Thai Journal of Mathematics
Volume 19 Number 3 (2021)
Pages 993–1003

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Implicit Fractional Differential Equation with Nonlocal Fractional Integral Conditions

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Abstract In this paper, we study and investigate the following implicit Caputo fractional derivative and nonlocal fractional integral conditions of the from:

$$^{C}D^{q}_{0+}u(t) = f(t, u(t), ^{c}D^{q}_{0+}u(t)),\ t \in [0, T]$$

$$u(0) = \eta,\ u(T) = _{RL} I^{p}_{0+}u(\kappa),\ \kappa \in (0, T)$$

where $1 < q \leq 2,\ 0 < p \leq 1,\ \eta \in \mathbb{R},\ ^{c}D^{q}_{0+}u(t)$ is the Caputo fractional derivative of order q, $_{RL}I^{p}_{0+}$ is the Riemann-Liouville fractional integral of order p and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function by using Krasnoselskii's fixed point theorem and Boyd-Wong non-linear contraction. Also, we study the existence and uniqueness of this problem. An example is established to support our main results.

MSC: 26A33; 34A34; 34B15; 47H01; 54H25

Keywords: Implicit fractional differential equation; Nonlocal fractional integral conditions; fixed point theorem.

Submission date: 27.04.2021 / Acceptance date: 21.07.2021

1. Introduction

The fractional differential equations have an important role in numerous fields of study carried out by mathematicians, physicists and engineers. They have used it basically to developed the mathematical modeling, many physical applications and engineering disciplines such as viscoelasticity, control, porous media, phenomena in eletromagnetics etc. (See [1–3]). The major differences between fractional order differential operator and classical calculus is it's nonlocal behavior, that is the feature future state based on the fractional differential operator depends on its current and past states. More details on the fundamental concepts of fractional calculus, fractional differential equations and fractional integral equations can be found in books like A. A. Kilbas, H. M Srivastava and

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J. J. Trujillo [1], K. S Miller and B. Ross [2], and J. Banas and K. Goebel [4]. Fractional integro-differential equations involving the Caputo-Fabrizio derivative have been studied by many researchers from differential points of view. (see, for example, [5–8] and the references therein) have been published. Qualitative theory of differential equations have significant application, and the existence of solutions and of positive solutions of fractional differential equations, which respect the initial and boundary value, have also received considerable attention. In order to study such type of problems different kind of techniques such as fixed point theorems [9–11], fixed point index [11, 12], upper and lower solutions method [13], coincidence theory [14], etc are in vogue. For instance, in [15, 16], the authors investigate the existence of solutions of initial value problems.

$$
\begin{aligned}
{}^{C}D^{\alpha}_{0+}u(t) &= f(t, u(t), {}^{C}D^{\beta}_{0+}u(t)), \quad t \in (0,1], \\
u^{(k)}(0) &= \eta_k, \quad k = 0, 1, ..., n-1,
\end{aligned}
$$

where $n-1 < \beta < \alpha < n$, $(n \in \mathbb{N})$, are the real number ${}^{C}D^{\alpha}_{0+}, {}^{C}D^{\beta}_{0+}$ are the Caputo fractional derivatives of order α, β and $f \in C([0,1] \times \mathbb{R})$.

In [17], the authors investigated the existence and uniqueness of solutions of the non-local fractional integral condition.

$$
\begin{aligned}
{}_{RL}D^{q}_{0+}x(t) &= f(t, x(t)), \ t \in [0,T], \\
x(0) = 0, \quad x(T) &= \sum_{i=1}^{n} \alpha_i \, {}_{H}I^{p_i}_{0+}x(\eta_i),
\end{aligned}
$$

where $1 < q \leq 2$, ${}_{RL}D^{q}_{0+}$ is the Riemann-Liouville fractional derivative of order q, ${}_{H}I^{p_i}_{0+}$ is Hadamard fractional integral of order $p_i > 0$, $\eta_i \in (0,T)$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, and $\alpha_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$ are real constants such that $\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{p_i}} \neq T^{q-1}$.

In [18], the authors study and investigate the following Caputo fractional derivative and Riemann-Liouville integral boundary value problems:

$$
\begin{cases}
{}^{C}D^{q}_{0+}u(t) = f(t, u(t)), \ t \in [0,T], \\
u^{(k)}(0) = \xi_k, \ u(T) = \sum_{i=1}^{m} \beta_i \, {}_{RL}I^{p_i}_{0+}u(\eta_i),
\end{cases}
$$

where $n-1 < q < n$, $n \geq 2$, $m, n \in \mathbb{N}$, $\xi_k, \beta_i \in \mathbb{R}$, $k = 0, 1, \ldots, n-2$, $i = 1, 2, \ldots, m$ and ${}^{C}D^{q}_{0+}$ is the Caputo fractional derivatives, $f : [0,T] \times C([0,T], E) \to E$, where E be Banach space ${}_{RL}I^{p_i}_{0+}$ is Riemann-Liouville fractional integral of order $p_i > 0$, $\eta_i \in (0,T)$ and $\sum_{i=1}^{m} \beta_i \eta_i^{p_i+n-1} \frac{\Gamma(n)}{\Gamma(n+p_i)} \neq T^{n-1}$.

Inspired by the above papers in [15–18], the objective of this paper is to derive the existence and uniqueness solution of implicit Caputo fractional derivative and nonlocal fractional integral conditions:

$$
\begin{cases}
{}^{C}D^{q}_{0+}u(t) = f(t, u(t), {}^{c}D^{q}_{0+}u(t)), \ t \in [0,T] \\
u(0) = \eta, \ u(T) = {}_{RL}I^{p}_{0+}u(\kappa), \ \kappa \in (0,T)
\end{cases}
\tag{1.1}
$$

where $1 < q \leq 2,\ 0 < p \leq 1,\ \eta \in \mathbb{R},\ {}^{C}D^{q}_{0+}u(t)$ is the Caputo fractional derivative of order q, ${}_{RL}I^{p}_{0+}$ is the Riemann-Liouville fractional integral of order p and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function.

Very recently, some existence results for an implicit fractional differential equation on compact intervals were investigated [19–27].

Our goal in this work is to give some existence and uniqueness results for implicit fractional differential equations by using the fixed point theorems of Krasnoselskii fixed-point theorem and Boyd-Wong non-linear contraction.

The current paper is organized as follows: Section 1 contains the introduction; in Section 2, some basic definitions of fractional differential equations are introduced. In Section 3, based on Krasnoselskii fixed point theorem and Boyd-Wong non-linear contraction together the main result is formulated and proved. Finally, a conclusion is presented in Section 4.

2. Preliminaries

We need the following lemmas that will be used to prove our main results.

Definition 2.1. [28] The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f : (0,\infty) \to \mathbb{R}$ is defined by

$${}_{RL}I^{q}_{0+}f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds,$$

where $\Gamma(\cdot)$ denotes the Gamma function defined by

$$\Gamma(q) = \int_0^\infty e^{-s}s^{q-1}ds.$$

Definition 2.2. [29] The Caputo fractional derivative of order $q > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$\left({}^{C}D^{q}_{0+}f\right)(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}f^{(n)}(s)ds,$$

where n is the smallest integer greater than or equal to q.

Lemma 2.3. [28] Let $n-1 < q < n$. If $f \in C^n([a,b])$, then

$${}_{RL}I^{q}_{0+}({}^{c}D^{q}_{0+}x)(t) = x(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, n is the smallest integer greater than or equal to q.

Proposition 2.4. [28] If $q, \rho > 0$ then

(1) If $f(t) = k \neq 0,\ k$ is a constant, then ${}^{C}D^{q}_{0+}k = 0$ and ${}_{RL}D^{q}_{0+}k = \frac{t^{-q}k}{\Gamma(1-q)}$.
(2) ${}_{RL}D^{q}_{0+}t^{q-1} = 0$.
(3) For $\rho > 1$, we have ${}_{RL}I^{q}_{0+}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(q+\rho+1)}t^{q+\rho}$.
(4) For $\rho > -1+q$, we have ${}_{RL}D^{q}_{0+}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(1+\rho-q)}t^{\rho-q}$.
(5) For $\rho > 0$, we have ${}^{C}D^{q}_{0+}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(1+\rho-q)}t^{\rho-q}$.

Lemma 2.5. [30] For $q, \rho, b > 0$ and $f(t) \in L^1(0,b)$ we have

(1) ${}_{RL}I^{q}_{0+}\,{}_{RL}I^{\rho}_{0+}f(t) = {}_{RL}I^{q+\rho}_{0+}f(t)$.
(2) ${}_{RL}I^{q}_{0+}\,{}_{RL}I^{\rho}_{0+}f(t) = {}_{RL}I^{\rho}_{0+}\,{}_{RL}I^{q}_{0+}f(t)$.

(3) ${}_{RL}I^q_{0+}(f(t)+g(t)) = {}_{RL}I^q_{0+}f(t) + {}_{RL}I^q_{0+}g(t)$.

Lemma 2.6. *Let $1 < q < 2$, Assume $h(t) \in C[0,1]$, then the following equation*

$$\begin{cases} {}^{C}D^q_{0+}u(t) = h(t), t \in [0,T] \\ u(0) = \eta, \ u(T) = {}_{RL}I^p_{0+}u(\kappa), \end{cases} \tag{2.1}$$

where $1 < q \leq 2$, $0 < p \leq 1$, $\eta \in \mathbb{R}$, ${}^{C}D^q_{0+}u(t)$ is the Caputo fractional derivative of order q, ${}_{RL}I^p_{0+}$ is the Riemann-Liouville fractional integral of order p and $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous function, has a unique solution

$$u(t) = {}_{RL}I^q_{0+}h(t) + \eta + \frac{t\Gamma(p+2)}{T\Gamma(p+2)-\kappa^{p+1}}\left\{ {}_{RL}I^{q+p}_{0+}h(\kappa) - {}_{RL}I^q_{0+}h(T) + \eta\left(\frac{\kappa^p}{\Gamma(p+1)} - 1\right)\right\}$$

Proof. We may apply Lemma 2.3 to reduce equation (2.1) to an equivalent integral equation.

$$u(t) = {}_{RL}I^q_{0+}h(t) + c_0 + c_1 t$$

By $u(0) = \eta$, we can get $c_0 = \eta$,

$$u(t) = {}_{RL}I^q_{0+}h(t) + \eta + c_1 t,$$

and second condition, we have

$${}_{RL}I^p_{0+}u(t) = {}_{RL}I^{q+p}_{0+} + \frac{\eta t^p}{\Gamma(p+1)} + c_1 {}_{RL}I^p_{0+}t,$$

and

$${}_{RL}I^p_{0+}u(\kappa) = {}_{RL}I^{q+p}_{0+}h(\kappa) + \frac{\eta\kappa^p}{\Gamma(p+1)} + \frac{c_1\kappa^{p+1}}{\Gamma(p+2)}$$

$$u(T) = {}_{RL}I^q_{0+}h(T) + \eta + C_1 T.$$

So;

$$c_1 = \frac{\Gamma(P+2)}{T\Gamma(p+2)-\kappa^{p+1}}\left\{ {}_{RL}I^{q+p}_{0+}h(\kappa) - {}_{RL}I^q_{0+}h(T) + \eta\left(\frac{\kappa^p}{\Gamma(p+1)} - 1\right)\right\}$$

Hence,

$$u(t) = {}_{RL}I^q_{0+}h(t) + \eta + \frac{t\Gamma(p+2)}{T\Gamma(p+2)-\kappa^{p+1}}\left\{ {}_{RL}I^{q+p}_{0+}h(\kappa) - {}_{RL}I^q_{0+}h(T) + \eta\left(\frac{\kappa^p}{\Gamma(p+1)} - 1\right)\right\}$$

■

Theorem 2.7. [31] **(Contraction Mapping Principle)**
Let E be a Banach space, $D \subset E$ be closed and $A : D \to D$ a contraction mapping (i.e. $\|Ax - Ay\| \leq k\|x-y\|$ for some $k \in (0,1)$) and for all $x, y \in D$. Then A has a unique fixed point.

Theorem 2.8. [31] **(Krasnoselskii's Fixed Point Theorem)**
Let M be closed, bounded, convex and non-empty subset of Banach space E. Let A_1, A_2 be the operators such that

(1) $A_1x + A_2y \in M$ whenever $x, y \in M$,
(2) A_1 is a compact and continuous,
(3) A_2 is a contraction mapping.

Then there exists $z \in M$ such that $z = A_1z + A_2z$.

Definition 2.9. [31] Let E be a Banach space and let $A : E \to E$ be a mapping. A is said to be a non-linear contraction if there exists a continuous non-decreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\epsilon) < \epsilon$ for all $\epsilon > 0$ with the property:

$$\|Ax - Ay\| \leq \Psi(\|x - y\|),\ \forall x, y \in E.$$

Theorem 2.10. [31] **(Boyd and Wong fixed point theorem)**
Let E be a Banach space and let $A : E \to E$ be a nonlinear contraction. Then A has a unique fixed point in E.

3. Main Results

Define the operator $A : [0,T] \times E \times E \to E$

$$\begin{aligned} Au(t) &= {}_{RL}I^q_{0^+} f(t, u(t), Ku(t)) + \eta + \frac{t\Gamma(p+2)}{T\Gamma(p+2) - \kappa^{p+1}} \Bigg\{ {}_{RL}I^{q+p}_{0^+} f(\kappa, u(\kappa), Ku(\kappa)) \\ &\quad - {}_{RL}I^q_{0^+} f(T, u(T), Ku(T)) + \eta \left(\frac{\kappa^p}{\Gamma(p+1)} - 1 \right) \Bigg\} \end{aligned}$$

then the equation (4.1) has a solution if and only if the operator A has a fixed point.

3.1. Existance Result via Krasnoselskii's fixed point theorem

Theorem 3.1. *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C[0,T]$ for any $u \in C[0,T]$ and there exist positive constants $M, N > 0, (0 < N < 1)$ such that*

$$|f(t, u, v) - f(t, u^*, v^*)| \leq M|u - u^*| + N|v - v^*| \tag{3.1}$$

for $u, v, u^*, v^* \in \mathbb{R}$ and $t \in [0,T]$. Suppose that there exist continuous functions $\varphi_1, \varphi_2, \varphi_3 : [0,T] \to \mathbb{R}^+$ with

$$\begin{aligned} \overline{\varphi_1} &= \sup_{t\in[0,T]} \varphi_1(t) < 1, \\ \overline{\varphi_2} &= \sup_{t\in[0,T]} \varphi_2(t) < 1, \\ \overline{\varphi_3} &= \sup_{t\in[0,T]} \varphi_3(t) < 1, \end{aligned}$$

with

$$|f(t, u, v)| \leq \varphi_1(t) + \varphi_2(t)|u| + \varphi_3(t)|v|,$$

for all $t \in [0,T]$ and $u, v \in \mathbb{R}$. If $\frac{TM\Gamma(p+2))}{(T\Gamma(p+2)-\kappa^{p+1})(1-N)} \left\{ \frac{\kappa^{q+p}}{\Gamma(q+p+1)} + \frac{T^q}{\Gamma(q+1)} \right\} < 1$ then the boundary value problem (4.1) has at least one solution.

Proof. Let

$$\begin{aligned} A_1u(t) &= {}_{RL}I^q_{0^+} f(t, u(t), Ku(t)), \\ A_2u(t) &= \eta + \frac{\Gamma(p+2)}{T\Gamma(p+2) - \kappa^{p+1}} \Bigg\{ {}_{RL}I^{q+p}_{0^+} f(\kappa, u(\kappa), Kv(\kappa)) \\ &\quad - {}_{RL}I^q_{0^+} f(T, u(T), Ku(T)) + \eta \left(\frac{\kappa^p}{\Gamma(p+1)} - 1 \right) \Bigg\} \end{aligned}$$

For any $u, v \in B_r = \{u \in E : \|u\| \leq r\}$, we get

$$\begin{aligned} |(A_1u)(t) + (A_2v)(t)| &\leq \sup_{t\in[0,T]} \Bigg\{ {}_{RL}I^q_{0^+}|f(s,u(s),Ku(s))|(t) + |\eta| \\ &+ \frac{t\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{p+1}|}\Bigg\{ {}_{RL}I^{q+p}_{0^+}|f(s,v(s),Kv(s))|(\kappa) \\ &+ {}_{RL}I^q_{0^+}|f(s,v(s),Kv(s))|(T) + \left|\eta\left(\frac{\kappa^p}{\Gamma(p+1)}-1\right)\right|\Bigg\}\Bigg\} \end{aligned}$$

$$\begin{aligned} |(A_1u)(t) + (A_2v)(t)| &\leq \frac{t^q}{\Gamma(q+1)}(\overline{\varphi_1}+\overline{\varphi_2}r) + \eta \\ &+ \frac{t\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{p+1}|}\Bigg\{\frac{t^{q+p}}{\Gamma(q+p+1)}(\overline{\varphi_1}+\overline{\varphi_2}r) \\ &+ \frac{t^q}{\Gamma(q+1)}(\overline{\varphi_1}+\overline{\varphi_2}r) + \eta\left(\frac{\kappa^p}{\Gamma(p+1)}-1\right) \\ &\leq \frac{T^q}{\Gamma(q+1)}(\overline{\varphi_1}+\overline{\varphi_2}r) + \eta \\ &+ \frac{T\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{p+1}|}\Bigg\{\frac{T^{q+p}}{\Gamma(q+p+1)}(\overline{\varphi_1}+\overline{\varphi_2}r) \\ &+ \frac{T^q}{\Gamma(q+1)}(\overline{\varphi_1}+\overline{\varphi_2}r) + \eta\left(\frac{\kappa^p}{\Gamma(p+1)}-1\right) \end{aligned}$$

$$|(A_1u)(t) + (A_2v)(t)| \leq r.$$

Setting:

$$\begin{aligned} \alpha &:= \overline{\varphi_1}\left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q+1}\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{p+1}|\Gamma(q+1)} + \frac{T^{p+q+1}\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{n+1}|\Gamma(q+p+1)}\right), \\ \beta &:= \eta\left(1 + \frac{T\Gamma(q+2)}{|T\Gamma(p+2)-\kappa^{p+1}|}\left(\frac{\kappa^p}{\Gamma(q+1)}-1\right)\right), \\ \gamma &:= 1 - \overline{\varphi_2}\left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q+1}\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{p+1}|\Gamma(q+1)} + \frac{T^{p+q+1}\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{n+1}|\Gamma(q+p+1)}\right), \end{aligned}$$

we obtain that

$$r \geq \frac{\alpha+\beta}{\gamma}.$$

This implies that $A_1u + A_2v \in B_r$. In order to prove that A_1 is a compact and continuous. The operator A_1 is continuous by the continuity of f. Since for $u \in E$, we have

$$\|A_1u\| \leq \frac{(\overline{\varphi_1}+\overline{\varphi_2}r)T^q}{\Gamma(q+1)},$$

then the operator A_1 is uniformly bounded on B_r. We next show that the operator A_1 is compact. We define $\sup_{(t,u,v)\in[0,T]\times B_r\times B_r}|f(t,u,v)| = \overline{\varphi} < \infty$ and for all $0 < \tau_1 < \tau_2 < T$, we get

$$\begin{aligned} |A_1u(\tau_2) - A_1u(\tau_1)| &= \frac{1}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]f(s, u(s), Ku(s))ds\right. \\ &\quad \left. + \int_0^{\tau_2}[(\tau_2 - s)^{q-1}f(s, u(s), Ku(s))ds\right| \\ &\le \frac{\overline{\varphi}(\tau_2^q - \tau_1^q)}{\Gamma(q+1)}. \end{aligned}$$

A consequence of these inequalities is that $\{A_1u : u \in B_r\}$ is a uniformly bounded and equicontinuous set in E. Thus by the Arzela-Ascoli Theorem, the operator A_1 is compact on B_r. Next step we show that A_2 is a contraction we take $u, v \in E$, and get

$$\begin{aligned} &|(A_2u)(t) - (A_2v)(t)| \\ \le\ & \frac{t\Gamma(p+2)}{T\Gamma(p+2) - \kappa^{p+1}}\left\{ {}_{RL}I_{0^+}^{q+p}|f(s, u(s), Ku(s) - f(s, v(s), Kv(s))|(\kappa)\right. \\ &+ {}_{RL}I_{0^+}^{q}|f(s, u(s), Ku(s) - f(s, v(s), Kv(s))|(T) \\ \le\ & \frac{t\Gamma(p+2)}{T\Gamma(p+2) - \kappa^{p+1}}\left\{ {}_{RL}I_{0^+}^{q+p}(M|u - v| + N|Ku - Kv|)(\kappa)\right. \\ &\left. + {}_{RL}I_{0^+}^{q}(M|u - v| + N|Ku - Kv|)(T)\right\} \\ \le\ & \frac{t\Gamma(p+2)}{T\Gamma(p+2) - \kappa^{p+1}}\left\{ {}_{RL}I_{0^+}^{q+p}(M|u - v| + \frac{NM|u - v|}{1 - N})(\kappa)\right. \\ &\left. + {}_{RL}I_{0^+}^{q}\left(M|u - v| + \frac{NM|u - v|}{1 - N}\right)(T)\right\} \\ \le\ & \frac{T\Gamma(p+2)}{T\Gamma(p+2) - \kappa^{p+1}}\left\{\frac{M\|u - v\|}{1 - N}\cdot\frac{\kappa^{p+q}}{\Gamma(q+p+1)} + \frac{M\|u - v\|}{1 - N}\cdot\frac{T^q}{\Gamma(q+1)}\right\} \\ \le\ & \frac{TM\Gamma(p+2)}{T\Gamma(p+2) - \kappa^{p+1}(1 - N)}\left\{\frac{\kappa^{q+p}}{\Gamma(q+p+1)} + \frac{T^q}{\Gamma(q+1)}\right\}\|u - v\|. \end{aligned}$$

This implies that $\|A_2u - A_2v\| \le k\|u - v\|$ where

$$k := \frac{TM\Gamma(p+2)}{T\Gamma(p+2) - \kappa^{p+1}(1 - N)}\left\{\frac{\kappa^{q+p}}{\Gamma(q+p+1)} + \frac{T^q}{\Gamma(q+1)}\right\} < 1.$$

Hence, A_2 is a contraction. A combination of this property of the operator A_1 with the inclusion property $A_1(B_r) + A_2(B_r) \subset B_r$ implies, by Krasnoselskii's theorem, that the problem (4.1) has at least one solution on $[0, T]$. ∎

3.2. Existence and Uniqueness Results via Boyd and Wong Fixed Point Theorem

Theorem 3.2. *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

$$|f(t,u,Ku) - f(t,v,Kv)| \leq \frac{\beta(t)|u-v|}{B+|u-v|}, \text{ for } t \in [0,T], u,v > 0$$

where $\beta(t) : [0,T] \to \mathbb{R}^+$ is continuous and B is the constant definded by

$$B :=_{RL} I^q_{0+}\beta(t) + \frac{T\Gamma(p+2)}{|T\Gamma(p+2) - \kappa^{p+1}|}\left\{ {}_{RL}I^{q+p}_{0+}\beta(\kappa) +_{RL} I^q_{0+}\beta(T)\right\} \neq 0.$$

Then the problem (4.1) has a unique solution on $[0,T]$.

Proof. Consider a non-decreasing function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ definded by $\sigma(\varepsilon) = \frac{B\varepsilon}{B+\varepsilon}$ for all $\varepsilon > 0$ such that $\sigma(0) = 0$, and $\sigma(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. For all $u,v \in C([0,T])$ and for each $t \in [0,T]$ yields

$$\begin{aligned}
& |Au(t) - Av(t)| \\
\leq\ & {}_{RL}I^q_{0+}|f(s,u(s),Ku(s)) - f(s,v(s),Kv(s))|(t) \\
& + \frac{t\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{p+1}|}\Bigg\{ {}_{RL}I^{q+p}_{0+}|f(s,u(s),Ku(s)) - f(s,v(s),Kv(s))|(\kappa) \\
& +_{RL}I^q_{0+}|f(s,u(s),Ku(s)) - f(s,v(s),Kv(s))|(T)\Bigg\} \\
\leq\ & {}_{RL}I^q_{0+}\frac{\beta(t)|u-v|}{B+|u-v|} + \frac{T\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{p+1}|}\Bigg\{ {}_{RL}I^{q+p}_{0+}\frac{\beta(\kappa)|u-v|}{B+|u-v|} \\
& +_{RL}I^{q+p}_{0+}\frac{\beta(T)|u-v|}{B+|u-v|}\Bigg\} \\
\leq\ & \frac{\sigma(\|u-v\|)}{B}\left[{}_{RL}I^q_{0+}\beta(t) + \frac{t\Gamma(p+2)}{|T\Gamma(p+2)-\kappa^{p+1}|}\left\{ {}_{RL}I^{q+p}_{0+}\beta(\kappa) + {}_{RL}I^q_{0+}\beta(T)\right\}\right] \\
\leq\ & \sigma(\|u-v\|).
\end{aligned}$$

This implies that $\|Au - Av\| \leq \sigma(\|u-v\|)$. Therefore A is a non-linear contraction. Hence, by theorem (Boyd and Wong). The operator A has a unique fixed point, which is the unique solution of the problem (4.1). ∎

Example 3.3. Consider the following fractional boundary value problems

$$\begin{cases} {}^C D^{\frac{7}{5}}_{0+}u(t) = \frac{\sin^2 t}{999+9^t}\left(\frac{|u(t)|}{1+|u(t)|}\right) + \frac{\cos^2 t}{999+e^t}\left(\frac{|{}^C D^{\frac{7}{5}}_{0+}u(t)|}{1+|{}^C D^{\frac{7}{5}}_{0+}u(t)|}\right) + \frac{1}{2},\ t \in [0,\pi], \\ u(0) = \pi,\ u(\pi) =_{RL} I^{\frac{1}{3}}_{0+}u(\frac{\pi}{2}). \end{cases} \tag{3.2}$$

By comparing problem (4.1) and (3.2), we obtain the following parameters: $q = 7/5$, $p = 1/3$, $\eta = \pi$, $\kappa = \pi/2$, $f(t,u(t),{}^C D^{\frac{7}{5}}_{0+}u(t)) = \frac{\sin^2 t}{999+9^t}\left(\frac{|u(t)|}{1+|u(t)|}\right) + \frac{\cos^2 t}{999+e^t}\left(\frac{|{}^C D^{\frac{7}{5}}_{0+}u(t)|}{1+|{}^C D^{\frac{7}{5}}_{0+}u(t)|}\right) + \frac{1}{2}$.

As, $\Big|f(t,u,v)-f(t,u^*,v^*)\Big| \leq \frac{1}{1{,}000}\Big|u-u^*\Big|+\frac{1}{1{,}000}\Big|v-v^*\Big|$ with $M=N=1/1,000$ and $f(t,u(t),^C D^{\frac{7}{5}}_{0+}u(t)) \leq \frac{1}{2}+\frac{1}{1{,}000}\Big|u(t)\Big|+\frac{1}{1{,}000}\Big|v(t)\Big|$. Therefore the condition of Theorem 3.1 is satisfied with $\frac{TM\Gamma(p+2))}{(T\Gamma(p+2)-\kappa^{p+1})(1-N)}\left\{\frac{\kappa^{q+p}}{\Gamma(q+p+1)}+\frac{T^q}{\Gamma(q+1)}\right\} \approx 0.0049 < 1$. Hence, the problem (3.2) has at least one solution on $[0,\pi]$, if we choose $\beta(t)=0.002$. Then, we find $B \approx 0.0202$, clealy clealy,

$$\begin{aligned}
\Big|f(t,u(t),^C D^{\frac{7}{5}}_{0+}u(t))-f(t,v(t),^C D^{\frac{7}{5}}_{0+}v(t))\Big| &\leq \frac{1}{1,000}\Big(\frac{\Big|u-v\Big|}{1+|u-v\Big|} \\
&\quad+\frac{\Big|^C D^{\frac{7}{5}}_{0+}u(t)-{^C D^{\frac{7}{5}}_{0+}}v(t)\Big|}{1+\Big|^C D^{\frac{7}{5}}_{0+}u(t)-{^C D^{\frac{7}{5}}_{0+}}v(t)\Big|}\Big) \\
&\leq \frac{1}{1,000}\Big(\frac{\Big|u-v\Big|}{1+|u-v\Big|}+\frac{\frac{1}{999}\Big|u-v\Big|}{1+\frac{1}{999}|u-v\Big|}\Big) \\
&\leq \frac{1}{1,000}\Big(\frac{\Big|u-v\Big|}{1+|u-v\Big|}+\frac{\Big|u-v\Big|}{999+|u-v\Big|}\Big) \\
&\leq \frac{1}{500}\Big(\frac{|u-v|}{0.0202+|u-v|}\Big).
\end{aligned}$$

Hence, by Theorem 3.2, problem (3.2) has a unique solution on $(0,\pi)$.

4. Conclusion

In conclusion, we extend the existence and uniqueness solution of implicit Caputo fractional derivative and nonlocal fractional integral conditions:

$$\begin{cases} {^C D^q_{0+}}u(t) = f(t,u(t),^c D^q_{0+}u(t)), \ t\in[0,T] \\ u(0)=\eta, \ u(T) =_{RL} I^p_{0+}u(\kappa), \ \kappa\in(0,T) \end{cases}$$

where $1<q\leq 2,\ 0<p\leq 1,\ \eta\in\mathbb{R},\ {^C D^q_{0+}}u(t)$ is the Caputo fractional derivative of order q, ${_{RL}}I^p_{0+}$ is the Riemann-Liouville fractional integral of order p and $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous function. By using Krasnoselskii's fixed point theorem and Boyd-Wong nonlinear contraction, we obtain the existence and uniqueness of this problem. An example is established to support our main results.

Acknowledgements

We would like to thank Valaya Alongkorn Rajabhat University under the Royal Potronage, Pathumthani and Rajamangala University of Technology Rattanakosin for giving us the opportunity to do research. Also, the authors are grateful to the referees for many useful comments and suggestions which have improved the presentation of this paper.

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