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A New Inertial Subgradient Extragradient Method for Solving Quasimonotone Variational Inequalities

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Abstract The main aim of this paper is to study the numerical solution of variational inequalities involving quasimonotone operators in infinite-dimensional real Hilbert spaces. We prove that the iterative sequence generated by the proposed algorithm for the solution of quasimonotone variational inequalities converges weakly to the solution. The main advantage of the proposed iterative scheme is that it employs an inertial scheme and a monotone stepsize rule based on operator knowledge rather than a Lipschitz constant or another line search method. Numerical results show that the proposed algorithm is effective for solving quasimonotone variational inequalities.?

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1. Introduction

Our main concern here is to study the iterative methods to estimate the solution of the *variational inequality problem* (shortly, VIP) involving quasimonotone operators in any real Hilbert space. In order to prove the weak convergence, it is considered that the following conditions have been satisfied:

($\mathcal{G}1$) The solution set of problem (VIP) is denoted by $VI(\mathcal{C}, \mathcal{G})$ is nonempty;
($\mathcal{G}2$) An operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is said to be quasimonotone if

$$\langle \mathcal{G}(y_1), y_2 - y_1 \rangle > 0 \Longrightarrow \langle \mathcal{G}(y_2), y_2 - y_1 \rangle \geq 0, \ \forall y_1, y_2 \in \mathcal{C}; \tag{QM}$$

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($\mathcal{G}3$) An operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is said to be *Lipschitz continuous* with constant $L > 0$ such that

$$\|\mathcal{G}(y_1) - \mathcal{G}(y_2)\| \leq L\|y_1 - y_2\|, \ \forall y_1, y_2 \in \mathcal{C}; \tag{LC}$$

($\mathcal{G}4$) An operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is *sequentially weakly continuous* if $\{\mathcal{G}(x_n)\}$ weakly converges to $\mathcal{G}(x)$ for any sequence $\{x_n\}$ weakly converges to x.

Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{H}$. Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be an operator. The problem (VIP) for $\mathcal{G}$ on $\mathcal{C}$ is defined in the following way [21]:

$$\text{Find } x^* \in \mathcal{C} \text{ such that } \left\langle \mathcal{G}(x^*), y - x^* \right\rangle \geq 0, \ \forall y \in \mathcal{C} \tag{VIP}$$

It is well-known that the problem (VIP) is an important problem in the field of nonlinear analysis. It is an important mathematical model that unifies many key concepts in applied mathematics, such as a nonlinear system of equations, optimization problems, complementarity problems, network equilibrium problems and finance (see [6, 9–13, 16, 22]). As a result, this notion has numerous applications in engineering, mathematical programming, network economics, transportation analysis, game theory, and computer science.

The regularized and projection methods are two important methods for determining a numerical solution to variational inequalities. It is worth noting that the first approach is most commonly used to solve variational inequalities accompanied by the class of monotone operators. In this method, the regularized subproblem is strongly monotone, and its unique solution is found to be more convenient than the initial problem. In this study, we studied the projection methods that are well known for their simpler numerical computation. The first well-proved projection method is the gradient projection method to solve variational inequalities, after which, several other projection methods have been established, including the well-known extragradient method [14] the subgradient extragradient method [3, 4] and others [5, 7, 15, 24, 33–36, 38] and others in [1, 8, 19, 20, 25–32, 37]. The methods mentioned above are used to study numerically variational inequalities involving monotone, strongly monotone, or inverse monotone. Furthermore, a common feature of these methods is that, when constructing approximation solutions and determining their convergence, fixed or variable stepsize is used depending on the Lipschitz constants of the operators. Because these parameters may be undefined or difficult to approximate in some situations, this can limit implementations.

The primary objective of this study is to examine quasimonotone variational inequalities in infinite-dimensional Hilbert spaces. We introduce an inertial-type method that can be used to improve the convergence rate of the iterative sequence in this context. Inertial methods have previously been established as a result of the oscillator equation with damping and conservative force restoration. This second-order dynamical system is called a heavy friction ball, which was originally studied by Polyak in [18]. The main feature of the inertial-type method is that it reuses the previous two iterations for the next iteration. We show that the iterative sequence generated by the subgradient extragradient algorithm for solving quasimonotone variational inequalities weakly converges to a solution.

The paper is organized in the following manner. In Sect. 2, some preliminary results were presented. Sect. 3 provides a new algorithm and its convergence analysis. Finally, Sect. 4 presents some numerical results to point out the practical efficiency of the proposed method.

2. Preliminaries

For all $x, y \in \mathcal{H}$, we have

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

A *metric projection* $P_{\mathcal{C}}(y_1)$ of $y_1 \in \mathcal{H}$ is defined by

$$P_{\mathcal{C}}(y_1) = \arg\min\{\|y_1 - y_2\| : y_2 \in \mathcal{C}\}.$$

Lemma 2.1. [2] *For any $y_1, y_2 \in \mathcal{H}$ and $\ell \in \mathbb{R}$. Then*

(i) $\|\ell y_1 + (1-\ell)y_2\|^2 = \ell\|y_1\|^2 + (1-\ell)\|y_2\|^2 - \ell(1-\ell)\|y_1 - y_2\|^2$;
(ii) $\|y_1 + y_2\|^2 \le \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle$.

Lemma 2.2. [17] *Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{H}$ and $\{x_n\}$ be a sequence in $\mathcal{H}$ such that*

(i) for each $x \in \mathcal{C}$, $\lim_{n\to\infty}\|x_n - x\|$ exists;
(ii) each sequentially weak cluster point of $\{x_n\}$ belongs to $\mathcal{C}$.

Then, $\{x_n\}$ converges weakly to a point in $\mathcal{C}$.

Lemma 2.3. [23] *Let $\{a_n\}$ and $\{t_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + t_n, \ \forall n \in \mathbb{N}.$$

If $\sum t_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

In this section, we present a new inertial method to solve quasimonotone variational inequalities in real Hilbert spaces and prove a weak convergence result for the proposed method. The main algorithm has been presented as follows.

Remark 3.1. It is clear from the expression (3.1) such that

$$\sum_{n=1}^{+\infty} \vartheta_n\|x_n - x_{n-1}\| \le \sum_{n=1}^{+\infty} \hat{\theta}_n\|x_n - x_{n-1}\| < +\infty, \tag{3.3}$$

which implies that

$$\lim_{n\to+\infty} \hat{\theta}_n\|x_n - x_{n-1}\| = 0. \tag{3.4}$$

Lemma 3.2. *A sequence $\{\lambda_n\}$ is generated by (3.1) is monotonically decreasing and convergent to $\lambda > 0$.*

Proof. It is given that $\mathcal{G}$ is Lipschitz-continuous with constant $L > 0$. Let $\mathcal{G}(w_n) \neq \mathcal{G}(y_n)$ such that

$$\begin{aligned}\frac{\mu\|w_n - y_n\|}{\|\mathcal{G}(w_n) - \mathcal{G}(y_n)\|} &\ge \frac{\mu\|w_n - y_n\|}{L\|w_n - y_n\|} \\ &\ge \frac{\mu}{L}.\end{aligned} \tag{3.5}$$

Thus, above expression implies that $\lim_{n\to\infty}\lambda_n = \lambda$. ∎

Algorithm 1 (Inertial Monotonic Explicit Subgradient Extragradient Method)

Step 0: Let $x_0, x_1 \in \mathcal{C}$, $\mu \in (0,1)$, $\theta \in [0,1)$, $\lambda_1 > 0$ and choose a sequence $\{\epsilon_n\} \subset [0,+\infty)$ such that

$$\sum_{n=1}^{+\infty} \epsilon_n < +\infty.$$

Step 1: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1})$$

where θ_n such that

$$0 \le \theta_n \le \hat{\theta}_n \quad \text{and} \quad \hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if} \quad x_n \ne x_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: Compute

$$y_n = P_{\mathcal{C}}(w_n - \lambda_n \mathcal{G}(w_n)).$$

If $w_n = y_n$, then STOP. Otherwise, go to **Step 3**.

Step 3: Firstly construct a half-space

$$\mathcal{H}_n = \{z \in \mathcal{H} : \langle w_n - \lambda_n \mathcal{G}(w_n) - y_n, z - y_n \rangle \le 0\}$$

and compute

$$x_{n+1} = P_{\mathcal{H}_n}(w_n - \lambda_n \mathcal{G}(y_n)).$$

Step 4: Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|w_n - y_n\|}{\|\mathcal{G}(w_n) - \mathcal{G}(y_n)\|}\right\} & \text{if} \quad \mathcal{G}(w_n) \ne \mathcal{G}(y_n), \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.2}$$

Set $n := n+1$ and go back to **Step 1**.

Lemma 3.3. *Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be an operator satisfies the condition ($\mathcal{G}$1)–($\mathcal{G}$4). For each $x^* \in VI(\mathcal{C}, \mathcal{G})$, we have*

$$\|x_{n+1} - x^*\|^2 \le \|w_n - x^*\|^2 - \Big(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\Big)\|w_n - y_n\|^2 - \Big(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\Big)\|x_{n+1} - y_n\|^2.$$

Proof. Consider that

$$\begin{aligned} \big\|x_{n+1} - x^*\big\|^2 &= \big\|P_{\mathcal{H}_n}[w_n - \lambda_n \mathcal{G}(y_n)] - x^*\big\|^2 \\ &= \big\|P_{\mathcal{H}_n}[w_n - \lambda_n \mathcal{G}(y_n)] + [w_n - \lambda_n \mathcal{G}(y_n)] - [w_n - \lambda_n \mathcal{G}(y_n)] - x^*\big\|^2 \\ &= \big\|[w_n - \lambda_n \mathcal{G}(y_n)] - x^*\big\|^2 + \big\|P_{\mathcal{H}_n}[w_n - \lambda_n \mathcal{G}(y_n)] - [w_n - \lambda_n \mathcal{G}(y_n)]\big\|^2 \\ &\quad + 2\big\langle P_{\mathcal{H}_n}[w_n - \lambda_n \mathcal{G}(y_n)] - [w_n - \lambda_n \mathcal{G}(y_n)], [w_n - \lambda_n \mathcal{G}(y_n)] - x^*\big\rangle. \end{aligned} \tag{3.6}$$

It is given that $x^* \in VI(\mathcal{C}, \mathcal{G}) \subset \mathcal{C} \subset \mathcal{H}_n$ such that

$$\begin{aligned} &\big\|P_{\mathcal{H}_n}[w_n - \lambda_n \mathcal{G}(y_n)] - [w_n - \lambda_n \mathcal{G}(y_n)]\big\|^2 \\ &\quad + \big\langle P_{\mathcal{H}_n}[w_n - \lambda_n \mathcal{G}(y_n)] - [w_n - \lambda_n \mathcal{G}(y_n)], [w_n - \lambda_n \mathcal{G}(y_n)] - x^*\big\rangle \\ &= \big\langle [w_n - \lambda_n \mathcal{G}(y_n)] - P_{\mathcal{H}_n}[w_n - \lambda_n \mathcal{G}(y_n)], x^* - P_{\mathcal{H}_n}[w_n - \lambda_n \mathcal{G}(y_n)]\big\rangle \le 0. \end{aligned} \tag{3.7}$$

Thus, above expression implies that

$$\begin{aligned}&\big\langle P_{\mathcal{H}_n}[w_n-\lambda_n\mathcal{G}(y_n)]-[w_n-\lambda_n\mathcal{G}(y_n)],[w_n-\lambda_n\mathcal{G}(y_n)]-x^*\big\rangle\\&\leq-\big\|P_{\mathcal{H}_n}[w_n-\lambda_n\mathcal{G}(y_n)]-[w_n-\lambda_n\mathcal{G}(y_n)]\big\|^2.\end{aligned}\tag{3.8}$$

Combining expressions (3.6) and (3.8), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\leq\big\|w_n-\lambda_n\mathcal{G}(y_n)-x^*\big\|^2-\big\|P_{\mathcal{H}_n}[w_n-\lambda_n\mathcal{G}(y_n)]-[w_n-\lambda_n\mathcal{G}(y_n)]\big\|^2\\&\leq\|w_n-x^*\|^2-\|w_n-x_{n+1}\|^2+2\lambda_n\big\langle\mathcal{G}(y_n),x^*-x_{n+1}\big\rangle.\end{aligned}\tag{3.9}$$

Since x^* is the solution of problem (VIP), we have

$$\langle\mathcal{G}(x^*),y-x^*\rangle\geq 0,\ \forall y\in\mathcal{C}.$$

Due to an operator $\mathcal{G}$ on $\mathcal{C}$, we obtain

$$\langle\mathcal{G}(y),y-x^*\rangle\geq 0,\ \forall y\in\mathcal{C}.$$

By substituting $y=y_n\in\mathcal{C}$, we obtain

$$\langle\mathcal{G}(y_n),y_n-x^*\rangle\geq 0.$$

Thus, we have

$$\big\langle\mathcal{G}(y_n),x^*-x_{n+1}\big\rangle=\big\langle\mathcal{G}(y_n),x^*-y_n\big\rangle+\big\langle\mathcal{G}(y_n),y_n-x_{n+1}\big\rangle\leq\big\langle\mathcal{G}(y_n),y_n-x_{n+1}\big\rangle.\tag{3.10}$$

Combining expressions (3.9) and (3.10), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\leq\|w_n-x^*\|^2-\|w_n-x_{n+1}\|^2+2\lambda_n\big\langle\mathcal{G}(y_n),y_n-x_{n+1}\big\rangle\\&\leq\|w_n-x^*\|^2-\|w_n-y_n+y_n-x_{n+1}\|^2+2\lambda_n\big\langle\mathcal{G}(y_n),y_n-x_{n+1}\big\rangle\\&\leq\|w_n-x^*\|^2-\|w_n-y_n\|^2-\|y_n-x_{n+1}\|^2+2\big\langle w_n-\lambda_n\mathcal{G}(y_n)-y_n,x_{n+1}-y_n\big\rangle.\end{aligned}\tag{3.11}$$

Note that $x_{n+1}=P_{\mathcal{H}_n}[w_n-\lambda_n\mathcal{G}(y_n)]$ and by the definition of λ_{n+1}, we have

$$\begin{aligned}&2\big\langle w_n-\lambda_n\mathcal{G}(y_n)-y_n,x_{n+1}-y_n\big\rangle\\&=2\big\langle w_n-\lambda_n\mathcal{G}(w_n)-y_n,x_{n+1}-y_n\big\rangle+2\lambda_n\big\langle\mathcal{G}(w_n)-\mathcal{G}(y_n),x_{n+1}-y_n\big\rangle\\&\leq\frac{\lambda_n}{\lambda_{n+1}}2\lambda_{n+1}\|\mathcal{G}(w_n)-\mathcal{G}(y_n)\|\|x_{n+1}-y_n\|\\&\leq\frac{\mu\lambda_n}{\lambda_{n+1}}\|w_n-y_n\|^2+\frac{\mu\lambda_n}{\lambda_{n+1}}\|x_{n+1}-y_n\|^2.\end{aligned}\tag{3.12}$$

Combining expressions (3.11) and (3.12), we obtain

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\&\leq\|w_n-x^*\|^2-\|w_n-y_n\|^2-\|y_n-x_{n+1}\|^2+\frac{\lambda_n}{\lambda_{n+1}}\left[\mu\|w_n-y_n\|^2+\mu\|x_{n+1}-y_n\|^2\right]\\&\leq\|w_n-x^*\|^2-\Big(1-\frac{\mu\lambda_n}{\lambda_{n+1}}\Big)\|w_n-y_n\|^2-\Big(1-\frac{\mu\lambda_n}{\lambda_{n+1}}\Big)\|x_{n+1}-y_n\|^2.\end{aligned}\tag{3.13}$$

■

Theorem 3.4. *Let $\mathcal{G}:\mathcal{H}\to\mathcal{H}$ be an operator satisfies the conditions ($\mathcal{G}1$)–($\mathcal{G}4$). Then, the sequence $\{x_n\}$ generated by the Algorithm 1 weakly converges to $x^*\in VI(\mathcal{C},\mathcal{G})$.*

Proof. Since $\lambda_n \to \lambda$ and there exists a fixed number $\epsilon \in (0, 1-\mu)$ such that

$$\lim_{n\to\infty}\left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}\right) = 1-\mu > \epsilon > 0.$$

Then, there exists a finite number $M_1 \in \mathbb{N}$ such that

$$\left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}\right) > \epsilon > 0, \ \forall\, n \geq M_1. \tag{3.14}$$

From expression (3.13), we obtain

$$\|x_{n+1}-x^*\|^2 \leq \|w_n - x^*\|^2, \ \forall\, n \geq M_1. \tag{3.15}$$

The above expression for all $n \geq M_1$, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq \|x_n + \theta_n(x_n - x_{n-1}) - x^*\| \\ &\leq \|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\|.\end{aligned} \tag{3.16}$$

By using Lemma 2.3 with the expressions (3.4) and (3.16) implies that

$$\lim_{n\to\infty}\|x_n - x^*\| = l, \ \text{for some finite } l \geq 0. \tag{3.17}$$

By using the definition of w_n in Algorithm 1, we have

$$\begin{aligned}\|w_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &= \|(1+\theta_n)(x_n - x^*) - \theta_n(x_{n-1}-x^*)\|^2 \\ &= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1}-x^*\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \\ &\leq (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1}-x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.18}$$

The above expression with (3.17) and (3.4) implies that

$$\lim_{n\to\infty}\|w_n - x^*\| = l. \tag{3.19}$$

From Lemma 3.3 and the expression (3.18), we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \\ &\leq (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1}-x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad - \left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 - \left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1}-y_n\|^2,\end{aligned} \tag{3.20}$$

which further implies that

$$\begin{aligned}&\left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 + \left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1}-y_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2 + \theta_n\big(\|x_n - x^*\|^2 - \|x_{n-1}-x^*\|^2\big) + 2\vartheta_n\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.21}$$

By taking the limit as $n \to +\infty$ in expression (3.21), we obtain

$$\lim_{n\to\infty}\|w_n - y_n\| = \lim_{n\to\infty}\|y_n - x_{n+1}\| = 0. \tag{3.22}$$

Thus, expressions (3.19) and (3.22) gives that

$$\lim_{n\to\infty}\|y_n - x^*\| = l. \tag{3.23}$$

This implies that, the sequences $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded. Now, we show that the sequence $\{x_n\}$ converges weakly to $x^* \in VI(\mathcal{C}, \mathcal{G})$. Indeed, since $\{x_n\}$ is bounded, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Since $\|x_n - w_n\| \to 0$, we have $w_{n_k} \rightharpoonup \hat{x}$. Since $\{w_{n_k}\}$ weakly convergent to $\hat{x}$ and due to $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ and sequence $\{y_{n_k}\}$ weakly convergent to $\hat{x}$. Next, we prove that $\hat{x} \in VI(\mathcal{C}, \mathcal{G})$. Indeed, we have

$$y_{n_k} = P_{\mathcal{C}}[w_{n_k} - \lambda_{n_k}\mathcal{G}(w_{n_k})]$$

that is equivalent to

$$\langle w_{n_k} - \lambda_{n_k}\mathcal{G}(w_{n_k}) - y_{n_k}, y - y_{n_k}\rangle \le 0, \ \forall y \in \mathcal{C}. \tag{3.24}$$

The above inequality implies that

$$\langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle \le \lambda_{n_k}\langle \mathcal{G}(w_{n_k}), y - y_{n_k}\rangle, \ \forall y \in \mathcal{C}. \tag{3.25}$$

Thus, we obtain

$$\frac{1}{\lambda_{n_k}}\langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle \mathcal{G}(w_{n_k}), y_{n_k} - w_{n_k}\rangle \le \langle \mathcal{G}(w_{n_k}), y - w_{n_k}\rangle, \ \forall y \in \mathcal{C}. \tag{3.26}$$

Since $\min\left\{\frac{\mu}{L}, \lambda_1\right\} \le \lambda \le \lambda_1$ and $\{w_{n_k}\}$ is a bounded sequence. By the use of $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$ and $k \to \infty$ in expression (3.26), we obtain

$$\liminf_{k\to\infty}\langle \mathcal{G}(w_{n_k}), y - w_{n_k}\rangle \ge 0, \ \forall y \in \mathcal{C}. \tag{3.27}$$

Moreover, we have

$$\begin{aligned} &\langle \mathcal{G}(y_{n_k}), y - y_{n_k}\rangle \\ &= \langle \mathcal{G}(y_{n_k}) - \mathcal{G}(w_{n_k}), y - w_{n_k}\rangle + \langle \mathcal{G}(w_{n_k}), y - w_{n_k}\rangle + \langle \mathcal{G}(y_{n_k}), w_{n_k} - y_{n_k}\rangle. \end{aligned} \tag{3.28}$$

Since $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$ and $\mathcal{G}$ is L-Lipschitz continuity on $\mathcal{H}$ implies that

$$\lim_{k\to\infty}\|\mathcal{G}(w_{n_k}) - \mathcal{G}(y_{n_k})\| = 0, \tag{3.29}$$

which together with expressions (3.28) and (3.29), we obtain

$$\liminf_{k\to\infty}\langle \mathcal{G}(y_{n_k}), y - y_{n_k}\rangle \ge 0, \ \forall y \in \mathcal{C}. \tag{3.30}$$

To prove further, let us take a positive sequence $\{\epsilon_k\}$ that is convergent to zero and decreasing. For each $\{\epsilon_k\}$, we denote by m_k the smallest positive integer such that

$$\langle \mathcal{G}(w_{n_i}), y - w_{n_i}\rangle + \epsilon_k > 0, \ \forall i \ge m_k, \tag{3.31}$$

where the existence of m_k follows from expression (3.30). Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{m_k\}$ is increasing.

Case I: If there exists a subsequence $\{w_{n_{m_{k_j}}}\}$ of $w_{n_{m_k}}$ such that $\mathcal{G}(w_{n_{m_{k_j}}}) = 0$ $(\forall j)$. Let $j \to \infty$, we obtain

$$\langle \mathcal{G}(\hat{x}), y - \hat{x}\rangle = \lim_{j\to\infty}\langle \mathcal{G}(w_{n_{m_{k_j}}}), y - \hat{x}\rangle = 0. \tag{3.32}$$

Thus, $\hat{x} \in \mathcal{C}$ and imply that $\hat{x} \in VI(\mathcal{C}, \mathcal{G})$.

Case II: If there exits $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \ge N_0$, $\mathcal{G}(w_{n_{m_k}}) \ne 0$. Consider that

$$\Upsilon_{n_{m_k}} = \frac{\mathcal{G}(w_{n_{m_k}})}{\|\mathcal{G}(w_{n_{m_k}})\|^2}, \ \forall n_{m_k} \ge N_0. \tag{3.33}$$

Due to the above definition, we obtain

$$\langle \mathcal{G}(w_{n_{m_k}}), \Upsilon_{n_{m_k}} \rangle = 1, \ \forall \, n_{m_k} \geq N_0. \tag{3.34}$$

Moreover, expressions (3.31) and (3.34), for all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{G}(w_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - w_{n_{m_k}} \rangle > 0. \tag{3.35}$$

Since $\mathcal{G}$ is quasimonotone, then

$$\langle \mathcal{G}(y + \epsilon_k \Upsilon_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - w_{n_{m_k}} \rangle > 0. \tag{3.36}$$

For all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{G}(y), y - w_{n_{m_k}} \rangle \geq \langle \mathcal{G}(y) - \mathcal{G}(y + \epsilon_k \Upsilon_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - w_{n_{m_k}} \rangle - \epsilon_k \langle \mathcal{G}(y), \Upsilon_{n_{m_k}} \rangle. \tag{3.37}$$

Due to $\{w_{n_k}\}$ weakly converges to $\hat{x} \in \mathcal{C}$ through $\mathcal{G}$ is sequentially weakly continuous on the set $\mathcal{C}$, we get $\{\mathcal{G}(w_{n_k})\}$ weakly converges to $\mathcal{G}(\hat{x})$. Suppose that $\mathcal{G}(\hat{x}) \neq 0$, we have

$$\|\mathcal{G}(\hat{x})\| \leq \liminf_{k \to \infty} \|\mathcal{G}(w_{n_k})\|. \tag{3.38}$$

Since $\{w_{n_{m_k}}\} \subset \{w_{n_k}\}$ and $\lim_{k \to \infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k \to \infty} \|\epsilon_k \Upsilon_{n_{m_k}}\| = \lim_{k \to \infty} \frac{\epsilon_k}{\|\mathcal{G}(w_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{G}(\hat{x})\|} = 0. \tag{3.39}$$

Next, consider $k \to +\infty$ in expression (3.37), we obtain

$$\langle \mathcal{G}(y), y - \hat{x} \rangle \geq 0, \ \forall \, y \in \mathcal{C}. \tag{3.40}$$

Let $x \in \ \mathcal{C}$ be an arbitrary element and for $0 < \lambda \ \leq 1$, let:

$$\hat{x}_\lambda = \lambda \ x + (1 - \lambda)\hat{x}. \tag{27}$$

Then $\hat{x}_\lambda \in \ \mathcal{C}$ and from expression (3.40) we have:

$$\lambda \ \left\langle \mathcal{G}(\hat{x}_\lambda), x - \hat{x} \right\rangle \geq 0. \tag{28}$$

Hence:

$$\left\langle \mathcal{G}(\hat{x}_\lambda), x - \hat{x} \right\rangle \geq 0. \tag{29}$$

Let $\lambda \ \to 0$. Then $\hat{x}_\lambda \to \hat{x}$ along a line segment. By the continuity of an operator, $\mathcal{G}(\hat{x}_\lambda)$ converges to $\mathcal{G}(\hat{x})$ as $\lambda \ \to 0$. It follows from expression (29) that:

$$\left\langle \mathcal{G}(\hat{x}), x - \hat{x} \right\rangle \geq 0. \tag{30}$$

Thus, we infer that $\hat{x} \in VI(\mathcal{C}, \mathcal{G})$. Therefore, we proved that:

(1) For every $x^* \in VI(\mathcal{C}, \mathcal{G})$, then $\lim_{n \to \infty} \|x_n - x^*\|$ exists;
(2) Every sequential weak cluster point of the sequence $\{x_n\}$ is in $VI(\mathcal{C}, \mathcal{G})$.

By Lemma 2.2, the sequence $\{x_n\}$ converges weakly to $x^* \in VI(\mathcal{C}, \mathcal{G})$.

■

4. Numerical Illustrations

The numerical results for the proposed method are described in this section. All computations are done in MATLAB R2018b and run on an HP i? 5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Let $\mathcal{H} = l_2$ be a real Hilbert space with sequences of real numbers satisfying the following condition

$$\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2 + \cdots < +\infty. \tag{4.1}$$

Assume that $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is defined by

$$G(x) = (5 - \|x\|)x, \ \forall\, x \in \mathcal{H},$$

where $C = \{x \in \mathcal{H} : \|x\| \leq 3\}$. It is easy to see that $\mathcal{G}$ is weakly sequentially continuous on $\mathcal{H}$ and $VI(\mathcal{C}, \mathcal{G}) = \{0\}$. For any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \big\|\mathcal{G}(x) - \mathcal{G}(y)\big\| &= \big\|(5 - \|x\|)x - (5 - \|y\|)y\big\| \\ &= \big\|5(x - y) - \|x\|(x - y) - (\|x\| - \|y\|)y\big\| \\ &\leq 5\|x - y\| + \|x\|\|x - y\| + \big|\|x\| - \|y\|\big|\|y\| \\ &\leq 5\|x - y\| + 3\|x - y\| + 3\|x - y\| \\ &\leq 11\|x - y\|. \end{aligned} \tag{4.2}$$

Hence $\mathcal{G}$ is L-Lipschitz continuous with $L = 11$. For any $x, y \in \mathcal{H}$ let $\big\langle \mathcal{G}(x), y - x \big\rangle > 0$ such that

$$(5 - \|x\|)\big\langle x, y - x \big\rangle > 0.$$

Since $\|x\| \leq 3$ implies that

$$\big\langle x, y - x \big\rangle > 0.$$

Thus, we have

$$\begin{aligned} \big\langle \mathcal{G}(y), y - x \big\rangle &= (5 - \|y\|)\big\langle y, y - x \big\rangle \\ &\geq (5 - \|y\|)\big\langle y, y - x \big\rangle - (5 - \|y\|)\big\langle x, y - x \big\rangle \\ &\geq 2\|x - y\|^2 \geq 0. \end{aligned} \tag{4.3}$$

Thus, we shown that $\mathcal{G}$ is quasimonotone on $\mathcal{C}$. A projection on the set C is computed explicitly as follows:

$$P_C(x) = \begin{cases} x \quad \text{if} \quad \|x\| \leq 3, \\ \frac{3x}{\|x\|}, \quad \text{otherwise}. \end{cases}$$

The control conditions have been taken as follows: (Algorithm 1):

$$\lambda_1 = 0.22, \mu = 0.44, \theta = 0.50, \epsilon_n = \frac{1}{(n+1)^2}$$

Table 1. Numerical results values for Example 4.1.

	Number of Iterations	Execution Time in Seconds
$x_0 = x_1$	**Algorithm 1**	**Algorithm 1**
$(1, 1, \cdots, 1_{1000}, 0, 0, \cdots)$	27	2.5648640000000
$(1, 2, \cdots, 1000, 0, 0, \cdots)$	38	3.5784638000000
$(5, 5, \cdots, 5_{10000}, 0, 0, \cdots)$	36	2.9547363930000
$(10, 10, \cdots, 10_{10000}, 0, 0, \cdots)$	44	4.1464846400000
$(100, 100, \cdots, 100_{10000}, 0, 0, \cdots)$	69	6.5639463000000

Conclusion

We formulated an explicit extragradient-type method to find a numerical solution to the quasimonotone variational inequalities problem in real Hilbert spaces. This method is considered to be a variant of the two-step gradient method. The proposed algorithm generates a weakly convergent iterative sequence. A numerical example is provided to evaluate the numerical solution of quasimonotone variational inequalities.

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