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Fixed Point Theorems in C^*-algebra Valued Fuzzy Metric Spaces with Application

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Abstract In this work, we establish a fixed point theorem for C^*-algebra-valued contractions in fuzzy metric spaces in the sense of George and Veeramani [1]. We also have an application in integral equations. Our results improve and generalize the corresponding results in the literature.

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1. Introduction and Preliminaries

The concept of fuzzy set was introduced by Zadeh [12] and fuzzy metric spaces initiated by Kramosil and Michàlet [6]. After that, the concept of fuzzy metric spaces was modified by George and Veeramani [1] as follows:

Definition 1.1. [1] Let X be an arbitrary nonempty set, $\triangle$ a continuous $t-norm$, and M a fuzzy set on $X \times X \times (0, \infty)$. The 3-tuple $(X, M, \triangle)$ is called a fuzzy metric space if satisfying the following conditions, for each $x, y, z \in X$ and $t, s > 0$,

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$ for all $t > 0$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) \triangle M(y, z, s) \leq M(x, z, t + s)$ for all $t, s > 0$,

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(v) $M(x,y,\cdot):(0,\infty)\to[0,1]$ is continuous.
In this case, we also say that $(X,M,\triangle)$ is a fuzzy metric space under $\triangle$. In the sequel, we will only consider fuzzy metric space satisfying :
(vi) $\lim\limits_{t\to\infty} M(x,y,t)=1$ for all $x,y\in X$.

Remark 1.2. Let (X,d) be a metric space. We define $a*b=ab$ for all $a,b\in[0,1]$ and $M_d(x,y,t)=\dfrac{kt^n}{kt^n+md(x,y)}$ for every $(x,y,t)\in X\times X\times[0,\infty)$, where k,m and n are positive real numbers then $(X,M_d,*)$ is a fuzzy metric space. Thus, every metric space induces a fuzzy metric space. The fuzzy metric given by $M_d(x,y,t)=\dfrac{t}{t+d(x,y)}$ for every $(x,y,t)\in X\times X\times[0,\infty)$ is called standard fuzzy metrics.

The concept of continuity is given by the following:

Definition 1.3. [8] Let $(X,M,\triangle)$ be a fuzzy metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to $x\in X$ if

$$\lim_{n\infty} M(x_n,x,t)=1$$

for all $t>0$.
A sequence $\{x_n\}$ in X is said to be a G-Cauchy sequence if

$$\lim_{n\infty} M(x_n,x_{n+p},t)=1$$

for all $t>0$ and $p\in\mathbb{N}$.

A fuzzy metric space is called G-complete if every G-Cauchy sequence converges in X

Lemma 1.4. *[8] Let $(X,M,\triangle)$ be a fuzzy metric space and $\{x_n\}$, $\{y_n\}$ are sequence in X such that $x_n\to x$, $y_n\to y$ then $M(x_n,y_n,t)\to M(x,y,t)$ for every continuity point t of $M(x,y,\cdot)$.*

Now, we recall some basic definitions of C^*-algebra. For more details, we refer to [5, 7, 9]. An algebra $\mathbb{A}$ is said to be a *complex algebra* with a conjugate linear involution mapping $*:\mathbb{A}\to\mathbb{A}$ defined by $a\mapsto a^*$, that is, for all $a,b\in\mathbb{A}$ and $z\in\mathbb{C}$ we have $(za+b)^*=\overline{z}a^*+b^*,(a^*)^*=a$ and $(ab)^*=b^*a^*$, is said to be a $*$-algebra. If $a\in\mathbb{A}$, then a^* is said to be the *adjoint* of a. Moreover, if $\mathbb{A}$ does have a unit, we write it as 1 or $1_{\mathbb{A}}$, then $\mathbb{A}$ is said to be an *unital $*$-algebra.* An unital $*$-algebra $\mathbb{A}$ with this norm, it is completely satisfying $\|a^*\|=\|a\|$ for all $a\in\mathbb{A}$ is said to be a *Banach $*$-algebra.* A Banach $*$-algebra $\mathbb{A}$ is said to be a *C^*-algebra* if it satisfies $\|a^*a\|=\|a\|^2$ for all $a\in\mathbb{A}$. Let $\mathbb{A}^+$ be a set of all positive elements and an element $a\in\mathbb{A}^+$ is said to be a positive element if $a=a^*$ and $r(a)\subset\mathbb{R}^+$, where $r(a)$ is the spectrum of a. If $a\in\mathbb{A}^+$ is positive, we write it as $0_{\mathbb{A}}\le a$. Using positive elements, one can define a partial ordering on $\mathbb{A}$ as follows: $a\le b$ if and only if $b-a\ge 0_{\mathbb{A}}$. For each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root.

Recently, Zhenhua Ma et al. [7] introduced a new concept of C^*-algebra-valued metric spaces.

Definition 1.5. [7] Let X be a nonempty set. Suppose that the mapping $d_A:X\times X\to\mathbb{A}$ satisfies:

(1) $d_A(x,y) > 0_A$,
(2) $d_A(x,y) = 0_A$ if and only if $x = y$,
(3) $d_A(x,y) = d_A(y,x)$,
(4) $d_A(x,y) \leq d_A(x,z) + d_A(z,y)$ for all $x,y,z \in X$.

Then, d is called a *C^*-algebra-valued metric* on X and $(X, \mathbb{A}, d_A)$ is called a *C^*-algebra-valued metric space.*

Definition 1.6. [7] Let $(X, \mathbb{A}, d_A)$ be a C^*-algebra-valued metric space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ with respect $\mathbb{A}$ if $\lim_{n\to\infty} d_A(x_n, x) = 0$.
(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence with respect $\mathbb{A}$ if for each $\epsilon > 0$, there exits $n_0 \in \mathbb{N}$ such that $||d_A(x_n, x_m)|| \leq \epsilon$ for each $n, m \geq n_0$.
(3) A C^*-algebra-valued metric space in which every Cauchy sequence is convergent with respect $\mathbb{A}$ is said to be complete.

Example 1.7. [7] Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$ define

$$d(x,y) = diag(|x-y|, \alpha|x-y|)$$

for any $x, y \in X$ and $\alpha \geq 0$ is a constant. It is easy to verify d is a C^*-algebra-valued metric space and $(X, M_2(\mathbb{R}), d)$ is a complete C^*-algebra-valued metric space by the completeness of $\mathbb{R}$.

The author [7] also defined a C^*-algebra-valued contraction and proved a fixed point theorem which generalizes the Banach contraction principle.

Definition 1.8. [7] Let $(X, \mathbb{A}, d_A)$ be a C^*-algebra-valued metric space. A mapping $T : X \to X$ is called a *C^*-algebra-valued contraction* mapping on X, if there exists $b \in \mathbb{A}$ with $\|b\| < 1$ such that

$$d_A(Tx, Ty) \leq b^* d_A(x,y) b, \quad \forall x, y \in X.$$

In this paper, we introduce a C^*-algebra-valued contraction mapping in fuzzy metric spaces and prove existence theorem of fixed point for such maps. Our results substantially generalize several comparable results in the literature (see [7, 9]).

2. Main Results

In this section, we first define a new notion of C^*-algebra-valued fuzzy metric spaces.

Definition 2.1. Let X be an arbitrary nonempty set, $\triangle$ is a continuous $t-norm$, and a fuzzy set $M_A : X \times X \times (0, \infty) \to [0_A, 1_A]$. The 4-tuple $(X, \mathbb{A}, M_A, \triangle)$ is called a *C^*-algebra valued fuzzy metric space* if satisfying the following conditions, for each $x, y, z \in X$ and $t, s > 0$,

(i) $M_A(x,y,t) > 0_A$,
(ii) $M_A(x,y,t) = 1_A$ if and only if $x = y$ for all $t > 0$,
(iii) $M_A(x,y,t) = M_A(y,x,t)$,
(iv) $M_A(x,y,t) \triangle M_A(y,z,s) \leq M_A(x,z,t+s)$ for all $t, s > 0$,
(v) $M_A(x,y,\cdot) : (0,\infty) \to [0_A, 1_A]$ is continuous.

In this case, we also say that $(X, \mathbb{A}, M_A, \triangle)$ is a C^*-algebra valued fuzzy metric space under $\triangle$. In the sequel, we will only consider C^*-algebra valued fuzzy metric space satisfying :

(vi) $\lim_{t\to\infty} M_A(x,y,t) = 1_A$ for all $x, y \in X$.

It is obvious that if X is a Banach space, then $(X, \mathbb{A}, M_A, \triangle)$ is a complete C^*-algebra valued fuzzy metric space if for $t > 0$, we set

$$M_A(x,y,t) = \left(\frac{t}{t+|x-y|}\right) I.$$

Next, we present a C^*-algebra valued contraction mapping in fuzzy metric spaces.

Definition 2.2. Let X be an arbitrary nonempty set. The 3-tuple $(X, \mathbb{A}, M_A, \triangle)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be a *C^*-algebra valued contraction mapping* if there exists an $b \in \mathbb{A}$ with $||b|| < 1$ such that

$$\frac{1}{M_A(Tx,Ty,t)} - 1 \leq b^* \left(\frac{1}{M_A(x,y,t)} - 1\right) b \tag{2.1}$$

for all $x, y \in X$ and $t > 0$.

Now, we ready to prove existence theorem for C^*-algebra valued contraction mapping in fuzzy metric spaces.

Theorem 2.3. *Let $(X, \mathbb{A}, M_A, \triangle)$ be a complete fuzzy metric space. A mapping $T : X \to X$ is a C^*-algebra valued contraction mapping. Then, T has a unique fixed point in X.*

Proof. It is obvious that if $b = 0_A$, then there is nothing to prove. Suppose that $b \neq 0_A$, let $x_0 \in X$ be a arbitrary. We define a sequence $\{x_n\}_{n\geq 0}$ by $x_{n+1} = Tx_n = T^n x_0$ and denote the element $M_A(x_0, x_1, t)$ in $\mathbb{A}$.

By C^*-algebra if $a_1, a_2 \in \mathbb{A}^+$ and $a_1 \leq a_2$, then $z^* a_1 z \leq z^* a_2 z$ for all $z \in \mathbb{A}$. Thus,

$$\begin{aligned}
\frac{1}{M_A(x_n,x_{n+1},t)} - 1 &= \frac{1}{M_A(Tx_{n-1},Tx_n,t)} - 1 \\
&\leq b^* \left(\frac{1}{M_A(x_{n-1},x_n,t)} - 1\right) b \\
&\leq (b^*)^2 \left(\frac{1}{M_A(x_{n-2},x_{n-1},t)} - 1\right) b^2 \\
&\ \vdots \\
&\leq (b^*)^n \left(\frac{1}{M_A(x_0,x_1,t)} - 1\right) b^n \\
&= (b^*)^n \mathbb{M} b^n, \quad \text{where } \mathbb{M} = \frac{1}{M_A(x_0,x_1,t)} - 1.
\end{aligned}$$

Suppose that, for $n+1 > m$, by the triangle inequality of fuzzy metric spaces. We have

$$\begin{aligned}
\frac{1}{M_A(x_m,x_{n+1},t)} - 1 &\leq \frac{1}{M_A(x_m,x_{m+1},t)} - 1 + \frac{1}{M_A(x_{m+1},x_{m+2},t)} - 1 + \ldots \\
&\quad + \frac{1}{M_A(x_{n-1},x_n,t)} - 1 + \frac{1}{M_A(x_n,x_{n+1},t)} - 1 \\
&\leq (b^*)^m \mathbb{M} b^m + (b^*)^{m+1} \mathbb{M} b^{m+1} + \ldots + (b^*)^n \mathbb{M} b^n \\
&= \sum_{i=m}^{n} (b^*)^i \mathbb{M} b^i
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=m}^{n} (b^*)^i \mathbb{M}^{\frac{1}{2}} \mathbb{M}^{\frac{1}{2}} b^i \\
&= \sum_{i=m}^{n} (\mathbb{M}^{\frac{1}{2}} b^i)^* (\mathbb{M}^{\frac{1}{2}} b^i) \\
&= \sum_{i=m}^{n} \left| \mathbb{M}^{\frac{1}{2}} b^i \right|^2 \\
&\leq \left\| \sum_{i=m}^{n} \left| \mathbb{M}^{\frac{1}{2}} b^i \right|^2 \right\| I \\
&\leq \sum_{i=m}^{n} \left\| \mathbb{M}^{\frac{1}{2}} \right\|^2 \cdot \left\| b^i \right\|^2 I \\
&\leq \left\| \mathbb{M}^{\frac{1}{2}} \right\|^2 \sum_{i=m}^{n} \left\| b^i \right\|^2 I \\
&\leq \|\mathbb{M}\| \cdot \frac{\|b\|^{2m}}{1 - \|b\|} I \to 0_A, \text{ where } m \to \infty.
\end{aligned}
$$

So $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in X with resect to $\mathbb{A}$. From $(X, M, \triangle)$ is complete, there exists $x \in X$ such that $\lim_{n \to \infty} M_A(x_n, x, t) = 1$, that is, $\lim_{n \to \infty} M_A(Tx_{n-1}, x, t) = 1$. Since

$$
\begin{aligned}
0_A &\leq \frac{1}{M_A(Tx, x, t)} - 1 \\
&\leq \frac{1}{M_A(Tx, Tx_n, t)} - 1 + \frac{1}{M_A(Tx_n, x, t)} - 1 \\
&\leq b^* \left(\frac{1}{M_A(x, x_n, t)} - 1 \right) b + \left(\frac{1}{M_A(x_{n+1}, x, t)} - 1 \right) \\
&\leq 0_A, \quad \text{where } n \to \infty.
\end{aligned}
$$

Therefore, $Tx = x$, that is, x is a fixed point of T.

Now, we will show that x is a unique fixed point. Suppose that $z \neq x$ be an another fixed point of T, then $Tz \neq Tx$. From contraction inequality (2.1) we have,

$$
\begin{aligned}
\frac{1}{M_A(z, x, t)} - 1 &= \frac{1}{M_A(Tz, Tx, t)} - 1 \\
&\leq b^* \left(\frac{1}{M_A(z, x, t)} - 1 \right) b \\
&\leq \left(\frac{1}{M_A(z, x, t)} - 1 \right) b^* b \\
&\leq \left(\frac{1}{M_A(z, x, t)} - 1 \right) \|b\|^2.
\end{aligned}
$$

Since $\|b\| < 1$, it is a contradiction. Hence, T has a unique fixed point. The proof is therefore completed. ∎

Remark 2.4. By Theorem 2.3, if we changing $\mathbb{A}^+ = \mathbb{R}^+$ on the C^*-algebra valued fuzzy metric space, then it becomes the fuzzy metric spaces from Theorem 2.3.

3. Application

In this section, we devote our main result to the existence of a solution of integral equations. Let $X = L^\infty(E)$ and consider $H = L^\infty(E)$ be a Hilbert space, where E be a set of Lebesgue measurable. For $f, g \in X$, we define $M_A(f,g,t) = \pi_{|\frac{t}{t+|f-g|}|}$, where $\pi_h(x) = h \cdot x$ for $x \in H$. Suppose that a function $F : E^2 \times \mathbb{R} \to \mathbb{R}$, there exists $g : E^2 \to \mathbb{R}$ is a continuous function, $\sup_{u\in E} \int_E \mid g(u,z) \mid dz \leq 1$ and $\alpha \in (0,1)$ for $u, v \in E$ and $y, z \in \mathbb{R}$ we have

$$\left| \frac{t}{t+F(u,v,y)} - \frac{t}{t+F(u,v,z)} \right| \leq \alpha \left| g(u,v) \left(\frac{t}{t+y} - \frac{t}{t+z} \right) \right|.$$

Then, $x^* \in X$ is a unique solution of the integral equation

$$x(u) = \int_E F(u,v,x(v))dv), \quad u \in E. \tag{3.1}$$

Proof. Let $(X, L(H), M, \triangle)$ be a complete C^*- algebra valued fuzzy metric space with respect to $L(H)$. Suppose that $T : X \to X$ be a self mapping, we obtain

$$Tx(u) = \int_E F(u,v,x(v))dv), \quad u \in E.$$

Now,

$$\begin{aligned}
&\|M_A(Tx,Ty,t)\| \\
&\quad = \sup_{\|h\|=1} \left(\pi_{|\frac{t}{t+|Tx-Ty|}|} h, h \right) \\
&\quad = \sup_{\|h\|=1} \int_E \left(\left| \int_E \left(\frac{t}{t+F(u,v,x(v))} - \frac{t}{t+F(u,v,y(v))} \right) dv \right| \right) h(u)\overline{h(u)} du \\
&\quad \leq \sup_{\|h\|=1} \int_E \left(\int_E \left| \left(\frac{t}{t+F(u,v,x(v))} - \frac{t}{t+F(u,v,y(v))} \right) \right| dv \right) |h(u)|^2 du \\
&\quad \leq \sup_{\|h\|=1} \int_E \left(\int_E \left| \alpha g(u,v) \left(\frac{t}{t+x(v)} - \frac{t}{t+y(v)} \right) \right| dv \right) |h(u)|^2 du \\
&\quad \leq \alpha \sup_{u\in E} \int_E |g(u,v)|\, dv \cdot \sup_{\|h\|=1} \int_E |h(u)|^2\, du \cdot \left\| \frac{t}{t+x(v)} - \frac{t}{t+y(v)} \right\|_\infty \\
&\quad \leq \alpha \left\| \frac{t}{t+x(v)} - \frac{t}{t+y(v)} \right\|_\infty \\
&\quad \leq \|b\| \, \|M_A(x,y,t)\|.
\end{aligned}$$

Since $\|b\| < 1$, then $x^* \in X$ is a unique solution of the integral equation. The proof is therefore completed. ∎

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References

[1] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.

[2] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988) 385–389.

[3] V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000) 485–489.

[4] V. Gregori, S. Morillas, A. Sapena, Examples of fuzzy metrics and applications, Fuzzy Sets and Systems 170 (2011) 95–111.

[5] T. Kamran, M. Postolache, A. Ghiura, S. Batul, R. Ali, The Banach contraction principle in C^*-algebra-valued metric spaces with application, Fixed Point Theory and Appl. 10 (2016) 1–7.

[6] O. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika, 11 (1975) 336–344.

[7] Z.H. Ma, L.N. Jiang, H.K. Sun, C^*-algebra-valued metric spaces and related fixed point theorems, Fixed Point Theory and Appl. 206 (2014) 1–11.

[8] D. Mihet, On fuzzy contractive mappings in fuzzy metric spaces, Fuzzy Sets and Systems 158 (2007) 915–921.

[9] G.J. Murphy, C^*-Algebras and Operator Theory, Academic Press, London, 1990.

[10] I. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 314–334.

[11] R. Vasuki, P. Veeramani, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, Fuzzy Sets and Systems 135 (2003) 415–417.

[12] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.