Thai Journal of Mathematics
*Volume 6 (2008) Number 2 : 315–322*
www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

# Approximating common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings

W. Cholamjiak and S. Suantai[1]

**Abstract :** In this paper, we introduce a new iterative scheme for finding a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in a Banach space and then obtain strong convergence theorems.

**Keywords :** Generalized asymptotically quasi-nonexpansive mapping; Iterative method; Common fixed point; Banach space; Strong convergence
**2000 Mathematics Subject Classification :** 47H09; 47H10

## 1 Introduction

Let $C$ be a nonempty closed convex subset of a real Banach space $X$. A mapping $T : C \to C$ is said to be

(i) *nonexpansive if* $\|Tx - Ty\| \leq \|x - y\|$ *for all* $x, y \in C$;

(ii) *quasi-nonexpansive if* $\|Tx - p\| \leq \|x - p\|$ *for all* $x \in C$ *and* $p \in F$;

(iii) *asymptotically nonexpansive if there exists a sequence* $\{r_n\}$ *in* $[0, \infty)$ *such that* $\lim_{n\to\infty} r_n = 0$ *and*

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|,$$

*for all* $x, y \in C$ *and* $n \geq 1$;

(iv) *asymptotically quasi-nonexpansive if there exists a sequence* $\{r_n\}$ *in* $[0, \infty)$ *such that* $\lim_{n\to\infty} r_n = 0$ *and*

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\|,$$

*for all* $x \in C, p \in F(T)$ *and* $n \geq 1$;

(v) *generalized quasi-nonexpansive if there exists a sequence* $\{s_n\}$ *in* $[0, 1)$ *such*

[1]Corresponding author.
*Email addresses:* scmti005@chiangmai.ac.th (Suthep Suantai), c-wchp007@hotmail.com (Watcharaporn Cholamjiak)

*that* $\lim_{n\to\infty} s_n = 0$ *and*

$$\|T^n x - p\| \leq \|x - p\| + s_n\|x - T^n x\|,$$

*for all* $x \in C, p \in F(T)$ *and* $n \geq 1$;

(vi) *generalized asymptotically quasi-nonexpansive if there exist two sequences* $\{r_n\}$ *and* $\{s_n\}$ *in* $[0,1)$ *such that* $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} s_n$ *and*

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n\|x - T^n x\|,$$

*for all* $x \in C, p \in F(T)$ *and* $n \geq 1$.

From the above definitions, it is clear that :

(i) a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(ii) a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(iii) an asymptotically nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(iv) an asymptotically quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping.

(v) a generalized quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping.

However, the converse of the statements is not true, we have the following example shows that a generalized asymptotically quasi-nonexpansive mapping may be not a generalized quasi-nonexpansive mapping and also may be not an asymptotically quasi-nonexpansive mapping.

**Example** Let $X = \ell_\infty$ with the norm $\|\cdot\|$ defined by

$$\|x\| = \sup_{i\in N} |x_i|, \ \forall x = (x_1, x_2, ..., x_n, ...) \in X,$$

and $C = \{x = (x_1, x_2, ..., x_n, ...) \in X : x_i \geq 0, x_1 \geq x_i, \forall i \in N \textit{ and } x_2 = x_1\}$. Then $C$ is a nonempty subset of $X$.

Now, for any $x = (x_1, x_2, ..., x_n, ...) \in C$, define a mapping $T : C \to C$ as follows

$$T(x) = (0, 2x_1, 0, ..., 0, ....).$$

It is easy to see that $T$ is a generalized asymptotically quasi-nonexpansive mapping. In fact, for any $x = (x_1, x_2, ..., x_n, ...) \in C$, taking $T(x) = x$ i.e.,

$$(0, 2x_1, 0, ..., 0, ....) = (x_1, x_2, ..., x_n, ...).$$

Then we have $F(T) = \{(0, 0, ..., 0, ...)\}$ and $T^n(x) = (0, 0, ..., 0, ...), \ \forall n = 2, 3, ....$

For all $r_1, s_1 \in [0, 1)$ with $r_1 + s_1 \geq 1$, we have

$\|T(x) - p\| - (1 + r_1)\|x - p\| - s_1\|x - T(x)\|$

$$\begin{aligned}
&= \|(0, 2x_1, 0, ..., 0, ...)\| - (1 + r_1)\|(x_1, x_2, ..., x_n, ...)\| - s_1\|(x_1, x_2, ..., x_n, ...)\| \\
&= 2x_1 - (1 + r_1)x_1 - s_1 x_1 \\
&\leq 0
\end{aligned}$$

and

$$\begin{aligned}\|T^n(x)-p\|-(1+r_n)\|x-p\|-s_n\|x-T^n(x)\| &= 0-(1+r_n)x_1-s_nx_1\\ &\leq 0,\end{aligned}$$

for all $n=2,3,...,\{r_n\}$ and $\{s_n\}\subset[0,1)$ with $r_n\to 0$ and $s_n\to 0$ as $n\to\infty$, and so $T$ is a generalized asymptotically quasi-nonexpansive mapping. However, $T$ is not a generalized quasi-nonexpansive mapping. Since

$$\begin{aligned}\|T(x)-p\|-\|x-p\|-s_1\|x-T(x)\| &= 2x_1-x_1-s_1x_1\\ &> 0\ ,\forall s_1\in[0,1).\end{aligned}$$

And $T$ is not an asymptotically quasi-nonexpansive mapping with respect to $\{r_n\}$. Since

$$\begin{aligned}\|T(x)-p\|-(1+r_1)\|x-p\| &= 2x_1-(1+r_1)x_1\\ &> 0\ ,\forall r_1\in[0,1).\end{aligned}$$

Since 1972, the weak and strong convergence problems of iterative sequences (with errors) for asymptotically nonexpansive types mapping in the condition of a Hilbert space or a Banach space have been studied by many authors (see, for example, [1],[3],[4]).

In this paper, we introduce a new iteration process for a finite family $\{T_i:\ i=1,2,...,m\}$ of generalized asymptotically quasi-nonexpansive mappings as follows :

Let $C$ be a closed convex subset of a Banach space $X$ and $x_0\in C$. Suppose that $\alpha_{in}\in[0,1]$, $i=0,1,2,...,m$ and $n\geq 1$. Let $\{T_i:i=1,2,...,m\}$ be a family of self-mappings of $C$. The iteration scheme is defined as follows :

$$x_{n+1}=S_nx_n,\quad \forall n\geq 1, \tag{1.1}$$

where $S_n=\alpha_{0n}I+\alpha_{1n}T_1^n+\alpha_{2n}T_2^n+...+\alpha_{mn}T_m^n$ with $\sum_{i=0}^m\alpha_{in}=1$.

Clearly, the iteration process (1.1) generalizes the modified Mann iteration from one mapping to the finite family of mappings $\{T_i:i=1,2,...,m\}$ . The main purpose of this paper is to establish a necessary and sufficient condition for strong convergence of the iteration scheme (1.1) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in a Banach space.

We need the following useful known lemmas for the development of our results.

**Lemma 1.1.** [5, Lemma 2.2] *Let the sequence $\{a_n\}$ and $\{u_n\}$ of real number satisfy :*

$$a_{n+1}\leq(1+u_n)a_n,\quad \forall n\geq 1,$$

*where $a_n\geq 0$, $u_n\geq 0$ and $\sum_{n=1}^\infty u_n<\infty$. Then*

(i) $\lim_{n\to\infty}a_n$ *exists;*

(ii) *if* $\liminf_{n\to\infty}a_n=0$, *then* $\lim_{n\to\infty}a_n=0$.

**Lemma 1.2.** [2, Lemma 2.2] *Let $C$ be a nonempty closed subset of a Banach space $X$ and $T : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping with the fixed point set $F(T) \neq \emptyset$. Then $F(T)$ is a closed subset in $C$.*

## 2 Convergent Theorem in a Banach Space

In this section, we prove strong convergence theorem of the iteration scheme (1.1) under some suitable conditions :

**Lemma 2.1.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $\{T_i : i = 1, 2, ..., m\}$ a family of generalized asymptotically quasi-nonexpansive self-mappings of $C$. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max_{1\le i\le m} r_{in}$ and $s_n = \max_{1\le i\le m} s_{in}$. Define the sequence $\{x_n\}$ as in (1.1). Then*

(i) *there exists a sequence $\{\delta_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\|x_{n+1} - p\| \le (1 + \delta_n)\|x_n - p\|$ for all $p \in F$ and $n \ge 1$;*

(ii) *there exists a constant $M > 0$ such that $\|x_{n+k} - p\| \le M\|x_n - p\|$ for all $p \in F$ and $n, k \ge 1$.*

*Proof.* (i) Let $p \in F$, $r_n = \max_{1\le i\le m} r_{in}$ and $s_n = \max_{1\le i\le m} s_{in}$.
Now, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|S_n x_n - p\| \\
&= \|\alpha_{0n}x_n + \alpha_{1n}T_1^n x_n + \alpha_{2n}T_2^n x_n + ... + \alpha_{mn}T_m^n x_n - p\| \\
&\le \alpha_{0n}\|x_n - p\| + \alpha_{1n}\|T_1^n x_n - p\| + \alpha_{2n}\|T_2^n x_n - p\| \\
&\quad + ... + \alpha_{mn}\|T_m^n x_n - p\| \\
&\le \alpha_{0n}\|x_n - p\| + \alpha_{1n}\big((1 + r_{1n})\|x_n - p\| + s_{1n}\|x_n - T_1^n x_n\|\big) \\
&\quad + \alpha_{2n}\big((1 + r_{2n})\|x_n - p\| + s_{2n}\|x_n - T_2^n x_n\|\big) \\
&\quad + ... + \alpha_{mn}\big((1 + r_{mn})\|x_n - p\| + s_{mn}\|x_n - T_m^n x_n\|\big) \\
&\le \alpha_{0n}\|x_n - p\| + \alpha_{1n}\big((1 + r_n)\|x_n - p\| + s_n\|x_n - T_1^n x_n\|\big) \\
&\quad + \alpha_{2n}\big((1 + r_n)\|x_n - p\| + s_n\|x_n - T_2^n x_n\|\big) \\
&\quad + ... + \alpha_{mn}\big((1 + r_n)\|x_n - p\| + s_n\|x_n - T_m^n x_n\|\big) \\
&= \big(\alpha_{0n} + \alpha_{1n}(1 + r_n) + \alpha_{2n}(1 + r_n) + ... + \alpha_{mn}(1 + r_n)\big)\|x_n - p\| \\
&\quad + s_n\big(\alpha_{1n}\|x_n - T_1^n x_n\| + \alpha_{2n}\|x_n - T_2^n x_n\| + ... + \alpha_{mn}\|x_n - T_m^n x_n\|\big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \big(\alpha_{0n}(1+r_n)+\alpha_{1n}(1+r_n)+\alpha_{2n}(1+r_n)\\
&\quad + ... + \alpha_{mn}(1+r_n)\big)\|x_n-p\| + s_n\big(\alpha_{1n}\|x_n-T_1^n x_n\|\\
&\quad + \alpha_{2n}\|x_n-T_2^n x_n\| + ... + \alpha_{mn}\|x_n-T_m^n x_n\|\big)\\
&= (\alpha_{0n}+\alpha_{1n}+\alpha_{2n}+...+\alpha_{mn})(1+r_n)\|x_n-p\|\\
&\quad + s_n\big(\alpha_{1n}\|x_n-T_1^n x_n\| + \alpha_{2n}\|x_n-T_2^n x_n\| + ... + \alpha_{mn}\|x_n-T_m^n x_n\|\big)\\
&\leq (1+r_n)\|x_n-p\| + s_n\big(\alpha_{1n}\|x_n-T_1^n x_n\| + \alpha_{2n}\|x_n-T_2^n x_n\| + ...\\
&\quad + \alpha_{mn}\|x_n-T_m^n x_n\|\big). \qquad (2.1)
\end{aligned}
$$

Now, we show that $\|x_n - T_i^n x_n\| \leq \frac{2+r_n}{1-s_n}\|x_n-p\|, \quad i = 1,2,...,m.$
For $i = 1,2,3,...,m$, we have

$$
\begin{aligned}
\|x_n - T_i^n x_n\| &\leq \|x_n-p\| + \|T_i^n x_n - p\|\\
&\leq \|x_n-p\| + (1+r_{in})\|x_n-p\| + s_{in}\|x_n - T_i^n x_n\|\\
&\leq \|x_n-p\| + (1+r_n)\|x_n-p\| + s_n\|x_n - T_i^n x_n\|\\
&= \frac{2+r_n}{1-s_n}\|x_n-p\|. \qquad (2.2)
\end{aligned}
$$

It follows from (2.1) and (2.2) that

$$
\begin{aligned}
\|x_{n+1}-p\| &\leq (1+r_n)\|x_n-p\| + s_n\big(\alpha_{1n}\frac{2+r_n}{1-s_n}\|x_n-p\| + \alpha_{2n}\frac{2+r_n}{1-s_n}\|x_n-p\|\\
&\quad + ... + \alpha_{mn}\frac{2+r_n}{1-s_n}\|x_n-p\|\big)\\
&= \big(1+r_n+(\alpha_{1n}+\alpha_{2n}+...+\alpha_{mn})s_n\frac{2+r_n}{1-s_n}\big)\|x_n-p\|\\
&\leq \big(1+r_n+s_n\frac{2+r_n}{1-s_n}\big)\|x_n-p\|. \qquad (2.3)
\end{aligned}
$$

Put $\delta_n = r_n + s_n\frac{2+r_n}{1-s_n}$. By assumption, it follows that $\sum_{n=1}^{\infty}\delta_n < \infty$. Hence above formula reduces to $\|x_{n+1}-p\| \leq (1+\delta_n)\|x_n-p\|$. This completes the proof of (i).

(ii) If $t \geq 0$ then $1+t \leq e^t$. Thus, from part (i), we get

$$
\begin{aligned}
\|x_{n+k}-p\| &\leq (1+\delta_{n+k-1})\|x_{n+k-1}-p\|\\
&\leq \exp\{\delta_{n+k-1}\}\|x_{n+k-1}-p\|\\
&\leq \exp\{\delta_{n+k-1}\}(1+\delta_{n+k-2})\|x_{n+k-2}-p\|\\
&\leq \exp\{\delta_{n+k-1}\}\exp\{1+\delta_{n+k-2}\}\|x_{n+k-2}-p\|\\
&\;\;\vdots\\
&\leq \exp\{\sum_{i=1}^{n+k-1}\delta_i\}\|x_n-p\|\\
&\leq \exp\{\sum_{i=1}^{\infty}\delta_i\}\|x_n-p\|.
\end{aligned}
$$

Setting $M = \exp\{\sum_{i=1}^{\infty}\delta_i\}$, then $\|x_{n+k}-p\| \leq M\|x_n-p\|$. This completes the proof of (ii).

□

**Theorem 2.2.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $\{T_i : i = 1,2,...,m\}$ a family of generalized asymptotically quasi-nonexpansive self-mappings of $C$. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max_{1\leq i\leq m} r_{in}$ and $s_n = \max_{1\leq i\leq m} s_{in}$. Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point of the family of mapping if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F}\|x-p\|$.*

*Proof.* We will only prove the sufficiency, the necessity is obvious. From Lemma 2.1 (i), we have $\|x_{n+1}-p\| \leq (1+\delta_n)\|x_n-p\|, \quad \forall n \geq 1$. Therefore

$$d(x_{n+1}, F) \leq (1+\delta_n)d(x_n, F) \quad \text{and} \quad \sum_{n=1}^{\infty}\delta_i < \infty.$$

By Lemma 1.1 and $\liminf_{n\to\infty} d(x_n, F) = 0$, we get that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 2.1 (ii), we have

$$\|x_{n+k}-p\| \leq M\|x_n-p\|, \tag{2.4}$$

for all $p \in F$ and $n, k \geq 1$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, for each $\epsilon > 0$, there exists a natural number $n_1$ such that $d(x_n, F) \leq \frac{\epsilon}{3M}, \quad \forall n \geq n_1$. Hence, there exists $z_1 \in F$ such that

$$\|x_{n_1} - z_1\| \leq \frac{\epsilon}{2M}. \tag{2.5}$$

From (2.4) and (2.5), $\forall n \geq n_1$, we have

$$\begin{aligned} \|x_{n+k}-x_n\| &\leq \|x_{n+k}-z_1\| + \|x_n - z_1\| \\ &\leq M\|x_{n_1}-z_1\| + M\|x_{n_1}-z_1\| \\ &\leq \epsilon. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in $X$. By the completeness of $X$, we also have that $\{x_n\}$ is a convergent sequence. Assume that $\{x_n\}$ converges to a point $p'$. Then $p' \in C$ because $C$ is a closed subset of $X$. By Lemma 1.2, we know that $F$ is closed. From the continuity of $d(x, F) = 0$ with $d(x_n, F) \to 0$ and $x_n \to p'$ as $n \to \infty$, we get $d(p', F) = 0$ and so $p' \in F$. This completes the proof. □

A generalized quasi-nonexpansive mapping must be a generalized asymptotically quasi-nonexpansive mapping, so we have

**Corollary 2.3.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $\{T_i : i = 1,2,...,m\}$ a family of generalized quasi-nonexpansive self-mappings of $C$. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $s_n = \max_{1\leq i\leq m} s_{in}$. Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point of the family of mapping if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F}\|x-p\|$.*

An asymptotically quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping, so we have

**Corollary 2.4.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$, and $\{T_i : i = 1, 2, ..., m\}$ a family of asymptotically quasi-nonexpansive self-mappings of $C$. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty} r_n < \infty$ where $r_n = \max_{1\leq i\leq m} r_{in}$. Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point of the family of mapping if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F} \|x - p\|$.*

**Corollary 2.5.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $\{T_i : i = 1, 2, ..., m\}$ a family of generalized quasi-nonexpansive self-mappings of $C$. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max_{1\leq i\leq m} r_{in}$ and $s_n = \max_{1\leq i\leq m} s_{in}$. Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a point $p \in F$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $p$.*

**Theorem 2.6.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $\{T_i : i = 1, 2, ..., m\}$ a family of generalized quasi-nonexpansive self-mappings of $C$. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ where $r_n = \max_{1\leq i\leq m} r_{in}$ and $s_n = \max_{1\leq i\leq m} s_{in}$.*
*Suppose that there exists a map $T_j$ which satisfies the following conditions:*

(i) $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$;

(ii) *there exists a constant $M$ such that* $\|x_n - T_j x_n\| \geq M d(x_n, F)$, $\forall n \geq 1$.

*Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a point $p \in F$.*

*Proof.* From (i) and (ii), it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 2.2, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings. □

**Acknowledgement(s) :** The authors would like to thank the Faculty of Science, Prince of Chiang Mai University, Thailand for the financial support during the preparation of this paper.

## References

[1] H. Fukhar-ud-din, Convergence of iterates with errors of asymptotically quasi-nonexpansive mapping and applications, J. Math. Anal. Appl. 328 (2007) 821-829.

[2] E.H. Lan, Common fixed-point iterative Process with errors for Generalized Asymptotically Quasi-Nonexpansive Mappings, Comput. Math. Appl. 52 (2006) 1403-1412.

[3] K. Nammanee, M.A. Noor, S. Suantai, Convergence criteria of modifies Noor iterations with errors for asymptotically nonexpansive Mappings, J. Math. Anal. Appl. 314 (2006) 320-334.

[4] W. Nilsrakoo, S. Saejung, A new three-step fixed point iteration scheme for asymptotically nonexpansive Mappings, Comput. Math. Appl. 181 (2006) 1026-1034.

[5] K.K. Tan and H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301-308.

(Received 30 May 2007)

Watcharaporn Cholamjiak
Department of Mathematics,
Faculty of Science,
Chiang Mai University,
Chiang Mai 10140 , THAILAND.
e-mail : c-wchp007@hotmail.com

Suthep Suantai
Department of Mathematics,
Faculty of Science,
Chiang Mai University,
Chiang Mai 10140 , THAILAND.
e-mail : scmti005@chiangmai.ac.th