Thai Journal of Mathematics
Volume 19 Number 3 (2021)
Pages 942–956
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Parallel Hybrid Extragradient-Proximal Methods for Split Equilibrium Problems in Hilbert Spaces

Ponkamon Kitisak[1], **Pronpat Peeyada**[1], **Natthakan Tewalok**[1], **Yollada Choiban**[1], **Siriluck Chaiwong**[1] **and Watcharaporn Cholamjiak**[1,*]

[1] *School of Science, University of Phayao, Phayao 56000, Thailand*
e-mail : pronkamonkitisak@gmail.com (P. Kitisak); pronpat.pee@gmail.com (P. Peeyada); tewalok2540@gmail.com (N. Tewalok); Bitey6125@gmail.com (Y. Choiban); Siriluckchaiwongying@gmail.com (S. Chaiwong); c-wchp007@hotmail.com (W. Cholamjiak)

Abstract In this paper, we introduce two different parallel hybrid extragradient-proximal methods for solving split equilibrium problems in Hilbert spaces. The algorithms combine the extragradient method, the proximal method and the hybrid projection method. The strong convergence theorems for iterative sequences generated by the algorithms are established under widely used assumptions for equilibrium bifunctions in Hilbert spaces.

MSC: 49K35; 47H10; 20M12
Keywords: extragradient-proximal method; split equilibrium problem; strong convergence; Hilbert space

Submission date: 27.04.2021 / Acceptance date: 03.08.2021

1. Introduction

Let H_1, H_2 be two real Hilbert space and C, Q be two nonempty closed convex subsets of H_1, H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : C \times C \to \mathbb{R}$ and $F : Q \times Q \to \mathbb{R}$ be two bifunctions with $f(x, x) = 0$ for all $x \in C$ and $F(y, y) = 0$ for all $y \in Q$. The split equilibrium problem (SEP) [34] is stated as follows:

$$\begin{cases} \text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \ \forall y \in C, \\ \text{and } u^* = Ax^* \in Q \text{ solves } F(u^*, u) \geq 0, \ \forall u \in Q. \end{cases} \tag{1.1}$$

Obviously, if $F = 0$ and $Q = H_2$, then SEP (1.1) becomes the following equilibrium problem (EP) [7].

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \ \forall y \in C. \tag{1.2}$$

The solution set of EP (1.2) for the bifunction f on C is denoted by $\text{EP}(f, C)$. A mentioned archetypal model in Section 2 of [31] is the split inverse problem (SIP), where there are a

*Corresponding author.

bounded linear operator A from a space H_1 to another space H_2 and two inverse problems IP1 and IP2 installed in H_1 and H_2, respectively. The SIP is stated as follows:

$$\begin{cases} \text{Find } x^* \in C \text{ that solves IP1 such that} \\ \text{the point } y^* = Ax^* \in H_2 \text{ that solves IP2.} \end{cases} \tag{1.3}$$

Many models of inverse problems in this framework can be solved by setting different inverse problems for IP1 and IP2. Two most notable examples are the split convex feasibility problem (SCFP) and the split optimization problem (SOP) in which IP1 and IP2 are two convex feasibility problems (CFP) or two constrained optimization problems (COP), see [3, 25].

It is also well known that EP (1.2) is a generalization of many mathematical models [7] involving variational inequality problem (VIP), constrained optimization problem (COP), convex feasibility problem (CFP) and fixed point problems (FPP). The EP is very important in the field of applied mathematics. Moreover, in recent years, the problem of finding a common solution to equilibrium problems (CSEP) has been widely and intensively studied by many authors, see in [8] and the reference therein.

We see that the problem of finding a common solution of EP1 and EP2 is on a same feasible set K and on a same space $\mathbb{R}^n$. As a generalization, when the feasible sets of EP1 and EP2 are different in a same space, or in more general, EP1 and EP2 are in two different spaces which originates from the model of SIP (1.3), i.e., a split equilibrium problem should enable us to split equilibrium solutions between two different subsets of spaces in which the image of a solution point of one problem, under a given bounded linear operator, is a solution point of another problem.

Moreover, the multi-objective split optimization problem (MSOP) has been considered by some authors in recent years, for examples, in [3, 25] and the references therein. This problem is stated as follows:

$$\begin{cases} \text{Find } x^* \in C \subset H_1 \text{ that solves } \min\{g_i(x) : x \in C\},\ i = 1, ..., N \text{ such that} \\ u^* = Ax^* \subset Q \subset H_2 \text{ solves } \min\{h_i(u) : u \in Q\},\ j = 1, ..., M, \end{cases} \tag{1.4}$$

where g_i, h_j are convex objective functions on C and Q, respectively. If the functions g_i and h_j are differentiable for all i, j then MSOP (1.4) can be solved by many different methods or reformulated equivalently to the multiple set SVIP ([31], Section 6.1) for derivative operators ∇g_i and ∇h_j. However, if g_i and h_j are only convex and not differentiable for some i, j then, by setting $f_i(x, y) = g_i(y) - g_i(x)$ and $F_j(u, v) = h_j(v) - h_j(u)$, MSOP (1.4) is equivalent to the SEP considered in this paper.

The interest is to cover many situations and some practical models are promosing in the future, for examples, decomposition methods for PDEs [9], game theory and equilibrium models [8] and intensity-dodulated radiation therapy [33]. Recently, SEP (1.1) and its special cases have been recieved a lot of attention by many authors and some methods for solving them can be found, for instance, in [1–3, 12–16, 18, 19, 21, 28, 30, 32, 34]. Almost proposed methods for SEPs based on the proximal method [11] which consists of solving a regularized equilibrium problem, i.e., at current iteration given x_k the next iterate x_{k+1} solves the following problem

$$\text{find } x \in C \text{ such that } f(x, y) + \frac{1}{r_k}\langle y - x, x - x_k\rangle \geq 0,\ \forall y \in C, \tag{1.5}$$

or $x_{k+1} = T^f_{r_k}(x_k)$ where $T^f_{r_k}$ is the resolvent of the bifunction f and $r_k > 0$, see [22].

In 2012, He [34] used the proximal method and proposed the following algorithm

$$\begin{cases} f_i(u^i_k, y) + \frac{1}{r_k}\langle y - u^i_k, u^i_k - x_k\rangle \geq 0, \ \forall y \in C, \ i = 1, ..., N, \\ \tau_k = \frac{u^1_k + ... + u^N_k}{N}, \\ F(w_k, z) + \frac{1}{r_k}\langle z - w_k, w_k - \tau_k\rangle \geq 0, \ \forall z \in Q, \\ x_{k+1} = P_C(\tau_k + \mu A^*(w_k - A\tau_k)), \end{cases}$$

for finding an element $\Omega = \{p \in \cap_{i=1}^N EP(f_i, C) : Ap \in EP(F, Q)\}$. Under the assumption of the monotonicity of $f_i : C \times C \to \mathbb{R}$, $F : Q \times Q \to \mathbb{R}$ and suitable conditions on the parameters r_k, μ, the author proved that $\{u^i_k\}, \{x_k\}$ converge weakly to some point in Ω

Very recently, for finding a common solution of a system of equilibrium problems for pseudomonotone and Lipschitz-type continuous bifunctions $\{f_i\}_{i=1}^N$ the authors in [5] have proposed the following parallel hybrid extragradient algorithm

$$\begin{cases} y^i_k = argmin\{\lambda f_i(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ z^i_k = argmin\{\lambda f_i(y^i_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ \bar{z}_k = argmax\{\|z^i_k - x_k\| : i = 1, ..., N\}, \\ C_k = \{v \in C : \|\bar{z}_k - v\| \leq |x_k - v\|\}, \\ Q_k = \{v \in C : \langle x_0 - x_k, v - x_k\rangle \leq 0\}, \\ x_{k+1} = P_{C_k \bigcap Q_k} x_0, \ k \geq 0. \end{cases}$$

It has been proved that $\{x_k\}, \{y^i_k\}, \{z^i_k\}$ converge strongly to the projection of the starting point x_0 onto the solution set $F := \bigcap_{i=1}^N EP(f_i, C)$ under certain conditions on the parameter λ. The advantages of the extragradient method are that it is used for the class of pseudomonotone bifunctions and two optimization programs are solved at each iteration which seems to be numerically easier than non-linear inequality (1.5) in the proximal method, see for instance [24, 26, 27] and the references therein.

In 2016, Hieua [6] introduced parallel extradient-proximal methods for solving split equilibrium problems. The algorithms combine the extragradient method, the proximal method and the shrinking projection method. The strong convergence theorems for iterative sequences generated by the algorithm established under widely used assumptions for equilibrium bifunctions. They also were presented an application to split variational inequality problems. The algorithm is generated as follows:

Algorithm 1.1. *Choose $x_0 \in C$, $C_0 = C$ the control parameters λ, r_k, μ satisfy the following conditions*

$$0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}, \ r_n \geq d > 0, \ 0 < \mu < \frac{2}{\|A\|^2}.$$

Step 1. *Solve $2N$ strongly convex optimization programs in parallel*

$$\begin{cases} y^i_k = argmin\{\lambda f_i(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \ i = 1, ..., N, \\ z^i_k = argmin\{\lambda f_i(y^i_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \ i = 1, ..., N. \end{cases}$$

Step 2. *Find among z^i_k the furthest element from x_k, i.e.,*

$$\bar{z}_k = argmax\{\|z^i_k - x_k\| : i = 1, ..., N\}.$$

Step 3. *Solve M regularized equilibrium programs in parallel*

$$w_k^j = T_{r_k}^{F_j}(A\bar{z}_k),\ j = 1, ..., M.$$

Step 4. *Find among w_k^j the furthest element from $A\bar{z}_k$, i.e.,*

$$\bar{w}_k = argmax\{\|w_k^j - A\bar{z}_k\| : j = 1, ..., M\}.$$

Step 5. *Compute*

$$t_k = P_C(\bar{z}_k + \mu A^*(\bar{w}_k - A\bar{z}_k)).$$

Step 6. *Set $C_{k+1} = \{v \in C_k : \|t_k - v\| \leq \|\bar{z}_k - v\| \leq \|x_k - v\|\}$. Compute*

$$x_{k+1} = P_{C_{k+1}}(x_0).$$

*Set $k = k + 1$ and go back **Step 1.***

Motivated and inspired by the recent works [16, 18, 19, 25, 31] and the results above, we consider SIP (1.3) in Hilbert spaces H1 and H2 in which IP1 and IP2 are common split equilibrium problems. We propose two different parallel extragradient-proximal methods for split equilibrium problems for a finite family of bifunctions $\{f_i\}_{i=1}^N : C \times C \to \mathbb{R}$ in H_1 and a system of bifunctions $\{F_i\}_{j=1}^M : Q \times Q \to \mathbb{R}$ in H_2. We use the extragradient method for pseudomonotone equilibrium problems in H_1 and the proximal method with CQ algorithm for monotone equilibrium problems in H_2 to obtain the strong convergence algorithm.

2. Preliminaries and lemmas

This section contains some definition and basic results that will be used in our subsequent analysis. We next recall some properties of the projection [10] for more details. For any point $u \in H$ there exists a unique point $P_C u \in C$ such that

$$\| u - P_C u \| \leq \| u - y \| \text{ for all } y \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \| P_C x - P_C y \|^2 \text{ for all } x, y \in H. \tag{2.1}$$

In particular, we get from (2.1) that

$$\langle x - y, x - P_C y\rangle \geq \| x - P_C y \|^2 \text{ for all } x \in C, y \in H.$$

Furthermore, $P_C x$ is characterized by the properties

$$P_C x \in C \text{ and } \langle x - P_C x, P_C x - y\rangle \geq 0 \text{ for all } y \in C.$$

For solving SEP (1.1), we set the following conditions for the bifunctions $f : C\times C \to \mathbb{R}$ and $F : Q \times Q \to \mathbb{R}$. Firstly, for establishing a weakly convergence algorithm, we assume that f satisfies the following condition.

Condition 2.1. A1. f is pseudomonotone on C and $f(x, x) = 0$ for all $x \in C$.
A2. f is Lipschitz-type continuous on C with the constants $c_1,\ c_2$.
A3. $f(\cdot, y)$ is weakly sequencially upper semicontinuous on C with every fixed $y \in C$, i.e., $\limsup_{k\to\infty} f(x_k, y) \leq f(x, y)$ for each sequence $\{x_k\} \subset C$ converging weakly to x.
A4. $f(x, \cdot)$ is convex and subdifferentiable on C for every fixed $x \in C$.

Next, for obtaining a strongly convergence algorithm, we replace the assumption (A3) in Condition 2.1 by the weaker one (A3a) below, i.e., the bifunction f satisfies the following condition.

Condition 2.2. The assumptions (A1), (A2), (A4) in Condition 2.1 hold, and (A3a), $f(\cdot, y)$ is sequencially upper semicontinuous on C with every fixed $y \in C$, i.e.,

$$\limsup_{k\to\infty} f(x_k, y) \leq f(x, y),$$

for each sequence $\{x_k\} \subset C$ converging strongly to x.

Throughout this paper, the bifunction F satisfies the following condition.

Condition 2.3. B1. F is monotone on C and $F(x, x) = 0$ for all $x \in C$.
B2. For all $x, y, z \in C$,

$$\limsup_{k\to 0^+} F(tz + (1-t)x, y) \leq F(x, y).$$

B3. For all $x \in C$, $F(x, \cdot)$ is convex and lower semicontinuous.

The following results concern with the monotone bifunction F.

Lemma 2.4 ([22], Lemma 2.12). *Let C be a nonempty, closed and convex subset of a Hilbert space H, F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying Condition 2.3 and let $r > 0$, $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.5 ([22], Lemma 2.12). *Let C be a nonempty, closed and convex subset of a Hilbert space H, F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying Condition 2.3. For all $r > 0$ and $x \in H$, define the mapping*

$$T_r^F x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}.$$

Then the followings hold:

C1. T_r^F is single-valued;

C2. T_r^F is a firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

C3. $Fix(T_r^F)$ =$EP(F, C)$, where $Fix(T_r^F)$ is the fixed point set of T_r^F;

C4. $EP(F, C)$ is closed and convex.

Lemma 2.6 ([34], Lemma 2.5). *For $r, s > 0$ and $x, y \in H$. Under the assumptions of Lemma 2.5, then*

$$\|T_r^F(x) - T_r^F(y)\|^2 \leq \|x - y\| + \frac{|\, s - r\,|}{s}\|T_r^F(y) - y\|.$$

The metric projection $P_C : H \to C$ is defined by $P_C x = argmin_{y\in C}\{\|y - x\|\}$. It is well-known that P_C has the following characteristic properties, see [17] for more details.

Lemma 2.7. [29] *Let $P_C : H \to C$ be the metric projection from H onto C. Then*

(i) For all $x \in C, y \in H$,

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2.$$

(ii) $z = P_C x$ if and only if $\langle x - z, z - y\rangle \geq 0, \ \forall y \in C$.

Any Hilbert space satisfies Opial's condition [35], *i.e., if* $\{x_k\} \subset H$ *converges weakly to* x *then*

$$\liminf_{k\to\infty} \|x_k - x\| < \liminf_{k\to\infty} \|x_k - y\|, \ \forall y \in H, \ y \neq x.$$

Lemma 2.8 ([23], Lemma 3.1)**.** *Suppose that* $x^* \in EP(f,C)$ *and* $\{x_k\}$, $\{y_k\}$, $\{z_k\}$ *are the sequences generated by Algorithm 1. Then*
(i) $\lambda(f(x_k,y) - f(x_k,y_k)) \geq \langle y_k - x_k, y_k - y\rangle, \ \forall y \in C.$
(ii) $\|z_k - x^*\|^2 \leq \|x_k - x^*\|^2 - (1-2\lambda c_1)\|y_k - x_k\|^2 - (1-2\lambda c_2)\|y_k - z_k\|^2.$

Lemma 2.9. [20] *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H *and* $P_C x : H \to C$ *be the metric projection from* H *onto* C*. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \ \forall x \in H, \ \forall y \in C.$$

Lemma 2.10. [4] *Let* H *be a real Hilbert space and let* $\{u_i\}_{i=1}^m \subseteq H$*. For* $\alpha_i \in (0,1)$, $i = 1,2,...,m$ *such that* $\sum_{i=1}^{m} \alpha_i = 1$, *the following identity holds:*

$$\|\sum_{i=1}^{m} \alpha_i u_i\|^2 = \sum_{i=1}^{m} \alpha_i \|u_i\|^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j \|u_i - u_j\|^2.$$

3. Main Results

In this section, we present two different hybrid algorithms for common split equilibrium problems and prove their strongly convergence theorems. Assume that all bifunctions $f_i : C \times C \to \mathbb{R}$ satisfy the Lipschitz-type continuous condition with same constants $c_1, \ c_2$, where $c_1 = \max\{c_1^i : i = 1,...,N\}$ and $c_2 = \max\{c_2^i : i = 1,...,N\}$ such that $c_1^i, \ c_2^i$ are two constants of Lipschitz-type continuous f_i. We denote the solution set of SEP for $\{f_i\}_{i=1}^N$ and $\{F_j\}_{j=1}^M$ by

$$\Omega = \{g^* \in \cap_{i=1}^N EP(f_i,C) : Ag^* \in \cap_{j=1}^M EP(F_j,Q)\}.$$

It is easy to show that if f_i satisfies Condition 2.1 or Condition 2.2 then the solution set $\mathrm{EP}(f_i,C)$ is closed and convex, see for instance [26]. Moreover, from Lemma 2.5 (C4), under Condition 2.3 the set of solutions $\mathrm{EP}(F_j,Q)$ is also closed and convex. Since the operator A is linear and bounded, Ω is closed and convex. In this paper, we assume that Ω is nonempty. We start with the following algorithm.

Algorithm 3.1. *Choose* $x_0 \in C, Q, \ C_0 = C$ *and* $Q_0 = C$ *the control parameters* $\lambda, \ r_k, \ \eta$ *satisfy the following conditions*

$$0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}, \ r_k \geq d > 0, \ 0 < \eta < \frac{2}{\|A\|^2}.$$

Step 1. *Solve* $2N$ *strongly convex optimization programs in parallel*

$$\begin{cases} h_k^i = argmin\{\lambda f_i(g_k,h) + \frac{1}{2}\|h - g_k\|^2 : h \in C\}, \ i = 1,...,N, \\ u_k^i = argmin\{\lambda f_i(h_k^i,h) + \frac{1}{2}\|h - h_k^i\|^2 : h \in C\}, \ i = 1,...,N. \end{cases}$$

Step 2. *Find among* u_k^i *the furthest element from* g_k*, i.e.,*

$$\bar{u}_k = argmax\{\|u_k^i - g_k\| : i = 1,...,N\}.$$

Step 3. *Solve M regularized equilibrium programs in parallel*

$$w_k^j = T_{r_k}^{F_j}(A\bar{u}_k),\ j = 1, ..., M.$$

Step 4. *Find among w_k^j the furthest element from $A\bar{u}_k$, i.e.,*

$$\bar{w}_k = argmax\{\|w_k^j - A\bar{u}_k\| : j = 1, ..., M\}.$$

Step 5. *Compute*

$$s_k = P_C(\bar{u}_k + \eta A^*(\bar{w}_k - A\bar{u}_k)).$$

Step 6. *Set $C_k = \{v \in H_1 : \|s_k - v\| \le \|\bar{u}_k - v\| \le \|g_k - v\|\}$ and $Q_k = \{v \in Q_{k-1} : \langle g_1 - g_k, g_k - u\rangle \ge 0$. Compute*

$$g_{k+1} = P_{C_k \cap Q_k}(g_0).$$

Set $k = k + 1$ and go back ***Step 1.***

Theorem 3.2. *Let C, Q be two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $\{f_i\}_{i=1}^N : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying Condition 2.2 and $\{F_j\}_{j=1}^M : Q \times Q \to \mathbb{R}$ be a finite family of bifunctions satisfying Condition 2.3. Let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint A^*. In addition the solution set Ω is nonempty. Then, the sequences $\{g_k\}, \{h_k^i\}, \{u_k^i\}, i = 1, ..., N$ generated by Algorithm 3.1 converge strongly to $Aw \in \cap_{j=1}^M EP(F_j, Q)$.*

Proof. We split the proof into five steps.

Claim 1. Show that $\{x_k\}$ is well-defined.

$$C_k^1 = \{v \in H_1 : \|s_k - v\| \le \|\bar{u}_k - v\|\},\ C_k^2 = \{v \in H_1 : \|\bar{u}_k - v\| \le \|g_k - v\|\},$$

then

$$C_k = C_k^1 \cap C_k^2.$$

Note that C_k^1, C_k^2 are either the halfspaces or the whole space H for all $k \ge 0$. Hence, they are closed and convex. Obviously, C_k is closed and convex.
Next, we show that $\Omega \subset C_k$ for all $k \ge 0$. From Lemma 2.8 (ii) and the hypothsis of λ, we have

$$\|u_k^i - g^*\| \le \|g_k - g^*\|\ \forall g^* \in \Omega.$$

Thus,

$$\|\bar{u}_k - g^*\| \le \|g_k - g^*\|. \tag{3.1}$$

Thus, from the definition of s_k and the nonexpansive of the projection,

$$\begin{aligned}
\|s_k - g^*\|^2 &= \|P_C(\bar{u}_k + \eta A^*(\bar{w}_k - A\bar{u}_k)) - P_C g^*\|^2 \\
&\le \|\bar{u}_k - g^* + \eta A^*(\bar{w}_k - A\bar{u}_k)\|^2 \\
&= \|\bar{u}_k - g^*\|^2 + \eta^2\|A^*(\bar{w}_k - A\bar{u}_k)\|^2 + 2\eta\langle \bar{u}_k - g^*, A^*(\bar{w}_k - A\bar{u}_k)\rangle \\
&\le \|\bar{u}_k - g^*\|^2 + \eta^2\|A^*\|^2\|\bar{w}_k - A\bar{u}_k\|^2 + 2\eta\langle A(\bar{u}_k - g^*), \bar{w}_k - A\bar{u}_k\rangle \\
&\le \|\bar{u}_k - g^*\|^2 + \eta^2\|A^*\|^2\|\bar{w}_k - A\bar{u}_k\|^2 - 2\eta\|\bar{w}_k - A\bar{u}_k\|^2 \\
&\le \|\bar{u}_k - g^*\|^2 - \eta(2 - \eta\|A^*\|^2)\|\bar{w}_k - A\bar{u}_k\|^2 \\
&\le \|\bar{u}_k - g^*\|^2.
\end{aligned} \tag{3.2}$$

From (3.1) and (3.2),

$$\|s_k - g^*\| \le \|\bar{u}_k - g^*\| \le \|g_k - g^*\|,\ \forall g^* \in \Omega.$$

Thus, by the definition of C_k and the induction, $\Omega \subset C_k$ for all $k \geq 0$. For $k = 1$, we have $g_1 = g \in C$ and $Q_1 = C$, hence $\Omega \subseteq C_1 \cap Q_1$. Suppose that g_k is given and $\Omega \subseteq C_k \cap Q_k$ for some $k \geq 1$. There exists a unique element $g_{k+1} \in C_k \cap Q_k$ such that $g_{k+1} \in P_{C_k \cap Q_k}(g_1)$, there holds $\langle g_{k+1} - u, g_1 - g_{k+1}\rangle \geq 0$ for each $u \in C_k \cap Q_k$ we get $\Omega \subseteq Q_{k+1}$. Therefore, we get $\Omega \subseteq C_{k+1} \cap Q_{k+1}$. This gives $\{g_k\}$ is well defined and $\Omega \subseteq C_k \cap Q_k$

Claim 2. Show that $\lim_{k\to\infty} \|g_k - g_1\|$ exists, since Ω is nonempty closed convex subset of C, there exists a unique element $u \in \Omega$ such that $u = P_\Omega(g_1)$. From $g_{k+1} = P_{Q_k}(g_1)$, we have

$$\|g_k - g_1\| \leq \|u - g_1\|, \tag{3.3}$$

for every $u \in Q_{k-1}$. Since $u \in \Omega \subseteq C_k \cap Q_k$, we obtain

$$\|g_k - g_1\| \leq \|g_{k+1} - g_1\|, \tag{3.4}$$

for each $k \in N$. It follows from (3.3) and (3.4) that thesequence $\{g_k\}$ is bounded and nondecreasing. There, $\lim_{k\to\infty} \|g_k - g_1\|$ exists.

Claim 3. Show that $g_k \to w \in C$ as $k \to \infty$. For $l > k$ by the definition of Q_k, we see that $g_l = P_{Q_l}(g_1) \in Q_k$. From Lemma 2.9, we have

$$\|g_l - g_k\|^2 \leq \|g_l - g_1\|^2 - \|g_k - g_1\|^2.$$

From Claim 2, we obtain that $\{g_k\}$ is a Cauchy sequence. Hence, there exists $w \in C$ such that $g_k \to w$ as $k \to \infty$. In particular, we have

$$\lim_{k\to\infty} \|g_{k+1} - g_k\| = 0. \tag{3.5}$$

Claim 4. Show that $h_k^i \to w$ as $k \to \infty$ for all $i = 1, ..., N$. From the definition of C_k and $g_{k+1} \in C_k$, we have

$$\|s_k - g_{k+1}\| \leq \|\bar{u}_k - g_{k+1}\| \leq \|g_k - g_{k+1}\|.$$

Thus, from the triangle inequality, one has

$$\begin{aligned}
\|s_k - g_k\| &\leq \|s_k - g_{k+1}\| + \|g_{k+1} - g_k\| \leq 2\|g_k - g_{k+1}\|,\\
\|\bar{u}_k - g_k\| &\leq \|\bar{u}_k - g_{k+1}\| + \|g_{k+1} - g_k\| \leq 2\|g_k - g_{k+1}\|,\\
\|\bar{u}_k - s_k\| &\leq \|\bar{u}_k - g_k\| + \|g_k - s_k\| \leq 4\|g_k - g_{k+1}\|.
\end{aligned}$$

Three last inequalities togeter with the relation (3.5) imply that

$$\lim_{k\to\infty} \|s_k - g_k\| = \lim_{k\to\infty} \|\bar{u}_k - s_k\| = \lim_{k\to\infty} \|\bar{u}_k - g_k\| = 0. \tag{3.6}$$

Hence, from the definition of $\bar{u}_k$, we also obtain

$$\lim_{k\to\infty} \|u_k^i - g_k\| = 0, \ \forall i = 1, .., N. \tag{3.7}$$

Since $\{g_k\}$ is a Cauchy sequence, $g_k \to w$ and

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \bar{u}_k = \lim_{k\to\infty} u_k^i = w, \ \forall i = 1, .., N,$$

and so

$$\lim_{k\to\infty} A\bar{u}_k = Aw. \tag{3.8}$$

From the relation (3.2) and the triangle inequality, we obtain

$$\begin{aligned}\eta(2-\eta\|A^*\|^2)\|\bar{w}_k - A\bar{u}_k\|^2 &\leq \|\bar{u}_k - g^*\|^2 - \|s_k - g^*\|^2 \\ &= (\|\bar{u}_k - g^*\| - \|s_k - g^*\|)(\|\bar{u}_k - g^*\| + \|s_k - g^*\|) \\ &\leq \|\bar{u}_k - s_k\|(\|\bar{u}_k - g^*\| + \|s_k - g^*\|).\end{aligned}$$

Thus, from $\eta(2-\eta\|A^*\|^2) > 0$, the boundedness of $\{s_k\}$, $\{\bar{u}_k\}$ and (3.6) we obtain

$$\lim_{k\to\infty} \|\bar{w}_k - A\bar{u}_k\| = 0.$$

From the defination of $\bar{w}_k$, we get

$$\lim_{k\to\infty} \|w_k^j - A\bar{u}_k\| = 0, \ \forall j = 1,..,M, \tag{3.9}$$

which follows from (3.8) that

$$\lim_{k\to\infty} w_k^j = Aw, \ \forall j = 1,..,M. \tag{3.10}$$

From Lemma 2.8 (ii) and the triangle inequality, we have

$$\begin{aligned}(1-2\lambda c_1)\|h_k^i - g_k\|^2 &\leq \|g_k - g^*\|^2 - \|u_k^i - g^*\|^2 \\ &= (\|g_k - g^*\| - \|u_k^i - g^*\|)(\|g_k - g^*\| + \|u_k^i - g^*\|) \\ &\leq \|g_k - u_k^i\|(\|g_k - g^*\| + \|u_k^i - g^*\|).\end{aligned}$$

Thus, from the hypothesis of λ, the boundedness of $\{g_k\}$, $\{u_k^i\}$ and (3.7) we obtain

$$\lim_{k\to\infty} \|h_k^i - g_k\| = 0.$$

Therefore, $h_k^i \to w$ as $k \to \infty$ for all $i = 1, ..., N$.

Claim 5. $w \in \Omega$ and $w = u \ : \ P_\Omega(g_0)$. The proof of Claim 3. By using Claim 2, we also obtain $w \in \cap_{i=1}^N EP(f_i, C)$. Moreover, from Lemma 2.6 for some $r > 0$ we have

$$\begin{aligned}\|T_r^{F_j}(Aw) - Aw\| &\leq \|T_r^{F_j}(Aw) - T_{r_k}^{F_j}(A\bar{u}_k) + \|T_{r_k}^{F_j}(A\bar{u}_k) - A\bar{u}_k\| + \|A\bar{u}_k - Aw\| \\ &\leq \|Aw - A\bar{u}_k\| + \frac{r_k - r}{r_k}\|T_{r_k}^{F_j}(A\bar{u}_k) - A\bar{u}_k\| + \|T_{r_k}^{F_j}(A\bar{u}_k) - A\bar{u}_k\| \\ &+ \|A\bar{u}_k - Aw\| \\ &= 2\|Aw - A\bar{u}_k\| + \frac{r_k - r}{r_k}\|w_k^j - A\bar{u}_k\| + \|w_k^j - A\bar{u}_k\| \to 0,\end{aligned}$$

which is followed from the relations (3.8), (3.9), (3.10) and $r_k \geq d > 0$. Thus, $T_r^{F_j}(Aw) = 0$ or Aw is a fixed point of $T_r^{F_j}$. From Lemma 2.5, we obtain $Aw \in \cap_{j=1}^M EP(F_j, Q)$. Thus, $w \in \Omega$. Finally, from (3.3), $\|g_k - g_1\| \leq \|u - g_1\|$ where $u = P_\Omega(x_1)$. Taking $k \to \infty$ in this inequality, one has $\|w - g_1\| \leq \|u - g_1\|$. From the definition of u, $w = u$. Theorem 3.2 is proved. ∎

Algorithm 3.3. *Choose $g_0 \in C, Q$, $C_0 = C$ and $Q_0 = C$ the control parameters λ, r_k, η satisfy the following conditions*

$$0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}, \ r_k \geq d > 0, \ 0 < \eta < \frac{2}{\|A\|^2}.$$

Step 1. *Solve $2N$ strongly convex optimization programs in parallel*

$$\begin{cases} h_k^i = argmin\{\lambda f_i(g_k, h) + \frac{1}{2}\|h - g_k\|^2 : h \in C\}, \ i = 1, ..., N, \\ u_k^i = argmin\{\lambda f_i(h_k^i, h) + \frac{1}{2}\|h - g_k\|^2 : h \in C\}, \ i = 1, ..., N. \end{cases}$$

Step 2. *Find among u_k^i the furthest element from g_k, i.e.,*

$$\bar{u}_k = \alpha_k^0 g_k + \sum_{i=1}^{N} \alpha_k^i u_k^i \ , \ \textit{where} \ \sum_{i=0}^{N} \alpha_k^i = 1 \ , \ \forall k \in \mathbb{N}.$$

Step 3. *Solve M regularized equilibrium programs in parallel*

$$w_k^j = T_{r_k}^{F_j}(A\bar{u}_k), j = 1, ..., M.$$

Step 4. *Find among w_k^j the furthest element from $A\bar{u}_k$, i.e.,*

$$\bar{w}_k = \beta_k^0 A\bar{u}_k + \sum_{i=1}^{M} \beta_k^i w_k^j \ , \ \textit{where} \ \sum_{i=0}^{M} \beta_k^i = 1 \ , \ \forall k \in \mathbb{N}.$$

Step 5. *Compute*

$$s_k = P_C(\bar{u}_k + \eta A^*(\bar{w}_k - A\bar{u}_k)).$$

Step 6. *Set $C_k = \{v \in H_1 : \|s_k - v\| \leq \|\bar{u}_k - v\| \leq \|g_k - v\|\}$ and $Q_k = \{v \in Q_{k-1} : \langle g_1 - g_k, g_k - u\rangle \geq 0$. Compute*

$$g_{k+1} = P_{C_k \cap Q_k}(g_0).$$

*Set $k = k + 1$ and go back **Step 1.***

Theorem 3.4. *Let C, Q be two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $\{f_i\}_{i=1}^N : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying Condition 2.2 and $\{F_j\}_{j=1}^M : Q \times Q \to \mathbb{R}$ be a finite family of bifunctions satisfying Condition 2.3. Let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint A^*. In addition the solution set Ω is nonempty. Assume that the following conditions hold:*
(i) $\liminf_{k\to\infty} \alpha_k^0 \alpha_k^i > 0$
(ii) $\liminf_{k\to\infty} \beta_k^0 \beta_k^i > 0.$
Then, the sequences $\{g_k\}, \{h_k^i\}, \{u_k^i\}, i = 1, ..., N$ generated by Algorithm 3.3 converge strongly to $Aw \in \cap_{j=1}^M EP(F_j, Q)$.

Proof. We split the proof into five steps.

Claim 1. Show that $\{g_k\}$ is well-defined.

$$C_k^1 = \{v \in H_1 : \|s_k - v\| \leq \|\bar{u}_k - v\|\}, \ C_k^2 = \{v \in H_1 : \|\bar{u}_k - v\| \leq \|g_k - v\|\},$$

then

$$C_k = C_k^1 \cap C_k^2.$$

Note that C_k^1, C_k^2 are either the halfspaces or the whole space H for all $k \geq 0$. Hence, they are closed and convex. Obviously, C_k is closed and convex. Next, we show that $\Omega \subset C_k$ for all $k \geq 0$. From Lemma 2.8 (ii) and the hypothsis of λ, we have

$$\|u_k^i - g^*\| \leq \|g_k - g^*\| \text{for all } g^* \in \Omega.$$

Thus,

$$\|\bar{u}_k - g^*\| \leq \|g_k - g^*\|. \tag{3.11}$$

Thus, from the definition of s_k and the nonexpansive of the projection,

$$\begin{aligned}\|s_k-g^*\|^2 &= \|P_C(\bar{u}_k+\eta A^*(\bar{w}_k-A\bar{u}_k))-P_Cg^*\|^2\\ &\le \|\bar{u}_k-g^*+\eta A^*(\bar{w}_k-A\bar{u}_k)\|^2\\ &= \|\bar{u}_k-g^*\|^2+\eta^2\|A^*(\bar{w}_k-A\bar{u}_k)\|^2+2\eta\langle\bar{u}_k-g^*,A^*(\bar{w}_k-A\bar{u}_k)\rangle\\ &\le \|\bar{u}_k-g^*\|^2+\eta^2\|A^*\|^2\|\bar{w}_k-A\bar{u}_k\|^2+2\eta\langle A(\bar{u}_k-g^*),\bar{w}_k-A\bar{u}_k\rangle\\ &\le \|\bar{u}_k-g^*\|^2+\eta^2\|A^*\|^2\|\bar{w}_k-A\bar{u}_k\|^2-2\eta\|\bar{w}_k-A\bar{u}_k\|^2\\ &\le \|\bar{u}_k-g^*\|^2-\eta(2-\eta\|A^*\|^2)\|\bar{w}_k-A\bar{u}_k\|^2\\ &\le \|\bar{u}_k-g^*\|^2. \end{aligned}\tag{3.12}$$

From (3.11) and (3.12),

$$\|s_k-g^*\|\le\|\bar{u}_k-g^*\|\le\|g_k-g^*\|,\ \forall g^*\in\Omega.$$

Thus, by the definition of C_k and the induction, $\Omega\subset C_k$ for all $k\ge 0$. For $k=1$, we have $g_1=g\in C$ and $Q_1=C$, hence $\Omega\subseteq C_1\cap Q_1$. Suppose that g_k is given and $\Omega\subseteq C_k\cap Q_k$ for some $k\ge 1$. There exists a unique element $g_{k+1}\in C_k\cap Q_k$ such that $g_{k+1}\in P_{C_k\cap Q_k}(g_1)$ there holds $\langle g_{k+1}-u,g_1-g_{k+1}\rangle\ge 0$ for each $u\in C_k\cap Q_k$ we get $\Omega\subseteq Q_{k+1}$. Therefore, we get $\Omega\subseteq C_{k+1}\cap Q_{k+1}$. This gives $\{g_k\}$ is well defined and $\Omega\subseteq C_k\cap Q_k$

Claim 2. Show that $\lim_{k\to\infty}\|g_k-g_1\|$ exists, since Ω is nonempty closed convex subset of C there exists a unique element $u\in\Omega$ such that $u=P_\Omega(g_1)$. From $g_{k+1}=P_{Q_k}(g_1)$, we have

$$\|g_k-g_1\|\le\|u-g_1\|,\tag{3.13}$$

for every $u\in Q_{k-1}$. Since $u\in\Omega\subseteq C_k\cap Q_k$, we obtain

$$\|g_k-g_1\|\le\|g_{k+1}-g_1\|,\tag{3.14}$$

for each $k\in N$. It follows from (3.13) and (3.14) that thesequence $\{g_k\}$ is bounded and nondecreasing. There, $\lim_{k\to\infty}\|g_k-g_1\|$ exists.

Claim 3. Show that $g_k\to w\in C$ as $k\to\infty$. For $l>k$ by the definition of Q_k, we see that $g_l=P_{Q_l}(g_1)\in Q_k$. From Lemma 2.9, we have

$$\|g_l-g_k\|^2\le\|g_l-g_1\|^2-\|g_k-g_1\|^2.$$

From Claim 2, we obtain that $\{g_k\}$ is a Cauchy sequence. Hence, there exists $w\in C$ such that $g_k\to w$ as $k\to\infty$. In particular, we have

$$\lim_{k\to\infty}\|g_{k+1}-g_k\|=0.\tag{3.15}$$

Claim 4. Show that $h_k^i\to w$ as $k\to\infty$ for all $i=1,...,N$. From the definition of C_k and $g_{k+1}\in C_k$, we have

$$\|s_k-g_{k+1}\|\le\|\bar{u}_k-g_{k+1}\|\le\|g_k-g_{k+1}\|.$$

Thus, from the triangle inequality, one has

$$\begin{aligned}\|s_k-g_k\| &\le \|s_k-g_{k+1}\|+\|g_{k+1}-g_k\| \le 2\|g_k-g_{k+1}\|,\\ \|\bar{u}_k-g_k\| &\le \|\bar{u}_k-g_{k+1}\|+\|g_{k+1}-g_k\| \le 2\|g_k-g_{k+1}\|,\\ \|\bar{u}_k-s_k\| &\le \|\bar{u}_k-g_k\|+\|g_k-s_k\| \le 4\|g_k-g_{k+1}\|.\end{aligned}$$

Three last inequalities togeter with the relation (3.15) imply that

$$\lim_{k\to\infty}\|s_k-g_k\|=\lim_{k\to\infty}\|\bar{u}_k-s_k\|=\lim_{k\to\infty}\|\bar{u}_k-g_k\|=0.\tag{3.16}$$

Hence, from Lemma 2.10 for $g^* \in \Omega$ we have

$$\begin{aligned}
\|\bar{u}_k - g^*\|^2 &= \|\alpha_k^0 g_k + \sum_{i=1}^N \alpha_k^i u_k^i - g^*\|^2 \\
&\le \alpha_k^0\|g_k - g^*\|^2 + \sum_{i=1}^N \alpha_k^i \|u_k^i - g^*\|^2 - \sum_{i=1}^N \alpha_k^0\alpha_k^i\|u_k^i - g_k\|^2 \\
&\le \alpha_k^0\|g_k - g^*\|^2 + \sum_{i=1}^N \alpha_k^i \|g_k - g^*\|^2 - \sum_{i=1}^N \alpha_k^0\alpha_k^i\|u_k^i - g_k\|^2 \\
&= \|g_k - g^*\|^2 - \sum_{i=1}^N \alpha_k^0\alpha_k^i\|u_k^i - g_k\|^2.
\end{aligned}$$

It follows that

$$\sum_{i=1}^N \alpha_k^0\alpha_k^i\|u_k^i - g_k\|^2 \le \|g_k - g^*\|^2 - \|\bar{u}_k - g^*\|^2.$$

By the condition $\liminf_{k\to\infty} \alpha_k^0\alpha_k^i > 0,\ \forall i = 1, ..., N$ and (3.16), we obtain

$$\lim_{k\to\infty} \|u_k^i - g_k\| = 0,\ \forall i = 1, .., N. \tag{3.17}$$

Since $\{g_k\}$ is a Cauchy sequence, $g_k \to w$ and

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \bar{u}_k = \lim_{k\to\infty} u_k^i = w,\ \forall i = 1, .., N,$$

and so

$$\lim_{k\to\infty} A\bar{u}_k = Aw. \tag{3.18}$$

Hence, from Lemma 2.10 for $g^* \in \Omega$ we have

$$\begin{aligned}
\|\bar{w}_k - g*\|^2 &= \|\beta_k^0 A\bar{u}_k + \sum_{i=1}^N \beta_k^i w_k^j - g^*\|^2 \\
&\le \beta_k^0\|A\bar{u}_k - g^*\|^2 + \sum_{i=1}^N \beta_k^i \|w_k^j - g^*\|^2 - \sum_{i=1}^N \beta_k^0\beta_k^i\|w_k^j - A\bar{u}_k\|^2 \\
&\le \beta_k^0\|A\bar{u}_k - g^*\|^2 + \sum_{i=1}^N \beta_k^i \|A\bar{u}_k - g^*\|^2 - \sum_{i=1}^N \beta_k^0\beta_k^i\|w_k^j - A\bar{u}_k\|^2 \\
&= \|A\bar{u}_k - g^*\|^2 - \sum_{i=1}^N \beta_k^0\beta_k^i\|w_k^j - A\bar{u}_k\|^2.
\end{aligned}$$

It follows that

$$\sum_{i=1}^N \beta_k^0\beta_k^j\|w_k^j - A\bar{u}_k\|^2 \le \|A\bar{u}_k - g^*\|^2 - \|\bar{w}_k - g^*\|^2.$$

By the condition $\liminf_{k\to\infty} \beta_k^0\beta_k^j > 0,\ \forall j = 1, ..., N$ and (3.18), we obtain

$$\lim_{k\to\infty} \|w_k^j - A\bar{u}_k\| = 0,\ \forall j = 1, .., N. \tag{3.19}$$

We get which follows from (3.18) that

$$\lim_{k\to\infty} w_k^j = Aw, \ \forall j = 1, .., M. \tag{3.20}$$

From Lemma 2.8 (ii) and the triangle inequality, we have

$$\begin{aligned}(1-2\lambda c_1)\|h_k^i - g_k\|^2 &\leq \|g_k - g^*\|^2 - \|u_k^i - g^*\|^2 \\ &= (\|g_k - g^*\| - \|u_k^i - g^*\|)(\|g_k - g^*\| + \|u_k^i - g^*\|) \\ &\leq \|g_k - u_k^i\|(\|g_k - g^*\| + \|u_k^i - g^*\|).\end{aligned}$$

Thus, from the hypothesis of λ, the boundedness of $\{g_k\}$, $\{u_k^i\}$ and (3.17) we obtain

$$\lim_{k\to\infty} \|h_k^i - g_k\| = 0.$$

Therefore, $h_k^i \to w$ as $k \to \infty$ for all $i = 1, ..., N$.

Claim 5. $w \in \Omega$ and $w = u \ : \ P_\Omega(g_0)$. The proof of Claim 3. By using Claim 2, we also obtain $w \in \cap_{i=1}^N EP(f_i, C)$. Moreover, from Lemma 2.6 for some $r > 0$ we have

$$\begin{aligned}\|T_r^{F_j}(Aw) - Aw\| &\leq \|T_r^{F_j}(Aw) - T_{r_k}^{F_j}(A\bar{u}_k) + \|T_{r_k}^{F_j}(A\bar{u}_k) - A\bar{u}_k\| + \|A\bar{u}_k - Aw\| \\ &\leq \|Aw - A\bar{u}_k\| + \frac{r_k - r}{r_k}\|T_{r_k}^{F_j}(A\bar{u}_k) - A\bar{u}_k\| + \|T_{r_k}^{F_j}(A\bar{u}_k) - A\bar{u}_k\| \\ &+ \|A\bar{u}_k - Aw\| \\ &= 2\|Aw - A\bar{u}_k\| + \frac{r_k - r}{r_k}\|w_k^j - A\bar{u}_k\| + \|w_k^j - A\bar{u}_k\| \to 0,\end{aligned}$$

which is followed from the relations (3.18), (3.19), (3.20) and $r_k \geq d > 0$. Thus, $T_r^{F_j}(Aw) = 0$ or Aw is a fixed point of $T_r^{F_j}$. From Lemma 2.5, we obtain $Aw \in \cap_{j=1}^M EP(F_j, Q)$. Thus, $w \in \Omega$. Finally, from (3.13), $\|g_k - g_1\| \leq \|u - g_1\|$ where $u = P_\Omega(g_1)$. Taking $k \to \infty$ in this inequality, one has $\|w - g_1\| \leq \|u - g_1\|$. From the definition of u, $w = u$. Theorem 3.4 is proved. ∎

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript. This work was supported by the revenue budget in 2021, School of Science, University of Phayao.

References

[1] A. Moudafi, A relaxed alternating CQ algorithm for convex feasibility problems. Nonlinear Anal. TMA 79 (2013) 117–121.

[2] A. Moudafi, E. Al-Shemas, simultaneously iterative methods for split equality problem. Trans. Math. Program. Appl. 1 (0) (2013) 1–11.

[3] A. Moudafi, Split monotone variational inclusions. J. Optim. Theory App. 150 (2) (2011) 275–283.

[4] C.E. Chidume, J.N. Ezeora, On krasonselskii-type algorithm for family of multi-valued strictly pseudo-contractive mappings. Fixed Point Theory and Applications 111 (2014).

[5] D.V. Hieu, L.D. Muu, P.K. Anh, Parallel hybrid extradient methods for pseudomonotone equilibrium problems and nonexpansive mappings. Numer. Algor. (2016) 1–21.

[6] D.V. Hieua, On parallel extradient-proximal methods for solving split equilibrium problems, Mathematical Modelling and Analysis 21 (4) (2016) 478–501.
[7] E. Blum, W. Oettli, From optimization and variational inequalities to eqyilibrium problems, Math. Program. 63 (1994) 123–145.
[8] F. Facchinei, J. S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer, Berlin, 2003.
[9] H. Attouch, A. Cabot A, F. Frankel, J. Peypouquet, Alternating proximal algorithms for constrained variational inequalities. application to domain decomposition for PDE's. Nonlinear Anal. TMA 74 (18) (2011) 7455–7473.
[10] H.H. Bauschke, P.L. Combettes, On Convex Analysis and Monotone Operator Theory in Hilbert Spaces, CMS Books in Mathematics, Springer, New York, 2011.
[11] I.V. Konnov, Combined Relaxation Methods for Variational Inequalities, Springer, Berlin, 2000.
[12] J. Deepho, J. Martínez-Moreno, K. Sitthithakerngkiet, P. Kumam, Convergence analysis of hybrid projection with Cesaro mean method for the split equilibrium and general system of finite variational inequalities, J. Comput. Appl. Math. 318 (2017) 658–673.
[13] Q.L. Dong, X.H. Li, D. Kitkuan, Y.J. Cho, P. Kumam, Some algorithms for classes of split feasibility problems involving paramonotone equilibria and convex optimization, Journal of Inequalities and Applications (2019) 1–23.
[14] J. Deepho, P. Kumam, The hybrid steepest descent method for split variational inclusion and constrain convex minimization problems, Abstract and Applied Analysis 2014 (2014) Article ID: 365203 (13 pages).
[15] J. Deepho, P. Kumam, The modified Mann's type extragradient for solving split feasibility and fixed point problems of Lipschitz asymptotically quasinonexpansive mappings, Fixed Point Theory Appl. 2013 (2013) 349.
[16] J. Deepho, W. Kumam, P. Kumam, A new hybrid projection algorithm for solving the split generalized equilibrium problems and the system of variational inequality problems, J. Math. Model. Algor. 13 (4) (2014) 405–423.
[17] K. Goebel, S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings. Marcel Dekker, New York, 1984.
[18] K.R. Kazmi, S.H. Rizvi, Implicit iterative method for approximating a common solution of split equilibrium problem and fixed point problem for a nonexpansive semigroup. Arab J. Math. Sci. 20 (1) (2014) 57–75.
[19] K.R. Kazmi, S.H. Rizvi, Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem. J. Egyptian Math. Society 21 (1) (2013) 44–51.
[20] K. Nakajo, W. Takahashi, Strongly convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372–379.
[21] K. Sitthithakerngkiet, J. Deepho, P. Kumam, A hybrid viscosity algorithm via modify the hybrid steepest descent method for solving the split variational inclusion and fixed point problems, Appl. Math. Comput. 250 (2015) 986–1001.
[22] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (1) (2005) 117–136.
[23] P.N. Anh, A hybrid extragradient method extended to fixed point problems and equilibrium problems. Optimization 62 (2) (2013) 271–283.
[24] P.T. Vuong, J.J. Strodiot, V.H. Nguyen, On extragradient-viscosity methods for solving

equilibrium and fixed point problems in a Hilbert space, Optimization 64 (2) (2015) 429–451.

[25] S. Chang, L. Wang, X.R. Wang, G. Wang, General split equality equilibrium problems with application to split optimization problems, J. Optim. Theory Appl. 166 (2) (2015) 377–390.

[26] T.D. Quoc, L.D. Muu, N.V. Hien, Extragradient algorithms extended equilibrium problems, Optimization 57 (6) (2008) 749–776.

[27] T.T.V. Nguyen, J.J. Strodiot, V.H. Nguyen, Hybrid methods for solving simultaneously an equilibrium problems and countably many fixed point problems in a Hilbert space, J. Optim. Theory Appl. 160 (3) (2014) 809–831.

[28] W. Kumam, J. Deepho, P. Kumam, Hybrid extradient method for finding a common solution of the split feasibiliy and a system of equilibrium problems, Dynamics of Continuous, Discrete and Impulsive System, DCDIS Series B:Applications Algorithms 21 (6) (2014) 367–388.

[29] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama (2000).

[30] X. Qin, Y.J. Cho, S.M. Kang, Convergence analysis on hybrid projection algorithms for equilibrium problems and variational inequality problems. Model. Math. Anal. 14 (3) (2009) 335–315.

[31] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, Numer. Algorithms 59(0) (2012) 301–323.

[32] Y. Censor, A. Segal, The split common fixed point problemfor directeg operators, J. Convex Anal. 16 (2009) 587–600.

[33] Y. Censor, T. Bortfeld, A. Trofimov, B. Martin, A unified approach for inversion problems in intensity-modulated radiation therapy, Phys. Med. Biol. 51 (10) (2006) 2353–2365.

[34] Z. He, The split equilibrium problems and its convergence algorithms, J. Inequal. Appl. 2012 (2012).

[35] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.