler $\lambda_n \in (0,\infty)$, $\gamma \in (0,2)$ and let $\xi_1 \in \mathcal{H}$ be a given stating point: Set $n = 1$.

Steps 1:

$$z_n = P_E\left(\xi_n - \lambda_n \mathcal{A}\xi_n\right) \tag{4.2}$$

If $\xi_n - z_n = 0$ STOP

Step 2 Compute

$$d(\xi_n, z_n) = \xi_n - z_n - \lambda_n\left(\mathcal{A}\xi_n - \mathcal{A}z_n\right), \tag{4.3}$$

Step 3 Compute

$$\xi_{n+1} = \alpha_n \xi_1 + (1-\alpha_n)\left(\xi_n - \gamma\rho_n d(\xi_n, z_n)\right),$$

where ρ_n is given by

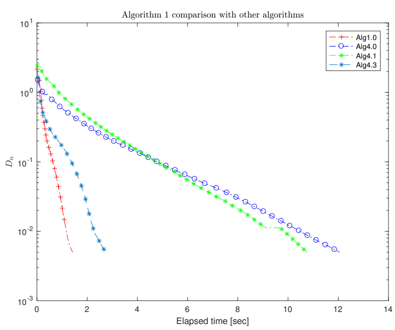
$$\rho_n = \begin{cases} \frac{\langle \xi_n - z_n, d(\xi_n, z_n)\rangle}{\|d(\xi_n, z_n)\|^2}, & \text{if} \quad d(\xi_n, z_n) \neq 0, \\ 0, & \text{if} \quad d(\xi_n, z_n) = 0, \end{cases}$$

Step 3. Set $n = n+1$ and go back to **Step 1.**

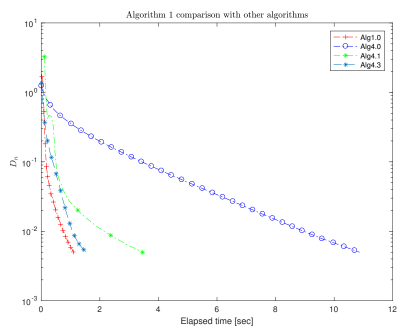
We compare the number of iterations and the execution time of these algorithms for different sizes of m and l in Table 3, Figure 1 and Figure 2.

TABLE 3. Comparison of Algorithm 1 with some existing Algorithms

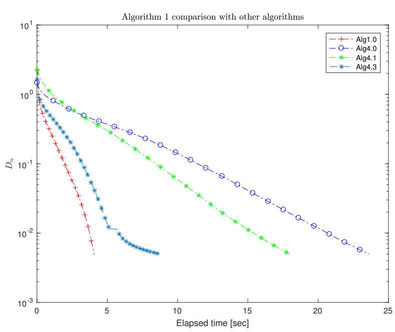
m, l .		Alg. 1	Alg. 4.0	Alg. 4.1	Alg. 4.3
20, 20	Iter.	110	1039	813	188
	cpu	1.4292	12.0576	10.7666	2.7607
20, 50	Iter.	93	947	558	129
	cpu	1.1522	10.8669	7.226187	1.5127
30, 70	Iter.	270	1726	1216	506
	cpu	4.0807	23.6044	17.9609	8.7087
30, 100	Iter.	258	1975	1443	368
	cpu	4.0943	34.2453	24.5201	6.2354



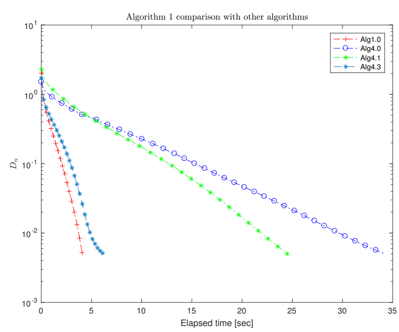
(A) $m = 20, l = 20$



(B) $m = 20, l = 50$



(C) $m = 30, l = 70$



(D) $m = 20, l = 100$

FIGURE 2. Comparison of error estimate with respect to the elapsed time of Alg. 1, Alg. 4.0, Alg. 4.1 and Alg. 4.3 with different sizes of m and l (see Example 4.3).

5. Conclusion

In this article, we presented a one-step inertial projection algorithm for solving VIP. The algorithm incorporates an inertial term with the projection-contraction method. We have shown that the sequence generated by our proposed algorithm converges strongly under mild assumptions imposed on the underlying operator. We also presented some numerical examples to illustrate the computational performance of our proposed algorithm. Moreover, we compared the proposed algorithm with other strong convergence algorithms [25, 33] and inertial projection and contraction algorithm presented in [14]. Our proposed algorithms performed better in both the number of iteration and computational time compared to these algorithms.

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