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Positive Solution of Boundary Value Problem Involving Fractional Pantograph Differential Equation

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Abstract In this paper, we study and integrate the positive solution of fractional pantograph differential equation with mixed conditions of the from:

$$ {}_{RL}D^{q}_{0^+}u(t) + f(t, u(t), u(\lambda t)) = 0,\ t \in (0,1),\ 0 < \lambda < 1, $$

$$ u(0) = 0,\ {}_{RL}D^{p}_{0^+}u(1) = \sum_{i=1}^{n} \alpha_i u(\eta_i),\ 0 < p \leq 1,\ \eta_i \in (0,1), $$

where $1 < q < 2,\ \alpha_i \in \mathbb{R},\ n \in \mathbb{N}$, and ${}_{RL}D^{q}_{0^+},\ {}_{RL}D^{p}_{0^+}$ are the Riemann-Liouville fractional derivative of order $q, p,\ f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By using the fixed point theorems, the main tools for finding positive solutions and uniqueness of this problem are obtained. We give one example of the main results.

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1. Introduction

In recent decades, fractional calculus and fractional differential equations have attracted the interest of many mathematicians and researchers. The fractional differential equations have an important role in numerous fields of study carried out by mathematicians, physicists and engineers. They have used it basically to developed the mathematical modeling, many physical applications and engineering disciplines such as viscoelasticity, control, porous media, phenomena in eletromagnetics etc. (See [1–3]). The major differences between fractional order differential operator and classical calculus is it's nonlocal behavior, that is the feature future state based on the fractional differential operator depends on its current and past states. More details on the fundamental concepts of

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fractional calculus, fractional differential equations and fractional integral equations can be found in books like A. A. Kilbas, H. M Srivastava and J. J. Trujillo [1], K. S Miller and B. Ross [2], and J. Banas and K. Goebel [4]. Fractional integro-differential equations involving the Caputo-Fabrizio derivative have been studied by many researchers from differential points of view. (see, for example, [5–8] and the references therein) have been published.

In the 1960s, the British Railways wanted to make the electric locomotive faster. An important construct was the pantograph, which collects current from an overhead wire. Therefore, J. R. Ockendon and A. B. Tayler studied the motion of the pantograph head on an electric locomotive in [16]. In the solution procedure of this problem, they came across a special delay differential equation of the form

$$x'(t) = ax(t) + bx(\lambda t), t > 0,$$

where a, b are real constants and $0 < \lambda < 1$ for $\in \mathbb{R}$. When the article was published in 1971, this kind of delay differential equation was called *pantograph equation*. The pantograph equation has many applications in electrodynamic and biology; see [9]. Several authors have solved the pantograph differential equations of integer-order such as, Jacobi operational method [10], Chebyshev polynomials [11], Bernoullie polynomials [12], variational iteration method [13], and so on. But there exist few methods applied to numerical solution of pantograph differential equations of fractional-order.

Boundary value problem for fractional differential equation has aroused much attention in the past few years; many professors devoted themselves to the solvability of fractional differential equations, especially to the study of the existence of solutions for boundary value problems of fractional differential equation.

Qualitative theory of differential equations have significant application, and the existence of solutions and of positive solutions of fractional differential equations, which respect the initial and boundary value, have also received considerable attention. In order to study such type of problems different kind of techniques such as fixed point theorems [14–16], fixed point index [16, 17], upper and lower solutions method [18], coincidence theory [19], etc are in vogue. For instance, in [20–23] the authors investigate the existence of positive solutions for boundary value problems.

$$_{RL}D^q_{0+}u(t) + f(t, u(t)) = 0, \ 0 < t < 1,$$

with one of the boundary conditions

$$u(0) = u(1) = 0,$$

$$u(0) + u'(0) = u(1) + u'(1) = 0,$$

where $1 < q \leq 2$, $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous function and $_{RL}D^q_{0+}$ is the Riemann-Liouville fractional derivative of order q.

In [24], the authors solve the problem:

$$_{RL}D^q_{0+}u(t) + \mu f(t, u(t)) = 0, \ 0 < t < 1,$$

$$u(0) = 0, \ u(1) = c,$$

where $1 < q \leq 2$, $\mu \in \mathbb{R}^+$, $c \in \mathbb{R}$, $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous function and $_{RL}D^q_{0+}$ is the Riemann-Liouville fractional derivative of order q.

In 2010 [25], the authors obtained the existence results of positive solutions for the following non-linear fractional boundary value problem:

$$_{RL}D^q_{0+}u(t) + f(t, u(t)) = 0, \ 0 < t < 1,$$

$$u(0) = 0,\ \beta u(\eta) = u(1),$$

where $1 < q \leq 2$, $\beta \in \mathbb{R} \setminus \{0\}$, $0 < \eta < 1$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous function and ${}_{RL}D^{q}_{0+}$ is the Riemann-Liouville fractional derivative of order q.
In 2013 [26], the authors study the existence and uniqueness of positive solutions for the following integral boundary value problem:

$${}_{RL}D^{q}_{0+}u(t) + f(t,u(t)) = 0,\ 0 < t < 1,$$

$$u(0) = 0,\ u(1) = \int_0^1 u(s)ds,$$

where $1 < q \leq 2$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous function and ${}_{RL}D^{q}_{0+}$ is the Riemann-Liouville fractional derivative of order q.
In 2020 [27], the authors study and consider the positive solution of fractional differential equation with nonlocal multi-point conditions of the form:

$$\begin{cases} {}_{RL}D^{q}_{0+}u(t) + g(t)f(t,u(t)) = 0,\ t \in (0,1), \\ u^{(k)}(0) = 0,\ u(1) = \displaystyle\sum_{i=1}^{m} \beta_i\ {}_{RL}D^{p_i}_{0+}u(\eta_i), \end{cases}$$

where $n-1 < q < n,\ n \geq 2,\ n-1 < p_i < n,\ q > p_i\ m, n \in \mathbb{N}, k = 0, 1, \ldots, n-2, 0 < \eta_1 < \eta_2 < \cdots < \kappa, \beta_i \leq 0, \kappa \in (0,1], {}_{RL}D^{q}_{0+}, {}_{RL}D^{p_i}_{0+}$ are the Riemann-Liouville fractional derivative of order $q, p_i, f : [0,1] \times C([0,1], E) \to E, E$ be Banach space and $g : (0,1) \to \mathbb{R}^+$ are continuous functions.

There have already been lots of books and papers involving the positive solutions for boundary value problems of fractional differential equation ; however, only a few papers cover that for positive solution of boundary value problem involving fractional pantograph differential equation. Inspired by the papers in [22–27], the objective of this paper is to derive the existence solution of the fractional pantograph differential equations and mixed conditions:

$$\begin{cases} {}_{RL}D^{q}_{0+}u(t) + f(t,u(t),u(\lambda t)) = 0,\ t \in [0,1],\ 0 < \lambda < 1, \\ u(0) = 0,\ {}_{RL}D^{p}_{0+}u(1) = \displaystyle\sum_{i=1}^{n} \alpha_i u(\eta_i),\ 0 < p \leq 1,\ \eta_i \in [0,1], \end{cases} \tag{1.1}$$

where $1 < q < 2$, $\alpha_i \in \mathbb{R}$, $n \in \mathbb{N}$, and ${}_{RL}D^{q}_{0+}$, ${}_{RL}D^{p}_{0+}$ are the Riemann-Liouville fractional derivative of order q, p, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The current paper is organized as follows: Section 1 contains the introduction; in Section 2, some basic definitions of fractional differential equations are introduced. In Section 3, the main results are devided into two parts; a positive solution for fractional pantograph differential equations is considered in Section 3.1; the study of existence and uniqueness result via Boyd and Wong fixed point theorem and one example of the main results are presented in Section 3.2. Finally, a conclusion is presented in Section 4.

2. Preliminaries

Some basic definitions and properties of fractional differential equations are presented in this section, which are used in this paper.

Definition 2.1. [28] The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$_{RL}I^q_{0+}f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds,$$

where $\Gamma(\cdot)$ denotes the Gamma function defined by

$$\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds.$$

Definition 2.2. [28] The Riemann-Liouville fractional derivative of order $q > 0$ of a function $f : (0, \infty) \to \mathbb{R}$, is defined by

$$\left({}_{RL}D^q_{0+}f\right)(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)ds,$$

where n is the smallest integer greater than or equal to q.

Lemma 2.3. [29] Let $b, q \geq 0$ and $x \in C(0,b) \cap L^1(0,b)$. Then

$$_{RL}I^q_{0+}\,{}_{RL}D^q_{0+}x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}, i = 1, 2, \ldots, n, n$ is the smallest integer greater than or equal to q.

Proposition 2.4. [30] If $0 < r < q$ then for $f \in L^p([a,b], \mathbb{R})$, $(1 \leq p \leq \infty)$, the relation $_{RL}D^r_{a+}({}_{RL}I^q_{a+}f(t)) ={}_{RL} I^{q-r}_{a+}f(t)$ hold almost every on $[a,b]$.

Proposition 2.5. [28] If $q, \rho > 0$, for $\rho > -1 + q$, we have $_{RL}D^q_{0+}t^\rho = \frac{\Gamma(\rho+1)}{\Gamma(1+\rho-q)} t^{\rho-q}$.

Lemma 2.6. [26] Let D be a subset of the cone P of semi-order Banach space E, $T : D \to E$ be non-decreasing. If there exist $x_0,\ y_0 \in D$ such that $x_0 \leq y_0$, $\langle x_0, y_0 \rangle \subset D$ and $x_0,\ y_0$ are the lower and upper solution of equation $x - T(x) = 0$, then the equation $x - T(x) = 0$ has maximum and minimum solution $x^*,\ y^*$ in $\langle x_0, y_0 \rangle$ such that $x^* \leq y^*$, when one of the following condition holds:

(1) P is normal and T is compact continuous;
(2) P is regular and T is continuous;
(3) E is reflexive, P is normal and T is continuous or weak continuous.

Definition 2.7. [31] Let E be a Banach space and let $A : E \to E$ be a mapping. A is said to be a non-linear contraction if there exists a continuous non-decreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\epsilon) < \epsilon$ for all $\epsilon > 0$ with the property:

$$\|Ax - Ay\| \leq \Psi(\|x - y\|),\ \forall x, y \in E.$$

Theorem 2.8. [31] **(Boyd and Wong fixed point theorem)**
Let E be a Banach space and let $A : E \to E$ be a nonlinear contraction. Then A has a unique fixed point in E.

3. Main Results

In this section, first we establish the following Lemma to support our main results:

Lemma 3.1. *Let $1 < q < 2$, Assume $h(t) \in C[0,1]$, then the following equation*

$$\begin{cases} {}_{RL}D^q_{0+}u(t) + h(t) = 0, \ t \in [0,1], \ 0 < \lambda < 1, \\ u(0) = 0, \ {}_{RL}D^p_{0+}u(1) = \sum_{i=1}^{n} \alpha_i u(\eta_i), \ 0 < p \leq 1, \ \eta_i \in [0,1], \end{cases} \tag{3.1}$$

has a unique solution $u(t) = \int_0^1 G(t,s)h(s)ds$, where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(q)\sigma(0)}\Big[t^{q-1}(1-s)^{q-p-1}\sigma(s) - (t-s)^{q-1}\sigma(0)\Big], & 0 \leq s \leq t \leq 1 \\ \frac{1}{\Gamma(q)\sigma(0)}\Big[t^{q-1}(1-s)^{q-p-1}\sigma(s)\Big], & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.2}$$

Proof. We may apply Lemma 2.3 to reduce equation (3.1) to an equivalent integral equation.

$$u(t) = -{}_{RL}I^q_{0+}h(t) + c_1 t^{q-1} + c_2 t^{q-2}.$$

By $u(0) = 0$, we can get $c_2 = 0$

$$u(t) \quad = \quad -{}_{RL}I^q_{0+}h(t) + c_1 t^{q-1},$$

and second condition, we have

$$\begin{aligned} {}_{RL}D^p_{0+}u(1) &= -{}_{RL}I^{q-p}_{0+}h(1) + \frac{c_1\Gamma(q)}{\Gamma(q-p)} \\ \sum_{i=1}^{n} \alpha_i u(\eta_i) &= -\sum_{i=1}^{n} \alpha_i {}_{RL}I^q_{0+}h(\eta_i) + c_1 \sum_{i=1}^{n} \alpha_i \eta_i^{q-1}. \end{aligned}$$

So,

$$c_1 = \frac{\Gamma(q-p)}{\Gamma(q-p)\sum_{i=1}^{n} \alpha_i \eta_i^{q-1} - \Gamma(q)}\Big[-{}_{RL}I^{q-p}_{0+}h(1) + \sum_{i=1}^{n} \alpha_i {}_{RL}I^q_{0+}h(\eta_i)\Big].$$

Hence,

$$u(t) = -{}_{RL}I^q_{0+}h(t) + \frac{t^{q-1}\Gamma(q-p)}{\Gamma(q-p)\sum_{i=1}^{n} \alpha_i \eta_i^{q-1} - \Gamma(q)}\Big[-{}_{RL}I^{q-p}_{0+}h(1) + \sum_{i=1}^{n} \alpha_i {}_{RL}I^q_{0+}h(\eta_i)\Big].$$

From Definition 2.1

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds - \frac{t^{q-1}\Gamma(q-p)}{\Gamma(q-p)\sum_{i=1}^n \alpha_i\eta_i^{q-1}-\Gamma(q)} \times \\
&\Big[\frac{1}{\Gamma(q-p)}\int_0^1 (1-s)^{q-p-1}h(s)ds - \sum_{i=1}^n \frac{\alpha_i}{\Gamma(q)}\int_0^{\eta_i}(\eta_i-s)^{q-1}h(s)\Big] \\
&= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds - \frac{t^{q-1}\Gamma(q-p)}{\Gamma(q-p)\sum_{i=1}^n \alpha_i\eta_i^{q-1}-\Gamma(q)} \times \\
&\Big[\frac{1}{\Gamma(q-p)}\int_0^t (1-s)^{q-p-1}h(s)ds + \frac{1}{\Gamma(q-p)}\int_t^1 (1-s)^{q-p-1}h(s)ds \\
&-\sum_{i=1}^n \frac{\alpha_i}{\Gamma(q)}\int_0^t (\eta_i-s)^{q-1}h(s)ds - \sum_{i=1}^n \frac{\alpha_i}{\Gamma(q)}\int_t^1 (\eta_i-s)^{q-1}h(s)ds\Big] \\
&= \int_0^t \Big[-\frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{t^{q-1}\Gamma(q-p)}{\Gamma(q-p)\sum_{i=1}^n \alpha_i\eta_i^{q-1}-\Gamma(q)}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} \\
&-\sum_{i=1}^n \frac{\alpha_i(\eta_i-s)^{q-1}}{\Gamma(q)}\Big)\Big]h(s)ds - \int_t^1 \Big[\frac{t^{q-1}\Gamma(q-p)}{\Gamma(q-p)\sum_{i=1}^n \alpha_i\eta_i^{q-1}-\Gamma(q)}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} \\
&-\sum_{i=1}^n \frac{\alpha_i(\eta_i-s)^{q-1}}{\Gamma(q)}\Big)h(s)\Big]ds \\
&= \int_0^t \frac{1}{\Gamma(q)\sigma(0)}\Big[-(t-s)^{q-1}(1-s)^{q-p-1}\sigma(0) + t^{q-1}\sigma(s)\Big]h(s)ds \\
&+\int_t^1 \frac{t^{q-1}(1-s)^{q-p-1}\sigma(s)}{\Gamma(q)\sigma(0)}h(s)ds.
\end{aligned}$$

Hence; $u(t) = \int_0^1 G(t,s)h(s)ds$, where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(q)\sigma(0)}\Big[t^{q-1}(1-s)^{q-p-1}\sigma(s) - (t-s)^{q-1}\sigma(0)\Big], & 0 \le s \le t \le 1 \\ \frac{1}{\Gamma(q)\sigma(0)}\Big[t^{q-1}(1-s)^{q-p-1}\sigma(s)\Big], & 0 \le t \le s \le 1, \end{cases}$$

$$\sigma(s) = \frac{1}{\Gamma(q-p)} - \sum_{i=1}^n \frac{\alpha_i(\eta_i-s)^{q-1}}{\Gamma(q)(1-s)^{q-p-1}},\ \sigma(0) = \frac{1}{\Gamma(q-p)} - \sum_{i=1}^n \frac{\alpha_i\eta_i^{q-1}}{\Gamma(q)}. \qquad \blacksquare$$

Lemma 3.2. *Suppose that* $\sigma(0) > 0,\ \alpha > 0,\ q-p-1 < 0$ *that function* $\sigma(s) > 0,\ s \in [0,1]$ *and* σ *is non-decreasing on* $[0,1]$.

Lemma 3.3. *The function* $G(t,s)$, *defined by* (3.2) *admits the following property* $G(t,s) \ge 0$.

Proof. For $0 \le s \le t \le 1$, noticing that $\sigma(0) > 0$. By Lemma 3.2, we get

$$\begin{aligned} G(t,s) &\ge \frac{1}{\Gamma(q)\sigma(0)}\Big[t^{q-1}(1-s)^{q-p-1}\sigma(0) - t^{q-1}(1-\frac{s}{t})^{q-1}\sigma(0)\Big] \\ &> \frac{t^{q-1}}{\Gamma(q)}\Big[(1-s)^{q-1} - (1-\frac{s}{t})^{q-1}\Big] \ge 0. \end{aligned}$$

In view of $0 \le t \le s \le 1$, we get $G(t,s) \ge 0$. Obviously, $G(t,s)$ is continuous on $[0,1]\times[0,1]$. ∎

Let $E = C[0,1]$ be the Banach space endowed with the sup-norm and define the cone $P \subset E$. $P = \{u \in E : u(t) \ge 0,\ 0 \le t \le 1\}$. Define the operator $T : P \to P$ as follows

$$Tu(t) := \int_0^1 G(t,s)f(s,u(s),u(\lambda s))ds,$$

then the equation (1.1) has a solution if and only if the operator T has a fixed point.

3.1. A positive solution for fractional pantograph differential equations

The definition of lower and upper solution of the operator T are given below.

Definition 3.4. [26] Let $v(t),\ w(t) \in E$, we say that $v(t)$ is called a lower solution of operator T if $v(t) \le Tv(t)$, and $w(t)$ is called an upper solution of operator T if $w(t) \ge Tw(t)$.

Theorem 3.5. *Assume that*
$(H1)$ $f : [0,1]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ *is continuous* $f(t,\cdot)$ *is non-decreasing for each* $t \in [0,1]$ *and there exists a positive constant* a *such that* $f(t,\cdot)$ *is strictly increasing on* $[0,a]$ *for each* $t \in [0,1]$*;*
$(H2)$ $0 < \lim\limits_{u\to+\infty} f(t,u(t),u(\lambda t)) < +\infty$ *for each* $t \in [0,1]$.
Then the equation (1.1) *has a positive solution.*

Proof. We will prove the theorem through four steps.
Step 1. $T : P \to P$ is completely continuous. The operator $T : P \to P$ is continuous in view of non-negativeness and continuity of $G(t,s)$ and $f(t,u(t),u(\lambda t))$, $0 < \lambda < 1$. Let $\Omega \subset P$ be bounded, which is to say there exists a positive constant $M > 0$ such that $\|u\| \le M$, $\forall u \in \Omega$. Let $L = \max\limits_{0\le t\le 1,\ 0\le u\le 1} |f(t,u(t),u(\lambda t))| + 1$. Then $\forall u \in \Omega$, we have

$$\begin{aligned} |Tu(t)| &\le \int_0^1 G(t,s)f(s,u(s),u(\lambda s))ds \\ &\le L\int_0^1 G(t,s)ds. \end{aligned}$$

Hence $T(\Omega)$ is bounded. For each $u \in \Omega,\ \forall \tau_1, \tau_2 \in [0,1]$ satisfy $\tau_1 < \tau_2$, we have

$$\begin{aligned}\Big|Tu(\tau_2) - Tu(\tau_1)\Big| &= \Big|\int_0^1 G(\tau_2,s)f(s,u(s),u(\lambda s))ds - \int_0^1 G(\tau_1,s)f(s,u(s),u(\lambda s))ds\Big| \\ &\leq \int_0^{\tau_1} \Big|G(\tau_2,s) - G(\tau_1,s)\Big| f(s,u(s),u(\lambda s))ds \\ &\quad + \int_{\tau_1}^{\tau_2} \Big|G(\tau_2,s) - G(\tau_1,s)\Big| f(s,u(s),u(\lambda s))ds \\ &\quad + \int_{\tau_2}^{1} \Big|G(\tau_2,s) - G(\tau_1,s)\Big| f(s,u(s),u(\lambda s))ds\end{aligned}$$

$$\begin{aligned}\Big|Tu(\tau_2) - Tu(\tau_1)\Big| &\leq \frac{1}{\Gamma(q)\sigma(0)} \int_0^{\tau_1} \Big[(\tau_2^{q-1} - \tau_1^{q-1})(1-s)^{q-p-1}\sigma(s) \\ &\quad -\Big\{(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\Big\}\sigma(0)\Big] f(s,u(s),u(\lambda s))ds \\ &\quad + \frac{1}{\Gamma(q)\sigma(0)} \int_{\tau_1}^{\tau_2} \Big[(\tau_2^{q-1} - \tau_1^{q-1})(1-s)^{q-p-1}\sigma(s) \\ &\quad -\Big\{(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\Big\}\sigma(0)\Big] f(s,u(s),u(\lambda s))ds \\ &\quad + \frac{1}{\Gamma(q)\sigma(0)} \int_{\tau_2}^{1} \Big[(\tau_2^{q-1} - \tau_1^{q-1})(1-s)^{q-p-1}\sigma(s) \\ &\quad -\Big\{(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\Big\}\sigma(0)\Big] f(s,u(s),u(\lambda s))ds.\end{aligned}$$

Since $\sigma(s)$ is non-decreasing on $[0,1]$ then $\sigma(s) < \sigma(1)$.

$$\begin{aligned}\Big|Tu(\tau_2) - Tu(\tau_1)\Big| &\leq \frac{L}{\Gamma(q)\sigma(0)}\Big[(\tau_2^{q-1} - \tau_1^{q-1})\sigma(1)\frac{[1-(1-\tau_1)^{q-1}]}{q-p} \\ &\quad -\sigma(0)\Big(\frac{\tau_2^q - \tau_1^q}{q} - \frac{(\tau_2 - \tau_1)^q}{q}\Big)\Big] \\ &\quad + \frac{L}{\Gamma(q)\sigma(0)}\Big[(\tau_2^{q-1} - \tau_1^{q-1})\sigma(1)\frac{[(1-\tau_1)^{q-p} - (1-\tau_2)^{q-p}]}{q-p} \\ &\quad -\sigma(0)\frac{(\tau_2 - \tau_1)^q}{q}\Big] + \frac{L}{\Gamma(q)\sigma(0)}(\tau_2^{q-1} - \tau_1^{q-1})\sigma(1)\frac{[1-(1-\tau_2)^{q-1}]}{q-p}\end{aligned}$$

As $\tau_2 \to \tau_1$, the right hand side tends to zero. The Arzela-Ascoli Theorem implies that $\overline{T(\Omega)}$ is compact. That is $T : P \to P$ is completely continuous.

Step 2. T is an increasing operator. In fact, by $(H1)$, let $u_1 \leq u_2$, we have

$$\begin{aligned}Tu_1(t) &= \int_0^1 G(t,s)f(s,u_1(s),u_1(s))ds \\ &\leq \int_0^1 G(t,s)f(s,u_2(s),u_2(s))ds \leq Tu_2(t).\end{aligned}$$

So; T is an increasing operator.

Step 3. By $(H2)$, $\exists M_1 > 0,\ N > 0$, such that $u \geq N_1$, it holds $f : [0,1] \times [0,N_1] \times [0,N_2]$, where $N_1 > N_2$, is continuous $\exists M_2 > 0$ such that $u \leq N_1$, it hold $f(t, u(t), u(\lambda t)) \leq M$, $\forall u \geq 0$. Now we consider the following equation

$$\begin{cases} {}_{RL}D^q_{0+} w(t) + M = 0, \ t \in [0,1], \ 1 < q \leq 2, \\ w(0) = 0, \ {}_{RL}D^p_{0+} w(1) = \sum_{i=1}^{n} \alpha_i w(\eta_i), \ 0 < p \leq 1, \ \eta_i \in [0,1]. \end{cases} \tag{3.3}$$

From Theorem 3.1, we have solution of (3.3) is an upper solution of the oprator T. On the other hand, its obvious that $v(t) \equiv 0$ is a lower solution of the operator T, and we have $v(t) \leq w(t)$.
Step 4. Since P is a normal cone. Lemma 2.6 implies that T has a fixed point $u \in \langle 0, w(t) \rangle$. Therefore, the equation (1.1) has a positive solution. ∎

3.2. Existence and Uniqueness Result Via Boyd and Wong Fixed Point Theorem.

Theorem 3.6. *Let $f : [0,1] \times [0, N_1] \times [0, N_2] \to \mathbb{R}$ be a continuous function satisfying the assumption:*

$$|f(t, u(t), u(\lambda t)) - f(t, v(t), v(\lambda t))| \leq \frac{\beta(t)|u - v|}{B + |u - v|}, \ \text{for } t \in [a,b], \ u, v \geq 0,$$

where $\beta(t) : [a,b] \to \mathbb{R}^+$ is continuous and B is the constant defined by

$$B := \int_0^1 G(s,t)\beta(s)ds < 1.$$

Then the problem (1.1) has a unique solution on $[0,1]$.

Proof. Consider a continuous non-decreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\epsilon) = \frac{B\epsilon}{B + \epsilon}$$

for all $\epsilon > 0$, such that $\Psi(0) = 0$ and $\Psi(\epsilon) < \epsilon$ for all $\epsilon > 0$. For any $u, v \in C[0,1]$ and for each $t \in [0,1]$, yields.

$$\begin{aligned} |Tu(t) - Tv(t)| &\leq \Big| \int_0^1 G(t,s) f(s, u(s), u(\lambda s))ds - \int_0^1 G(t,s) f(s, v(s), v(\lambda s))ds \Big| \\ &\leq \int_0^1 G(t,s) \Big| f(s, u(s), u(\lambda s))ds - f(s, v(s), v(\lambda s)) \Big| ds \\ &\leq \int_0^1 G(s,t) \frac{\beta(s)|u - v|}{B + |u - v|} ds \\ &\leq \frac{\Psi(\|u - v\|)}{B} \Big[\int_0^1 G(s,t)\beta(s)ds \Big] \\ &\leq \Psi(\|u - v\|). \end{aligned}$$

This implies that $\|Tu - Tv\| \leq \Psi(\|u - v\|)$. There for A is a non-linear contraction. Hence, by theorem (Boyd and Wong). The operator T has a unique fixed point, which is the unique solution of the problem (1.1).

∎

Example 3.7. Consider the following fractional boundary value problems

$$\begin{cases} {}_{RL}D_{0^+}^{\frac{5}{4}}u(t) + \frac{(\sqrt{t}+3)}{7}\Big(1 - \frac{1}{2|1+u(t)|} - \frac{1}{3|1+u(\frac{2t}{3})|}\Big) = 0, \ t \in (0,1), \\ u(0) = 0, \ {}_{RL}D_{0^+}^{\frac{1}{2}}u(1) = \frac{1}{3}u(\frac{1}{3}) + \frac{1}{4}u(\frac{1}{4}) + \frac{1}{5}u(\frac{1}{5}). \end{cases} \tag{3.4}$$

By comparing problem (3.1) and (3.4), we obtain the following parameters: $q = 5/4, \ p = 1/2, \ \alpha_1 = 1/3, \ \alpha_2 = 1/4, \ \alpha_3 = 1/5, \ \eta_1 = 1/3, \eta_2 = 1/4, \ \eta_3 = 1/5, \ f(t, u(t), u(\lambda t)) = \frac{(\sqrt{t}+3)}{7}\Big(1 - \frac{1}{2|1+u(t)|} - \frac{1}{3|1+u(\frac{2t}{3})|}\Big)$. Note that $-\frac{1}{2|1+u(t)|} \le 0, \ -\frac{1}{3|1+u(\frac{2t}{3})|} \le 0$. We now obtain

$$\begin{aligned} \lim_{u\to+\infty} f(t,u(t),u(\lambda t)) &= \lim_{u\to+\infty} \frac{(\sqrt{t}+3)}{7}\Big(1 - \frac{1}{2|1+u(t)|} - \frac{1}{3|1+u(\frac{2t}{3})|}\Big) \\ &\le \frac{(\sqrt{t}+3)}{7} < +\infty. \end{aligned}$$

Then, the condition $(H1)$ and $(H2)$ hold. If fact, the solution of (3.4) is equivalent to a fixed point of the operator T, here

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s) f(s,u(s),u(\lambda s)) ds \\ &= \int_0^1 G(t,s)\frac{(\sqrt{s}+3)}{7}\Big(1 - \frac{1}{2|1+u(s)|} - \frac{1}{3|1+u(\frac{2s}{3})|}\Big) ds. \end{aligned}$$

Setting $w(t) = \int_0^1 G(t,s)\frac{(\sqrt{s}+3)}{7} ds$, and $v(t) \equiv 0$, then

$$w(t) \ge \int_0^1 G(t,s) f(s,w(s),w(\lambda s)) ds = Tw(t),$$

which implies $w(t)$ is an upper solution of the operator T. It is obvious that $v(t) \equiv 0$ is a lower solution of the operator T. Thus, by Theorem 3.5, we can get that the problem (3.7) has a positive solution. By choosing $\beta(t) = 5/6$ we then have $B = \int_0^1 G(s,t)\beta(s) ds \approx 0.4906$. Consider

$$\begin{aligned} \Big| f(t,u_1(t),u_1(\lambda t)) - f(t,u_2(t),u_2(\lambda t)) \Big| &= \Big| -\frac{1}{2|1+u_1(t)|} + \frac{1}{2|1+u_2(t)|} \\ &\quad -\frac{1}{3|1+u_1(\frac{2t}{3})|} + \frac{1}{3|1+u_2(\frac{2t}{3})|}\Big| \\ &\le \Big| -\frac{5}{6|1+u_1(t)|} + \frac{5}{6|1+u_2(t)|}\Big| \\ &\le \frac{\frac{5}{6}|u_1 - u_2|}{1 + |u_1 - u_2|} \\ &\le \frac{\frac{5}{6}|u_1 - u_2|}{0.4906 + |u_1 - u_2|}. \end{aligned}$$

Hence, by Theorem 3.6, problem (3.4) has a unique solution on $(0,1)$.

4. Conclusion

In conclusion, a positive solution for fractional pantograph differential equations is obtained and existence and uniqueness result via Boyd and Wong fixed point theorem is presented. Also we give example as an application to illustrate the results obtained.

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