Thai Journal of Mathematics
Volume 19 Number 3 (2021)
Pages 1028–1036

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

New best proximity points theorem for $F-$Suzuki proximal contractions

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Abstract In this paper, we establish a new best proximity point theorem for $F-$Suzuki proximal contractions in complete metric spaces. Also we give some illustrative examples of our main results and show that our result contain some recent results.

MSC: 47H09; 47H10

Keywords: Best proximity point; $F-$Suzuki proximal contractions

Submission date: 28.04.2021 / Acceptance date: 18.07.2021

1. Introduction

Several problems can be modeled as equations of the form $Tx = x$ where T is a given self-mapping defined on a subset of a metric space, a normed linear space, topological vector spaces or some suitable spaces. However, if T is a nonself-mapping from A to B, then the aforementioned equation does not necessarily admit a solution. In this case, it is contemplated to find an approximate solution x in A such that the error $d(x, Tx)$ is minimum, where d is the distance function. In view of the fact that $d(x, Tx)$ is at least $d(A, B)$, a best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution x satisfies the condition $d(x, Tx) = d(A, B)$. Such optimal approximate solutions are called best proximity points of the mapping T. Interestingly, best proximity theorems also serve as a natural generalization of fixed point theorems, for a best proximity point serves as an optimal approximate solution to the equation $Tx = x$.

A classical best approximation theorem was introduced by Fan [1], that is, if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including Prolla [3], Reich [4], Sehgal and Singh [5, 6], have derived extensions of Fan's Theorem in many directions. Other

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Published by The Mathematical Association of Thailand.

works of of the existence of a best proximity point for contractions can be seen in [7, 8, 15]. On the other hand, in 2008, Suzuki [16] defined generalized versions of Edelstein's mapping which called Suzuki contraction mapping and proved fixed point results for this mapping in compact metric spaces. Later, in 2012, Wardoski [11] introduced a new contraction called $F-$contraction and acquired a fixed point result which generalize Banach contraction principle in many ways. Afterward, many authors use this mapping for studied the existences of fixed points and best proximity point and applied to several problems concern F-contraction mapping for example see [2, 10, 12, 14, 17].

The aim of this paper is to combine the Suzuki contraction and F-contraction mapping for define $F-$Suzuki proximal contractions and establish a new best proximity point for such the mapping in complete matric spaces. Also we give some illustrative example of our main results.

2. Preliminaries

2.1. Best proximity point

Let A and B be nonempty subsets of a metric space (X, d), we recall the following notations and notions that will be used in what follows.

$$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\},$$
$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are nonempty. Further, it is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B respectively, provided A and B are closed subsets of a normed linear space such that $d(A, B) > 0$ (see [13]).

Definition 2.1. A point $x \in A$ is said to be a best proximity point of the mapping $T : A \to B$ if it satisfies the following condition

$$d(x, Tx) = d(A, B).$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

2.2. F-contraction mapping

In 2012, Wardowski [11], introduced the following concept:

Definition 2.2. We denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:

- (F1) F is strictly increasing, that is $s < t \Rightarrow F(s) < F(t)$ *for all* $s, t \in \mathbb{R}^+$,
- (F2) for every sequence $\{s_n\}_{n=1}^{\infty}$ in $\mathbb{R}^+$ we have $\lim\limits_{n\to\infty} s_n = 0 \; iff \; \lim\limits_{n\to\infty} F(s_n) = -\infty$,
- (F3) there exists a number $k \in (0, 1)$ such that $\lim\limits_{s\to 0^+} s^k F(s) = 0$.

Example 2.3. ([11])The following functions $F_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3, 4$, belong to $\mathcal{F}$:

- (i) $F_1(t) = \ln t$, with $t > 0$,
- (ii) $F_2(t) = \ln t + t$, with $t > 0$,
- (iii) $F_3(t) = \ln(t^2 + t)$, with $t > 0$,

(iv) $F_4(t) = -\frac{1}{\sqrt{t}}$, with $t > 0$.

Definition 2.4 ([11]). Let (X, d) be a metric space. A mapping $f : X \to X$ is called an F-contraction on X if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(fx, fy) > 0$, we have

$$\tau + F(d(fx, fy)) \leq F(d(x, y)). \tag{2.1}$$

Remark 2.5. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfies (F1)-(F3) for any $k \in (0, 1)$. Each mapping $T : X \to X$ satisfying (2.1) is an F-contraction such that

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, \ Tx \neq Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds, i.e. T is Banach contraction.

Remark 2.6. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln(\alpha^2 + \alpha)$. It is clear that F satisfies (F1)-(F3) for any $k \in (0, 1)$. Each mapping $T : X \to X$ satisfying (2.1) is an F-contraction such that

$$\frac{d(Tx, Ty)(d(Tx, Ty) + 1)}{d(x, y)(d(x, y) + 1)} \leq e^{-\tau}, \text{ for all } x, y \in X, \ Tx \neq Ty. \tag{2.2}$$

Remark 2.7. Let $F \in \mathcal{F}$. Then

(a) From (F1) and (2.1) every F-contraction, we get T is a contractive mapping, i.e.

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X, \ Tx \neq Ty.$$

(b) Every F-contraction is a continuous mapping.

Now, by using the concept of Suzuki contraction and $F-$contraction, we extend to $F-$Suzuki proximal contractions as follow:

Definition 2.8. Let A and B be nonempty of metric space (X, d). A mapping $T : A \to B$ is said to be an $F-$Suzuki proximal contraction, if there exist $\tau > 0$ such that for all $x, y \in A$ with $Tx \neq Ty$,

$$\frac{1}{2} d^*(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{2.3}$$

where $F \in \mathcal{F}$ and $d^*(x, Tx) = d(x, Tx) - d(A, B)$.

3. Main result

Theorem 3.1. *Let A and B be nonempty closed subsets of a complete metric space X such that $A \neq \emptyset$. Let $T : A \to B$ be a mapping satisfies the following conditions :*

(1) T is an $F-$Suzuki proximal contraction

(2) $T(A_0) \subseteq B_0$.

Then, T has a best proximity point in A. Moreover, for each x_0 in A_0 there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, $d(x_n, Tx_n) = d(A, B)$ and the sequence $\{x_n\}_{n=1}^{\infty}$ converges to the best proximity point of T. Furthermore, if a pair (A, B) satisfies weak $P-$property, the best proximity point of T is unique.

Proof. Let x_0 a fixed element in A_0. By the fact that $T(A_0) \subseteq B_0$ then there exists an element $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Again, by $T(A_0) \subseteq B_0$ then there exists an element $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. By similar fashion we can find sequence $\{x_n\}_{n=1}^{\infty}$ in A_0 such that for each $n \in \mathbb{N}$,

$$d(x_n, Tx_{n-1}) = d(A, B). \tag{3.1}$$

So, for each $n \in \mathbb{N}$, we can write

$$\begin{aligned} \frac{1}{2}d^*(x_{n-1}, Tx_{n-1}) &= \frac{1}{2}[d(x_{n-1}, Tx_{n-1}) - d(A, B)] \\ &\leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) - d(A, B)] \\ &= \frac{1}{2}d(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n). \end{aligned} \tag{3.2}$$

Then, by (2.3) and (3.2)

$$\tau + F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n)) \tag{3.3}$$

and hence

$$\begin{aligned} F(d(Tx_{n-1}, Tx_n)) &\leq F(d(x_{n-1}, x_n)) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &\vdots \\ &\leq F(d(x_0, x_1)) - n\tau. \end{aligned} \tag{3.4}$$

Then, we obtain that $\lim_{n\to\infty} F(d(x_n, Tx_n)) = -\infty$, by $(F2)$, we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.5}$$

If there exits $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, we observe that

$$d(x_{n_0}, Tx_{n_0}) \ = d(x_{n_0+1}, Tx_{n_0}) \ = d(A, B),$$

that is x_{n_0} is a best proximity point of T. Thus, we suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$, putting $\alpha_n := d(x_n, x_{n+1})$ by (3.5) and $(F3)$, there exist $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0. \tag{3.6}$$

So, for any $n \in \mathbb{N}$, we have

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq \left(\alpha_n^k F(\alpha_0) - \alpha_n^k n\tau\right) - \alpha_n^k F(\alpha_0) = -\alpha_n^k n\tau \leq 0. \tag{3.7}$$

Letting, $n \to \infty$, and using (3.6)and (3.7) we obtain

$$\lim_{n\to\infty} n\alpha_n^k = 0. \tag{3.8}$$

Hence, there exists $n_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_1$. Therefore for any $n \geq n_1$,

$$\alpha_n \leq \frac{1}{n^{1/k}}. \tag{3.9}$$

So, we have the series $\sum_{n=1}^{\infty} \alpha_n$ is convergent. Let $m, n \in \mathbb{N}$ with $m \geq n \geq n_1$, by the triangular inequality, we get

$$d(x_n, x_m) \leq \alpha_n + \alpha_n + \cdots + \alpha_{m-1} \leq \sum_{i=n}^{\infty} \alpha_i.$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A. Since A is closed subset of a complete metric space X, then there exists $x_* \in A$ such that $x_n \to x_*$ as $n \to \infty$.

Now, we will show that

$$\frac{1}{2}d^*(x_n, Tx_n) < d(x_n, x_*) \quad \text{or} \quad \frac{1}{2}d^*(Tx_n, T^2x_n) < d(Tx_n, x_*) \tag{3.10}$$

for all $n \in \mathbb{N}$. On the contrary, suppose that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}d^*(x_m, Tx_m) \geq d(x_m, x_*) \quad \text{and} \quad \frac{1}{2}d^*(Tx_m, T^2x_m) \geq d(Tx_m, x_*). \tag{3.11}$$

Hence, we have

$$\begin{aligned} 2d(x_m, x_*) &\leq d^*(x_m, Tx_m) \\ &= d(x_m, Tx_m) - d(A, B) \\ &\leq d(x_m, x_*) + d(x_*, Tx_m) - d(A, B) \\ &= d(x_m, x_*) + d^*(x_*, Tx_m) \end{aligned} \tag{3.12}$$

its follow that

$$d(x_m, x_*) \leq d^*(x_*, Tx_m) \leq d(x_*, Tx_m). \tag{3.13}$$

Using (3.11) and (3.13), we have

$$d(x_m, x_*) \leq \frac{1}{2}d^*(Tx_m, T^2x_m). \tag{3.14}$$

Since

$$\frac{1}{2}d^*(x_m, Tx_m) < d(x_m, Tx_m) - d(A, B) \leq d(x_m, Tx_m).$$

By T is an the $F-$Suzuki proximal contraction, there exists $\tau > 0$ such that

$$\tau + F(d(Tx_m, T^2x_m)) \leq F(d(x_m, Tx_m)).$$

Since, $\tau > 0$ implies that

$$F(d(Tx_m, T^2x_m)) < F(d(x_m, Tx_m)).$$

By $(F1)$, we get

$$d(Tx_m, T^2x_m) < d(x_m, Tx_m). \tag{3.15}$$

Using (3.11), (3.14) and (3.15),

$$\begin{aligned} d(Tx_m, T^2x_m) &< d(x_m, Tx_m) \\ &\le d(x_m, x_*) + d(x_*, Tx_m) \\ &\le \frac{1}{2}d^*(Tx_m, T^2x_m) + \frac{1}{2}d^*(Tx_m, T^2x_m) \\ &= d^*(Tx_m, T^2x_m) \end{aligned} \tag{3.16}$$

which is a contradiction. Hence, we obtain (3.10) hold, and for every $n \in \mathbb{N}$ with $\tau > 0$, either

$$\tau + F(d(Tx_n, Tx_*)) \le F(d(x_n, x_*)) \tag{3.17}$$

or

$$\tau + F(d(T^2x_n, Tx_*)) \le F(d(Tx_n, x_*)) = F(d(x_{n+1}, x_*)). \tag{3.18}$$

Suppose that (3.17) is true, by $(F2)$, we have

$$\lim_{n\to\infty} F(d(Tx_n, Tx_*)) = -\infty,$$

again by $(F2)$ which implies that $\lim_{n\to\infty} d(Tx_n, Tx_*) = 0$ and hence $Tx_n \to Tx_*$ as $n \to \infty$. Therefore,

$$d(x_*, Tx_*) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B).$$

If (3.18) is true, by $(F2)$, we have

$$\lim_{n\to\infty} F(d(T^2x_n, Tx_*)) = -\infty,$$

again by $(F2)$ which implies that $\lim_{n\to\infty} d(T^2x_n, Tx_*) = 0$ and hence $T^2x_n \to Tx_*$ as $n \to \infty$. Therefore,

$$d(x_*, Tx_*) = \lim_{n\to\infty} d(x_{n+2}, T^2x_n) = \lim_{n\to\infty} d(x_{n+2}, Tx_{n+1}) = d(A, B).$$

From, all case we get that x_* is a best proximity point of T in A. That is

$$d(x_*, Tx_*) = d(A, B). \tag{3.19}$$

Suppose that $x_\star$ is another best proximity point of T and a pair (A, B) satisfies the weak

$P-$property, so that

$$d(x_\star, Tx_\star) = d(A, B). \tag{3.20}$$

From (3.19), (3.20) and a pair (A, B) satisfies the weak $P-$property, we get

$$d(x_*, x_\star) = d(Tx_\star, Tx_*). \tag{3.21}$$

Since,

$$\frac{1}{2}d^*(x_*, Tx_*) < d(x_*, x_\star). \tag{3.22}$$

By (3.21), (3.22) and T is an $F-$Suzuki proximal contraction with $\tau > 0$, we have

$$\begin{aligned} F(d(x_*, x_\star)) &= F(d(Tx_*, Tx_\star)) \\ &< \tau + F(d(Tx_*, Tx_\star)) \\ &\leq F(d(x_*, x_\star)) \end{aligned} \tag{3.23}$$

which is a contradiction and hence the he best proximity point of T is unique. ■

Remark 3.2. Interesting, in Theorem 3.1 we can prove existence of a best proximity point by using only properties of a mapping T. However, for the uniqueness of a best proximity point of a mapping T, we add the weak P-property in the assumption.

Since, $F-$Suzuki proximal contraction $\Rightarrow$ $F-$ contraction for non-self, then we obtain following result:

Corollary 3.3. *Let A and B be nonempty closed subsets of a complete metric space X such that $A \neq \emptyset$. Let $T : A \to B$ be a mapping satisfies the following condition :*

(1) T is an $F-$contraction
(2) $T(A_0) \subseteq B_0$.

Then, T has a best proximity point in A. Moreover, for each x_0 in A_0 there is a sequence $\{x_n\}$ such that for each $n \in \mathbb{N}$, $d(x_n, Tx_n) = d(A, B)$ and the sequence $\{x_n\}$ converges to the best proximity point of T. Furthermore, if a pair (A, B) satisfies weak $P-$property, the best proximity point of T is unique.

Corollary 3.4. *[12] Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be an $F-$contraction non-self mapping such that $T(A_0) \subseteq B_0$. Assume that a pair (A, B) satisfies $P-$property.*

Then, there exists a unique $x_* \in A$ such that $d(x_*, Tx_*) = d(A, B)$.

4. An Example

Now, below we give an example to illustrate Theorem 3.1.

Example 4.1. Recall the complete metric space in[11], $X = \{S_n : n \in \mathbb{N}\}$ with the metric $d(x, y) = |x - y|$ for all $x, y \in X$, where the sequence $\{S_n\}_{n=1}^{\infty}$, defined by

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1+2 \\ S_3 &= 1+2+3 \\ &\vdots \\ S_n &= 1+2+3+\ldots+n. \end{aligned}$$

Let $A = \{S_{2n} : n \in \mathbb{N}\}$ and $B = \{S_{2n-1} : n \in \mathbb{N}\}$. It is easy to see that $d(A, B) = 2$, $A_0 = A$ and $B_0 = B$. Define a mappings $T : A \to B$, by $T(S_{2n}) = S_{2n-1}$ for all $n \geq 1$. We will show that T is an $F-$Suzuki proximal contraction with $F \in \mathcal{F}$ defined in Example 2.3(i), that is $F(\alpha) = \ln \alpha$ and $\tau = 2$. Observe that,

$$\frac{1}{2} d^*(S_{2n}, T(S_{2n})) < d(S_{2n}, S_{2m}) \Leftrightarrow [(n < m) \vee (m < n)].$$

With out of generality, we may assume that $n < m$, and since

$$\begin{aligned} S_{2n-1} &= 1+2+3+\ldots+2n-1, \\ S_{2m-1} &= 1+2+3+\ldots+(2n-1)+2n+\ldots+(2m-1), \\ S_{2n} &= 1+2+3+\ldots+(2n-1)+2n, \\ S_{2m} &= 1+2+3+\ldots+2n+(2n+1)+\ldots+2m. \end{aligned}$$

It follow that,

$$\begin{aligned} d(T(S_{2n}), T(S_{2m})) &= |S_{2n-1} - S_{2m-1}| \\ &= 2n + (2n+1) + \ldots + (2m-1), \end{aligned}$$

$$\begin{aligned} d(S_{2n}, S_{2m}) &= |S_{2n} - S_{2m}| \\ &= (2n+1) + \ldots + (2m-1) + 2m, \end{aligned}$$

and

$$\begin{aligned} d(T(S_{2n}), T(S_{2m})) - d(S_{2n}, S_{2m}) &= |S_{2n-1} - S_{2m-1}| - |S_{2n} - S_{2m}| \\ &= 2n - 2m. \end{aligned}$$

So that,

$$\begin{aligned} \frac{d(T(S_{2n}), T(S_{2m}))}{d(S_{2n}, S_{2m})} e^{d(T(S_{2n}), T(S_{2m})) - d(S_{2n}, S_{2m})} &= \frac{2n + \ldots + (2m-1)}{(2n+1) + \ldots + 2m} e^{|S_{2n-1} - S_{2m-1}| - |S_{2n} - S_{2m}|} \\ &< e^{2(n-m)} \\ &< e^{-2}. \end{aligned}$$

There fore $\tau + F(d(T(S_{2n}), T(S_{2m}))) \leq F(d(S_{2n}, S_{2m}))$ with $\tau = 2$. Hence T is an $F-$Suzuki proximal contraction and there exist $S_2 \in A$ such that

$$d(S_2, TS_2) = d(S_2, S_1) = 2 = d(A, B).$$

Acknowledgements

The authors would like to thank Department of Mathematics, Statistics and Computer Sciences and Faculty of Liberal Arts and Science, Kasetsart University for financial supported.

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