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Solving Common Nonmonotone Equilibrium Problems Using a Parallel Inertial Type Algorithm with Armijo Linesearch

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Abstract In this work, we study an iterative method with an inertial term extrapolation step and parallel algorithm for solving common equilibrium problems of nonmonotone bifunctions in real Hilbert spaces. The inertial term extrapolation step is introduced to speed up the rate of convergence of the iteration process. We obtain weak convergence results under some continuity and convexity assumptions on the bifunction and the condition that the common solution set of the associated Minty equilibrium problems is nonempty. We also give some numerical experiments of our algorithm which are defined by parallel inertial type algorithm to show the efficiency and implementation for LASSO problem in signal recovery in situation different types of blurred matrices.

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1. Introduction

Let C be a nonempty closed and convex subset of Ω, where Ω is an open convex subset of a Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $f:\Omega\times\Omega\to\mathbb{R}$ on. The equilibrium problem (see [1]) is to find $x^*\in C$ such that

$$f(x^*,x)\geq 0,\ \text{for all } x\in C. \tag{1.1}$$

In this work, we shall assume that the bifunction f satisfies $f(x,x)=0$ and shall denote by $EP(C,f)$, the solutions set of (1.1). Associated with the equilibrium problem (1.1) is the Minty equilibrium problem (see [2]) which is to find $\hat{x}\in C$ such that

$$f(x,\hat{x})\leq 0,\ \text{for all } x\in C. \tag{1.2}$$

Let the solution set of the Minty equilibrium problem be represented as S_M.

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The equilibrium problems include as particular cases, the scalar and vector optimization problems, saddle-point problems, variational inequality problems, Nash equilibria, complementarity problems and have in recent years engaged the interest of many mathematicians, see, for example, [1, 3–15, 17] and some of the references therein.

Recently, Dinh and Kim [18] proposed the following projection algorithms with linesearch for solving equilibrium problem where the bifunction is not required to be pseudomonotone.

Algorithm 1.1. Initialization: pick $x_0 \in C$, choose parameters $\eta \in (0,1), \rho > 0$ and $C_0 = C$. **Iteration** $n(n = 0, 1, 2, 3, ...)$. Having x_n do the following steps:
Step 1 : Solve the strongly convex program

$$argmin\{f(x,y) + \frac{\rho}{2}\|y - x_n\|^2 : y \in C\}, \tag{1.3}$$

to obtain its unique solution y_n. If $y_n = x_n$, then stop. Otherwise do go to the next step
Step 2 : (Armijo linesearch rule) Find m_n as the smallest nonnegative integer m satisfying

$$\begin{cases} z_{n,m} = (1 - n^m)x_n + n^m y_n, \\ u_{n,m} \in \partial_2 f(z_{n,m}, z_{n,m}), \\ \langle u_{n,m}, x_n - y_n \rangle \geq \frac{\rho}{2} \|y_n - x_n\|^2 . \end{cases} \tag{1.4}$$

Set $\eta_n := \eta^{m_n}, z_n = z_{n,m_n}, u_n = u_{n,m_n}$. Take

$$H_n = \{x \in H : \langle u_n, x - z_n \rangle \leq 0\}, C_{n+1} = C_n \cap H_n. \tag{1.5}$$

Step 3 :Compute

$$x_{n+1} = P_{C_{n+1}}(x_n), \tag{1.6}$$

and go to step 1 with n replaced by $n + 1$.

Using Algorithm 1.1, Dinh and Kim [18] obtained weak convergence result for solving the equilibrium problem (1.1) under the assumption that the bifunction f is continuous, convex and not required to satisfy any monotonicity property and the nonemptyness of the solution set of the Minty equilibrium problem (1.2).

In 2017, Van et.al. [19] study a quasi-equilibrium problem with a nonmonotone bifunction is considered in a finite-dimensional space. The following algorithm is presented for obtaining weak convergence theorem:

Algorithm 1.2. Step 0 : Let $x_0 \in C, C \in [0,1]$ and $\eta \in [0,1]$. Let also $\mu_k \subseteq [a,b]$ where $0 < a \leq b < 1$. Set $n = 0$.
Step 1 : Compute $y_n = \underset{y \in C(w_n)}{argmin}\{f(w_n, y) + \frac{1}{2}\|y - w_n\|^2\}$.
If $y_n = w_n$, then Stop. Otherwise go to Step 2.
Step 2 : Find m the smallest nonnegative integer such that

$$\begin{cases} \langle u_{n,m}, w_n - y_n \rangle \geq c\|w_n - y_n\|^2 \\ \text{where } z_{n,m} = (1 - \eta^m)w_n + \eta^m y_n \text{ and } u_{n,m} \in \partial_2 f(z_{n,m}, z_{n,m}) \end{cases} \tag{1.7}$$

and set $\eta_n = \eta^m$, $z_n = z_{n,m}$ and $y_n = y_{n,m}$. Consider the half-space

$$B_n = \{x \in \mathbb{R}^n | \langle u_n, x - z_n \rangle \leq 0\}. \tag{1.8}$$

Step 3 : Find $x_n = P_{C_n}(x_n)$ where C_n denotes the convex closed set

$$C_n = C \cap (\bigcap_{i=0}^{n} B_i). \tag{1.9}$$

Calculate $x_{n+1} = \mu_n w_n + (1-\mu_n)v_n$ where $v_n = P_{K(u_n)}u_n$.
Set $n := n+1$ and go back to Step 1.

In 2018, Iyiola et.al. [20] motivated the inertial-type algorithms and the work of Dinh and Kim [18], they obtained convergence theorems and presented the following inertial type iterative method with step-size called, Armijo linesearch which is faster and more efficient than the Algorithm 1.1 by Dihn and Kim [18].

Algorithm 1.3. Step 1 : Choose sequence $\{\epsilon_n\}_{n=1}^{\infty} \in l_1$ and take $\eta \in (0,1), \rho > 0$. Select arbitrary points $x_0 \in C_0, x_1 \in C_1; C_0 = C_1 = C$ and $\theta \in [0,1)$. Set $n := 1$.
Step 2 : Given the iterates x_{n-1} and $x_n, n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta_n} = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|^2}\}, & x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases}$$

Step 3 : Compute

$$w_n := x_n + \theta_n(x_n - x_{n-1}).$$

Step 4 : Compute

$$y_n := argmin\{f(w_n, y) + \frac{\rho}{2}\|y - w_n\|^2 : y \in C\},$$

if $y_n = w_n$, then stop. Otherwise go to Step 5.
Step 5 : Find m_n as the smallest nonnegative integer m satisfying

$$\begin{cases} z_{n,m} = (1+n^m)x_n + n^m y_n, \\ u_{n,m} \in \partial_2 f(z_{n,m}, z_{n,m}), \\ \langle u_{n,m}, x_n - y_n\rangle \geq \frac{\rho}{2}\|y_n - x_n\|^2 \end{cases} \tag{1.10}$$

Set $\eta_n := \eta^{m_n}, z_n = z_{n,m_n}, u_n = u_{n,m_n}$.
Step 6 : Compute

$$x_{n+1} = P_{C_{n+1}}(w_n), \tag{1.11}$$

where $C_{n+1} = C_n \cap B_n, B_n = \{x \in H \ : h_n(x) \leq 0\}$ and

$$h_n(x) := \langle u_n, x - y_n\rangle. \tag{1.12}$$

Step 7 : Set n $\leftarrow$ n+1 and go to 2.
Here, θ_n is an extrapolation factor and the inertial is represented by the term $\theta_n(x_n - x_{n-1})$. It is remarkable that the inertial term extrapolation step greatly improves the performance of the algorithm and has a nice convergence properties [21, 22]

In this work, we study the common equilibrium problem (CEP) which is to find $x^* \in C$ such that

$$f_i(x^*, x) \leq 0 \text{ for all } x \in C. \tag{1.13}$$

For the set of common solution of the Minty equilibrium problem respert with f_i, we denote it CS_M

Motivated by the recent interest on inertial-type algorithms and the work of Dinh, Kim [18], Van [23] and Iyiola [20], we propose an algorithm which is a combination of Algorithm 1.1, Algorithm 1.2 and Algorithm 1.3 above and inertial extrapolation step for solving equilibrium problem (1.1) in infinite dimensional Hilbert spaces. Under with parallel algorithm the same conditions imposed on the bifunction f_i.

2. Preliminaries and Lemmas

We next recall some properties of the projection, cf. [24] for more details. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \le \|u - y\|, \forall y \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a nonexpensive mapping of H onto C. It is also known that P_C satises

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \forall x, y \in H. \tag{2.1}$$

In particular, we get

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \forall x \in C, y \in H. \tag{2.2}$$

Furthermore, $P_C x$ is characterized by the properties

$$P_C x \in C \text{ and } \langle x - P_C x, P_C x - y\rangle \ge 0, \forall y \in C. \tag{2.3}$$

Further properties of the metric projection can be found, for example, in Section 3 of [25].

Lemma 2.1. Let C be a nonempty closed and convex subset of H, $y := P_C(x)$ and $x^* \in C$. Then

$$\|y - x^*\|^2 \le \|x - x^*\|^2 - \|x - y\|^2. \tag{2.4}$$

We shall denote by $dist(., K)$ distance function to K, i.e.,

$$dist(x, K) = \inf\{\|x - y\| : y \in K\}.$$

Lemma 2.2. [26] Let h be a real-valued function on H and define $K := \{x \in H : h(x)) \le 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then

$$dist(x, K) \ge \theta^{-1} \max\{h(x), 0\}, \forall x \in C.$$

Definition 2.3. A bifunction $f : C \times C \to \mathbb{R}$ is said to be jointly weakly continuous on $C \times C$ if for all $x, y \in C$ and $\{x_n\}, \{y_n\}$ are two sequences in C converging weakly to x and y respectively, then $f(x_n, y_n)$ converges to $f(x, y)$.

We now state the followings assumptions which is required in the sequel.
(A1) $f(x, \cdot)$ is convex on H forevery $x \in H$;
(A2) f is jointly weakly continuous on $H \times H$.

For each $x, z \in H$, by $\partial_2 f(z, x)$ we denote the subdifferential of convex function $f(z, \cdot)$ at x, i.e.,

$$\partial_2 f(z, x) := \{u \in H : f(z, y) \ge f(z, x) + \langle u, y - x\rangle, \forall y \in H\}.$$

In particular,

$$\partial_2 f(z, z) = \{u \in H : f(z, y) \ge \langle u, y - z\rangle, \forall y \in H\}.$$

We now state the following lemmas which will be used in the convergence analysis in the sequel.

Lemma 2.4. Let $f : H \times H \to \mathbb{R}$ be a function satisfying conditions (A1) and (A2). Let $\bar{x}, \bar{y} \in H$ and $\{x_n\}, \{y_n\}$ be two sequences in Hconverging weakly to $\bar{x}, \bar{y}$, respectively. Then, for any $\epsilon > 0$, there exists $\eta > 0$ and $n_\epsilon \in \mathbb{N}$ such that

$$\partial_2 f(x_n, y_n) \subset \partial_2 f(\bar{x}, \bar{y}) + \tfrac{\epsilon}{\eta} B,$$

for every $n \geq n_\epsilon$, where B denote the closed unit ball in H.

Lemma 2.5. [[18], Lemma 2.19] Under the assumptions (A1) *and* (A2) , if $\{z_n\} \subset H$ is a sequence such that $\{z_n\}$ converges strongly to $\bar{z}$ and the sequence $\{z_n\}$, with $u_n \in \partial_2 f(z_n, z_n)$, converges weakly to $\bar{u}$, then $\bar{u} \in \partial_2 f(\bar{z}, \bar{z})$.

Lemma 2.6. [[18], Lemma2.20] Suppose the bifunction f satisfies the assumptions (A1) and (A2). If $\{x_n\} \subset C$ is bounded, $\rho > 0$ and $\{y_n\}$ is a sequence such that

$$y_n = argmin \left\{ f(x_n, y) + \tfrac{\rho}{2} \|y - x_n\|^2 : y \in C \right\},$$

then $\{y_n\}$ is bounded.

Lemma 2.7. The following well-known result holds in a real Hilbert space:

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H.$$

Lemma 2.8. [27] Assume $\varphi_n \in [0, \infty)$ and $\delta_n \in [0, \infty)$ satisfy:
1. $\varphi_{n+1} - \varphi_n \leq \theta_n(\varphi_n - \varphi_{n-1}) + \delta_n$,
2. $\sum_{n=1}^{\infty} \delta_n < \infty$,
3. $\{\theta_n\} \subset [0, \theta]$, where $\theta \in (0, 1)$.

Then the sequence $\{\varphi_n\}$ is convergent with $\sum_{n=1}^{\infty} [\varphi_{n+1} - \varphi_n]_+ < \infty$, where $[t]_+ := \max\{t, 0\}$ (for any $t \in R$).

3. Main results

In this section, we introduce our proposed method for solving the equilibrium problem (1.1) and give some comments regarding the iterative parameters.

Algorithm 3.1. Step 1 : Choose sequence $\{\epsilon_n\}_{n=1}^{\infty} \in l_1$ and take $\eta \in (0, 1), \rho > 0$. Select arbitrary points $x_0 \in C_0, x_1 \in C_1 : C_0 = C_1 = C$ and $\theta \in [0, 1)$. Set $n := 1$.
Step 2 : Given the iterates x_{n-1} and $x_n, n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|^2}\right\} & , x_n \neq x_{n-1} \\ \theta, \text{ otherwise.} \end{cases}$$

Step 3 : Compute

$$w_n := x_n + \theta_n(x_n - x_{n-1}).$$

Step 4 : Compute

$$y_n^i := argmin \left\{ f_i(w_n, y) + \tfrac{\rho}{2} \|y - w_n\|^2 : y \in C \right\}, \ \forall\ i = 1, 2, ..., N.$$

If $y_n^i = w_n$, $\forall i = 1, 2, ..., N$, then stop. otherwise go to Step 5.
Step 5 : Find m_n^i as the smallest nonnegative integer m^i satisfying

$$\begin{cases} z_{n,m}^i = (1 - n^{m^i}) w_n + \eta^{m^i} y_{n^i}, \\ u_{n,m}^i \in \partial_2 f_i(z_{n,m}^i, z_{n,m}^i), \\ \left\langle u_{n,m}^i, w_n - y_n^i \right\rangle \geq \frac{\rho}{2} \left\| y_n^i - w_n \right\|^2. \end{cases} \tag{3.1}$$

set $\eta_n^i := \eta^{m_n}, z_n^i = z_{n,m_n}^i, u_n^i = u_{n,m}^i$.
Step 6 : Compute

$$x_{n+1} = Pc_{n+1}(w_n), \tag{3.2}$$

where $C_{n+1} = C_n \bigcap \left(\bigcap_{i=1}^{N} B_n^i \right)$, $B_n^i = \left\{ x \in H : h_n^i(x) \leq 0 \right\}$ and

$$h_n^i(x) := \left\langle u_n^i, x - y_n^i \right\rangle. \tag{3.3}$$

Step 7 : Set $n \leftarrow n+1$ and go to Step 2.

It is clear that if $y_n^i = w_n$, then w_n is a solution of the equilibrium problem. In our convergence theory, we will implicitly assume that this does not occur after finitely many iterations, so that Algorithm 3.1 generate an infinite sequences satisfying, in particular, $y_n^i - w_n \neq 0$ for all $n \in N$.

Remark 3.2. 1. We remark here that Step 2 in Algorithm 3.1 is easily implemented in numerical computation since the value of $\|x_n - x_{n-1}\|$ is a priori known before choosing θ_n. Furthermore, observe that by the assumption that $\{\epsilon_n\}_{n=1}^{\infty} \subset l_1$, we have that $\sum_{n=0}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$.

2. It is well known that the projection of a point on the intersection is very hard to compute. However for computation purposes, this can alternatively be written as the following optimization problem:

$$P_{C^*}(x) := \min_{y \in C^*} \|y - x\|^2,$$

where

$$C^* := C_n \bigcap \left(\bigcap_{i=1}^{N} B_n^i \right).$$

Kindly see [28] for many other ways to computationally handle projection onto intersection of sets. We give the following result which is very similar to algorithm 3.1 of [18] but the proof is given for the sake of completeness.

Lemma 3.3. Let the solution set CS_M of the Minty equilibrium problem (1.2) be nonempty. Then the following hold.

1. There exist an integer number $m^i > 0$ satisfying the inequality $\langle u_{n,m}^i, w_n - z_n^i \rangle \geq \frac{\rho}{2} \|w_n - y_n^i\|^2$ for every $u_{n,m}^i \in \partial_2 f_i(z_{n,m}^i, z_{n,m}^i)$.
2. C_n is nonempty closed convex.

Proof. We start with showing that at each iteration n, there exists a positive integer $m_0^i, \forall_i = 1, 2, ..., N$. such that

$$\left\langle u_{n,m_0^i}^i, w_n - y_n^i \right\rangle \geq \frac{\rho}{2} \left\| w_n - y_n^i \right\|^2, \forall u_{n,m_0^i}^i \in \partial f_i(z_{n,m_0^i}^i, z_{n,m_0^i}^i), \forall_i = 1, 2, ..., N.$$

Thus the linesearch is well defined. Since $\langle u_n^i, w_n - z_n^i \rangle = \eta_n \langle u_n^i, w_n - y_n^i \rangle$ combining with the linesearch rule (3.1), we get

$$\langle u_n^i, w_n - z_n^i \rangle \geq \frac{\eta_n \rho}{2} \left\| y_n^i - w_n \right\|^2.$$

Next, we show that C_n is nonempty. Indeed, by the assumption $CS_M \neq \phi$, then for each $x^* \in CS_M$, we get $f_i(y_n^i, x^*) \leq 0, \forall y \in C$. $\forall_i = 1, 2, ..., N$. So, $f_i(z_n^i, x^*) \leq 0, \forall_n = 1, 2, ..., N$. From the convexity of $f_i\left(z_n^i, .\right)$, we have

$$f_i\left(z_n^i, y\right) \geq f_i\left(z_n^i, z_n^i\right) + \left\langle u_n^i, y - y_n^i \right\rangle, \forall y \in C.$$

Therefore,

$$0 \geq f_i\left(z_n^i, x^*\right) \geq \left\langle u_n^i, x^* - y_n^i\right\rangle.$$

Hence, $x^* \in B_n^i, \forall_i = 1, 2..., N$. This means x_n generated by Algorithm 3.1 is well defined. This implies that $x^* \in C_n, \forall n$. ∎

Our main contribution in this paper is given in the next result. In this next result, we establish the convergence analysis of the sequence of iterates generated by our proposed Algorithm 3.1 to the solution of the equilibrium problem (1.1). Here, $x_n \rightharpoonup x^*$ means x_n converges weakly to x^*.

Theorem 3.4. Let $CS_M \neq \phi$ and let $f_i : \Omega \times \Omega \to \mathbb{R}$ satify Assumptions (A1), (A2) hold for all $i = 1, 2, ..., N$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to $z \in \bigcap_{i=1}^N EP\left(C, f_i\right)$.

Proof. We break our proof into several steps below for the sake of clarity.

Step (i) : We first show that $\{x_n\}$ is bounded and there exists a weak cluster point of $\{x_n\}$. Let $x^* \in CS_M$, then from Lemma 3.3, we have that $x^* \in C_n$.

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|P_{c_{n+1}}(w_n) - x^*\|^2 \\ &\leq \|w_n - x^*\|^2. \end{aligned} \tag{3.4}$$

But

$$\begin{aligned} \|w_n - x^*\|^2 &= \|x_n + \theta_n\left(x_n - x_{n-1}\right) - x^*\|^2 \\ &= \|x_n - x^*\|^2 + 2\theta_n\left\langle x_n - x^*, x_n - x_{n-1}\right\rangle + \theta_n^2\left\|x_n - x_{n-1}\right\|^2. \end{aligned} \tag{3.5}$$

Observe that

$$2\left\langle x_n - x^*, x_n - x_{n-1}\right\rangle = \|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2 + \|x_n - x_{n-1}\|^2. \tag{3.6}$$

Thus, from (3.4) and (3.5) and noting that $\theta_n^2 \leq \theta_n$, we have

$$\begin{aligned} \|w_n - x^*\|^2 &= \|x_n - x^*\|^2 + \theta_n\left(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\right) \\ &\quad + \left(\theta_n + \theta_n^2\right)\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - x^*\|^2 + \theta_n\left(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\right) \\ &\quad + 2\theta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.7}$$

Hence, it follows from (3.5) and (3.6) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 &+ \theta_n\left(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\right) \\ &+ 2\theta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.8}$$

Now, since $\epsilon_n \in l_1$, it follows that

$$\sum_{n=1}^{\infty} \theta_n \left\|x_n - x_{n-1}\right\|^2 < \infty.$$

Therefore, letting $\delta_n = 2\theta_n\|x_n - x_n\|^2$ and $\varphi_n = \|x_n - x^*\|^2$, we deduce from Lemma 2.8 that the sequence $\{\|x_n - x^*\|\}$ is convergent. Thus, $\{x_n\}$is bounded and $\sum_{n=1}^{\infty}[\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2] < \infty$. Furthermore, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup p \in H$.

Step (ii) : We now show that any weak accumulation point p of the sequence $\{x_n\}$ belongs to C_n for all n. Suppose that $\{x_{n_j}\} \subset \{x_n\}, x_{n_j} \rightharpoonup p$ as $j \to \infty$, and there exists n_0 such that $p \notin C_{n_0}$. Then by the closedness and convexity of C_{k_0}, C_{k_0} is also weakly closed. Hence, there exists $n_{j_0} > n_0$ such that $\{x_{n_j}\} \notin C_{n_0}$ for all $n_j \geq n_{j_0}$, especially $x_{n_{j_0}} \notin C_{n_0}$. This contradicts the fact that $x_{n_{j_0}} \in C_{n_{j_0}-1} \subset \cdots \subset C_{k_0+1} \subset C_{k_0}$. Therefore $p \in C_n, \forall n$ or $p \in \bigcap_{n=0}^{\infty} C_n$. Since $C_n \subset B_n, \forall n$. then we have

$$p \in \bigcap_{n=0}^{\infty} B_n.$$

Step (iii) : Next, we show that $p \in \bigcap_{i=1}^{N} EP(C, f_i)$. Using Algorithm 3.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \left\|P_{C_{n+1}}(w_n) - x^*\right\|^2 \\ &= \left\|\left(P_{C_{n+1}}(w_n) - w_n\right) + (w_n - x^*)\right\|^2 \\ &= \left\|P_{C_{n+1}}(w_n) - w_n\right\|^2 + \|w_n - x^*\|^2 \\ &\quad + 2\langle P_{C_{n+1}}(w_n) - w_n, w_n - x^*\rangle \\ &= \left\|P_{C_{n+1}}(w_n) - w_n\right\|^2 + \|w_n - x^*\|^2 \\ &\quad + 2\langle P_{C_{n+1}}(w_n) - w_n, w_n - P_{C_{n+1}}(w_n)\rangle \\ &\quad + 2\langle P_{C_{n+1}}(w_n) - w_n, P_{C_{n+1}}(w_n) - x^*\rangle. \end{aligned} \tag{3.9}$$

Hence

$$\begin{aligned} \left\|P_{C_{n+1}}(w_n) - x^*\right\|^2 &= \|w_n - x^*\|^2 - \left\|P_{C_{n+1}}(w_n) - w_n\right\|^2 \\ &\quad + 2\langle P_{C_{n+1}}(w_n) - w_n, P_{C_{n+1}}(w_n) - x^*\rangle. \end{aligned} \tag{3.10}$$

From (3.7) and (3.10), we have

$$\begin{aligned} \left\|P_{C_{n+1}}(w_n) - w_n\right\|^2 &= \|x_n - x^*\|^2 + \theta_n \left(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\right) \\ &\quad - \|x_{n+1} - x^*\|^2 + 2\theta_n \|x_n - x_{n-1}\|^2 \\ &\quad + 2\langle P_{C_{n+1}}(w_n) - w_n, P_{C_{n+1}}(w_n) - x^*\rangle. \\ &\leq \left(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\right) \\ &\quad + \theta_n \left(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\right)_+ + 2\theta_n \|x_n - x_{n-1}\|^2 \\ &\quad + 2\langle P_{C_{n+1}}(w_n) - w_n, P_{C_{n+1}}(w_n) - x^*\rangle. \end{aligned} \tag{3.11}$$

Clearly, by (2.2) we get

$$\langle P_{C_{n+1}}(w_n) - w_n, P_{C_{n+1}}(w_n) - x^*\rangle \leq 0 \text{ and } \lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\|^2 = 0$$

from Algorithm 3.1. Thus, from (3.11), we conclude that

$$\lim_{n\to\infty} \left\|P_{C_{n+1}}(w_n) - w_n\right\| = 0.$$

From $w_n = x_n + \theta_n (x_n - x_{n-1})$, we get

$$\|w_n - x_n\|^2 \leq \theta_n \|x_n - x_{n-1}\|^2 \to 0$$

and hence

$$\|w_n - x_n\| \to 0, n \to \infty. \tag{3.12}$$

Furthermore,

$$\|x_{n+1} - w_n\| = \left\|P_{C_{n+1}}(w_n) - w_n\right\| \to 0, n \to \infty. \tag{3.13}$$

From (3.12) and (3.13), we get

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - w_n\| + \|x_n - w_n\| \to 0, n \to \infty. \tag{3.14}$$

Since $x_{n_j} \rightharpoonup p$, then it follows that $w_{n_j} \rightharpoonup p$. Observe that h_n is Lipschitz continuous with modulus $M > 0$. Combining Lemma 2.1, 2.2 and 3.3, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 = \left\|P_{C_{n+1}}(w_n) - x^*\right\| &\leq \|w_n - x^*\|^2 - \left\|P_{C_{n+1}}(w_n) - w_n\right\|^2 \\
&= \|w_n - x^*\|^2 - dist^2(w_n, C_{n+1}) \\
&\leq \|w_n - x^*\|^2 - \left(\frac{1}{M} h_n^i(w_n)\right)^2 \\
&\leq \|w_n - x^*\|^2 - \left(\frac{1}{2M} \rho\eta_n \left\|w_n - y_n^i\right\|^2\right)^2. \qquad (3.15)
\end{aligned}$$

Then from (3.15), we obtain that

$$\begin{aligned}
(\frac{1}{2M}\rho\eta_n\|w_n - y_n^i\|^2)^2 &\leq \|w_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&= (\|w_n - x^*\| - \|x_{n+1} - x^*\|)(\|w_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\leq (\|w_n - x^*\| - \|x_{n+1} - x^*\|)M_1 \\
&\leq \|w_n - x_{n+1}\|M_1 \to 0, n \to \infty, \qquad (3.16)
\end{aligned}$$

where $M_1 := \sup_{n\geq 1}\{\|w_n - x^*\| + \|x_{n+1} - x^*\|\}$. Thus $\eta_n \left\|w_n - y_n^i\right\|^2 \to 0, n \to \infty$.

Case I : Suppose that $\liminf_{n\to\infty} \eta_n > 0$. Then

$$0 \leq \left\|w_n - y_n^i\right\|^2 = \frac{\eta_n \left\|w_n - y_n^i\right\|^2}{\eta_n},$$

which implies that

$$\begin{aligned}
\limsup_{n\to\infty} \left\|w_n - y_n^i\right\|^2 &\leq \limsup_{n\to\infty} \left(\eta_n \left\|w_n - y_n^i\right\|^2\right) \left(\limsup_{n\to\infty} \frac{1}{\eta_n}\right) \\
&= \limsup_{n\to\infty} \left(\eta_n \left\|w_n - y_n^i\right\|^2\right) \left(\frac{1}{\liminf_{n\to\infty} \eta_n}\right) = 0, \qquad (3.17)
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \left\|w_n - y_n^i\right\| = 0. \tag{3.18}$$

Moreover,

$$\left\|x_n - y_n^i\right\| \leq \|w_n - x_n\| + \left\|w_n - y_n^i\right\| \to 0, n \to \infty. \tag{3.19}$$

Since $x_{n_j} \rightharpoonup p$ and (3.19), it follows that $y_{n_j}^i \rightharpoonup p$ as $j \to \infty$. By the definition of $y_{n_j}^i$ such that

$$y_{n_j}^i = \text{argmin}\left\{f_i\left(w_{n_j}, y\right) + \tfrac{\rho}{2}\left\|y - w_{n_j}\right\|^2 : y \in C\right\},$$

we have

$$0 \in \partial_2 f_i\left(w_{n_j}, y^i_{n_j}\right) + \rho\left(y^i_{n_j}, w_{n_j}\right) + Nc(y^i_{n_j}),$$

so there exists $v^i_{n_j} \in \partial_2 f\left(w_{n_j}, y^i_{n_j}\right)$ such that

$$\left\langle v^i_{n_j}, y - y^i_{n_j}\right\rangle + \rho\left\langle y^i_{n_j} - w_{n_j}, y - y^i_{n_j}\right\rangle \geq 0, \forall y \in C.$$

Combining with

$$f_i\left(w_{n_j}, y\right) - f_i\left(w_{n_j}, y^i_{n_j}\right) \geq \left\langle v^i_{n_j}, y - y^i_{n_j}\right\rangle, \forall y \in C,$$

we have

$$f_i\left(w_{n_j}, y\right) - f_i\left(w_{n_j}, y^i_{n_j}\right) + \left\langle y^i_{n_j} - w_{n_j}, y - y^i_{n_j}\right\rangle \geq 0, \forall y \in C. \tag{3.20}$$

Moreover,

$$\left\langle y^i_{n_j} - w_{n_j}, y - y^i_{n_j}\right\rangle \leq \left\|y^i_{n_j} - w_{nj}\right\| \left\|y - y^i_{n_j}\right\|.$$

Thus, we get

$$f_i\left(w_{n_j}, y\right) - f_i\left(w_{n_j}, y^i_{n_j}\right) + \rho\left\|y^i_{n_j} - w_{nj}\right\| \left\|y - y^i_{n_j}\right\| \geq 0.$$

Letting $j \to \infty$, by the jointly weak continuity of f, (3.12) and (3.17), we obtain in the limit that

$$f_i\left(p, y\right) - f_i\left(p, p\right) \geq 0.$$

Hence

$$f_i\left(p, y\right) \geq 0, \forall y \in C,$$

which gives that $p \in \bigcap_{i=1}^N EP\left(C, f_i\right), \forall i = 1, 2, ..., N$.

Case II : On the other hand, suppose $\lim\limits_{n\to\infty} \eta_n = 0$. Then from the boundedness of $\{y^i_n\}$, it deduces that there exists $\{y^i_{n_k}\} \subset \{y^i_n\}$ such that $\{y^i_{n_k}\} \rightharpoonup \overline{y^i}$ as $k \to \infty$. Replacing y by w_{n_k} in (3.20), we have

$$f_i(w_{n_k}, y^i_{n_k}) + \rho\left\|y_{n_k} - w_{n_k}\right\|^2 \leq 0. \tag{3.21}$$

Moreover, by the Armijo linesearch rule (3.1) , for m_{n_k-1}, there exists $u^i_{n_k}, m_{n_k-1} \in \partial_2 f_i(z_{n_i}, m_{n_k-1}, z_{n_i}, m_{n_k-1})$ such that

$$\left\langle u^i_{n_k,m_{n_k}-1}, w_{n_k} - y_{n_k}\right\rangle < \frac{\rho}{2}\left\|y^i_{n_k} - w_{n_k}\right\|^2. \tag{3.22}$$

By the convexity of $f_i(z^i_{n_k}, m_{n_k-1})$ and (3.22), we have

$$\begin{aligned} f_i\left(z^i_{n_k,m_{n_k}-1}, y_{n_k}\right) &\geq f_i\left(z^i_{n_k,m_{n_k}-1}, z^i_{n_k,m_{n_k}-1}\right) + \left\langle u^i_{n_k,m_{n_k}-1}, y^i_{n_k} - z^i_{n_k,m_{n_k}-1}\right\rangle \\ &= (1 - \eta_{n_k,m_k-1})\left\langle u^i_{n_k,m_k-1}, y^i_{n_k} - w_{n_k}\right\rangle \\ &> -\left(1 - \eta_{n_k,m_{n_k}-1}\right)\frac{\rho}{2}\left\|y^i_{n_k} - w_{n_k}\right\|^2. \end{aligned} \tag{3.23}$$

From (3.21) and (3.23), we obtain

$$f_i\left(z^i_{n_k,m_{n_k}-1}, y^i_{n_k}\right) > -\left(1-\eta_{n_k,m_{n_k}-1}\right)\frac{\rho}{2}\left\|y^i_{n_k}-w_{n_k}\right\|^2$$
$$\geq \frac{1}{2}\left(1-\eta_{n_k,m_{n_k}-1}\right) f_i\left(w_{n_k}, y^i_{n_k}\right). \tag{3.24}$$

By (3.1), $z^i_{n_k,m_{n_k}-1} = \left(1-\eta^{m_{n_k}-1}\right)w_{n_k} + \eta^{m_{n_k}-1}y^i_{n_k}, \eta_{n_k,m_{n_k}-1} \to 0$ and w_{n_k} converges weakly to p , $y^i_{n_k}$ converges weakly to $\bar{y}$, it implies that $z^i_{n_k,m_{n_k}-1} \rightharpoonup p$ as $i \to \infty$. Beside that $\left\{\left\|y^i_{n_k} - w_{n_k}\right\|^2\right\}$ is bounded, without loss of generality, we may assume that $\lim\limits_{i\to\infty}\left\|y^i_{n_k} - w_{n_k}\right\|^2$ exists. Hence, we get in the limit (3.24) that

$$f_i(p,\bar{y}) \geq -\frac{\rho}{2}\lim_{k\to\infty}\left\|y^i_{n_k}-w_{n_k}\right\|^2 \geq \frac{1}{2}f_i(p,\bar{y}). \tag{3.25}$$

Therefore, $f_i(p,\bar{y}) = 0$ and $\lim\limits_{k\to\infty}\left\|y^i_{n_k}-w_{n_k}\right\|^2 = 0$. By the Case I, it is immediate that $p \in \bigcap_{i=1}^N EP(C,f_i), \forall i = 1,2,...,N$.

step(iv) : We finally show that $\{x_n\}$ converges weakly to a point in $EP(c,f_i), \forall i = 1,2,..,N$. Now , let x^* and p be two accumulation points of $\{x_n\}$. Then there exist $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup p$ and $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Using similar arguments as in Step (ii) above, we can show that x^* , $p \in \bigcap_{n=0}^\infty C_n$. Let $\lim\limits_{n\to\infty}\|x_n - x^*\|^2 = \alpha$ and $\lim\limits_{n\to\infty}\|x_n - p\|^2 = \beta$. Then

$$\begin{aligned}
\alpha &= \lim_{n\to\infty}\|x_n - x^*\|^2 = \lim_{k\to\infty}\|x_{n_k} - x^*\|^2 \\
&= \lim_{k\to\infty}\left[\|x_{n_k}-p\|^2 + 2\langle x_{n_k}-p, p-x^*\rangle + \|p-x^*\|^2\right] \\
&= \lim_{k\to\infty}\left[\|x_{n_k}-p\|^2 + \|p-x^*\|^2\right] \\
&= \lim_{n\to\infty}\left[\|x_{n_k}-p\|^2 + \|p-x^*\|^2\right] \\
&= \lim_{n\to\infty}\left[\|x_{n_k}-x^*\|^2 + 2\|p-x^*\|^2\right] \\
&= \alpha + \|p-x^*\|^2.
\end{aligned}$$

Therefore, $\|p - x^*\| = 0$ and so $\{x_n\}$ convregés weakly to p. This completes the proof. ■

4. Numerical Experiments

In this section, we present some numerical examples to the signal recovery. We consider our algorithm defined by projection method. In this case, we set $\text{prox}_{\lambda g}(w_n - \lambda\nabla f(w_n)) = argmin\{f_i(w_n,y) + \lambda\|y-w_n\|^2 : y \in C\}$ when $\lambda \in (0, 2/L)$ and L is the Lipschitz constant of ∇f. Compressed sensing can be modeled as the following underdeterminated linear equation system:

$$t = Ax + \epsilon, \tag{4.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $t \in \mathbb{R}^M$ is the observed or measured data with noisy ϵ, and $A : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear operator. It is known that to solve (4.1) can be seen as solving the LASSO problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|t - Ax\|_2^2 + \lambda\|x\|_1, \tag{4.2}$$

where $\lambda > 0$. So we can apply our method for solving (4.2) in case $f(x) = \frac{1}{2}\|t - Ax\|_2^2$ and $g(x) = \lambda\|x\|_1$. It is noted that $\nabla f(x) = A^T(Ax - t)$.

The goal in this paper is to remove noise without knowing the type of noise. Thus, we focus in the following problem

$$\begin{aligned} \min_{x\in\mathbb{R}^N} \frac{1}{2}\|A_1x - t_1\|_2^2 &\quad + \quad \lambda_1\|x\|_1, \\ \min_{x\in\mathbb{R}^N} \frac{1}{2}\|A_2x - t_2\|_2^2 &\quad + \quad \lambda_2\|x\|_1, \\ &\vdots \\ \min_{x\in\mathbb{R}^N} \frac{1}{2}\|A_Nx - t_N\|_2^2 &\quad + \quad \lambda_N\|x\|_1. \end{aligned} \tag{4.3}$$

where x is original signal, A_i is a bounded linear operator and t_i is observed signal with noisy for all $i = 1, 2, ..., N$. In our experiment, the sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval [-2,2] with n nonzero elements. The matrix $A_1, A_2 \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and invariance one. The observation t_1, t_2 is generated by with Gaussian noise white signal-to-noise ratio SNR. The initial point x_1 is picked randomly. The restoration accuracy is measured by the mean squared error as follows:

$$\text{MSE} = \frac{1}{N}\|x_n - x^*\|_2^2 < 10^{-3},$$

where x^* is an estimated signal of x.

In what follows, let $\eta_n^i = 0.5$ for all $i = 1, 2$ and let the step size $\lambda_1 = \frac{1}{\|A_1\|^2}$ and $\lambda_2 = \frac{1}{\|A_2\|^2}$. Next, we aim to find the common solutions of signal recovery problem (4.3) with $N = 2$ by using the proposed algorithm is also tested (input A_1 and A_2 on the proposed algorithm) shown in as follows:

dimension	Case	input	Iter	CPU
512×256	1.1	A_1	31539	350.7598
	1.2	A_2	27329	273.0109
	1.3	A_1A_2	27236	297.3950
1536×768	2.1	A_1	24498	1.3632e+03
	2.2	A_2	28809	1.9419e+03
	2.3	A_1A_2	25296	1.2823e+03

TABLE 1. The convergence behavior of inputting A_i, $i = 1, 2$, stop condition(mean squared error) $< 10^{-3}$.

From Table 1: we observe that the number of iterations and CPU time of our proposed algorithm small reduction when $N = 512,\ M = 256$.

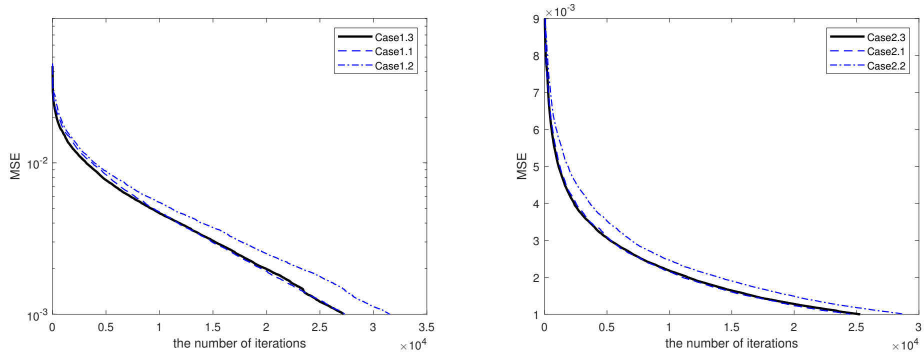

FIGURE 1. Graph of number of iterations versus MSE

Next, we provide performance of the studied proposed algorithm with the following two original signal are tested.

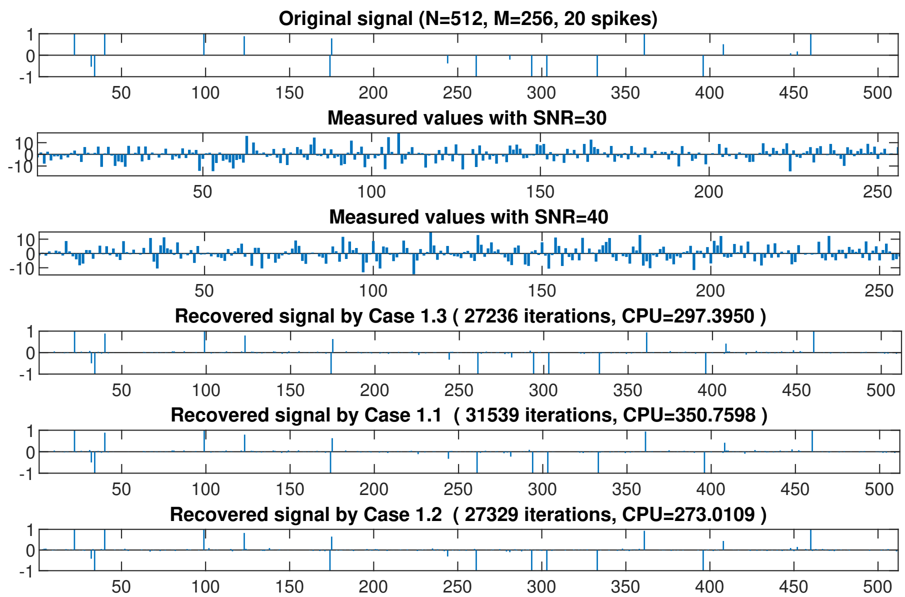

FIGURE 2. From top to bottom: original signal, observation data SNR=30 and SNR=40, recovered signal by Case 1.3, Case 1.1 and Case 1.2, respectively.

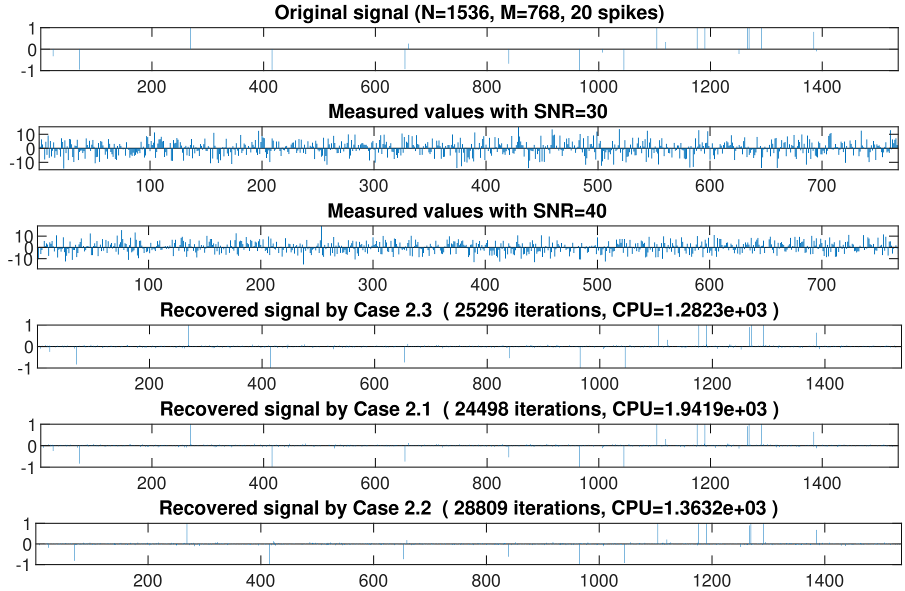

Figure 3. From top to bottom: original signal, observation data SNR=30 and SNR=40, recovered signal by Case 3.3, Case 3.1 and Case 3.2, respectively.

From Table 1 and Figure 1-3, we see that Case 1.3 and Case 2.3 have a fewer number of iterations and CPU time that Case 1.1-1.2 and Case 2.2-2.3 8.07% and 16.87% averages, respectively.

5. Conclusion

In this work, we introduce an iterative method with an inertial extrapolation step and parallel algorithm for solving common equilibrium problems of nonmonotone bifunctions in real Hilbert spaces. We then prove weak convergence theorems under some continuity and convexity assumptions on the bifunction and the condition that the common solution set of the associated Minty equilibrium problems is nonempty. Moreover, we apply our main result to signal processing and demonstrate its computational performance.

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