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The Tseng's Extragradient Method for Quasimonotone Variational Inequalities

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Abstract In this paper, we examine the weak convergence of a method to solve classical variational inequality problems involving quasimonotone and Lipschitz continuous operators in a real Hilbert space. The proposed method is inspired by Tseng's extragradient method and uses a simple self-adaptive step size rule that is independent of the Lipschitz constant. We established a weak convergence theorem for a new method without involving any additional projections or knowledge of the Lipschitz constant of a operator. Finally, we present some numerical experiments that show the efficiency and advantages of the proposed method.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. The weak converge of the sequence $\{x_n\}$ to a point x are denoted by $x_n \rightharpoonup x$. For a given closed and convex subset $\mathcal{C} \subset \mathcal{H}$, the variational inequality problem denoted by $VI(\mathcal{C}, \mathcal{G})$ is to find $x^* \in \mathcal{C}$ such that

$$\left\langle \mathcal{G}(x^*), y - x^* \right\rangle \geq 0, \ \forall\, y \in \mathcal{C}, \tag{VIP}$$

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where $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be an operator. For a closed and convex $\mathcal{C} \subset \mathcal{H}$, the metric projection $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is defined, for all $x \in \mathcal{H}$ such that

$$P_{\mathcal{C}}(x) = \arg\min\{\|x - y\| : y \in \mathcal{C}\}.$$

Furthermore, $\mathbb{R}$, $\mathbb{N}$ is the set of real and natural numbers, respectively. It is useful to note that the problem (VIP) is equivalent to solving the following problem:

$$\text{Find } x^* \in \mathcal{C} \text{ such that } x^* = P_{\mathcal{C}}[x^* - \lambda \mathcal{G}(x^*)],$$

where λ can be any positive real number. The theory of variational inequalities has been used as an important tool to study a wide range of topics, i.e., physics, engineering, economics and optimization theory. Stampacchia [27] presented this problem in 1964, and it is also well-established that the problem (VIP) is a crucial problem in non-linear analysis. This is a significant mathematical problem that encompasses several important topics in applied mathematics, including network equilibrium problems, necessary optimality conditions, complementarity problems, and systems of nonlinear equations (for more details [6, 9–11, 17]). On the other hand, the projection methods are important iterative methods to solve variational inequalities. Many iterative methods for solving variational inequalities have been proposed and analyzed (see for more details [3, 4, 8, 12–15, 18–20, 29–31, 41]) and others in [5, 7, 16, 22–26, 32, 33, 35–40]. The extragradient method was introduced by Korpelevich [12] and Antipin [1]. The method is of the form

$$\begin{cases} x_0 \in \mathcal{C}, \\ y_n = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(x_n)], \\ x_{n+1} = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(y_n)], \end{cases} \tag{1.1}$$

where $0 < \lambda < \frac{1}{L}$ and L is Lipschitz constant of an operator $\mathcal{G}$. In view of this method, we use two projections on the underlying set $\mathcal{C}$ over each iteration. Of course, if the feasible set $\mathcal{C}$ has a complicated structure, this can have an impact on the computational effectiveness of the method used. Here, we restrict our interest to presenting some methods which can address this drawback. The first is the following subgradient extragradient method due to Censor et al. [3]. This method takes the form

$$\begin{cases} x_0 \in \mathcal{C}, \\ y_n = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(x_n)], \\ x_{n+1} = P_{\mathcal{H}_n}[x_n - \lambda \mathcal{G}(y_n)]. \end{cases} \tag{1.2}$$

where $0 < \lambda < \frac{1}{L}$ and

$$\mathcal{H}_n = \{z \in \mathcal{H} : \langle x_n - \lambda \mathcal{G}(x_n) - y_n, z - y_n \rangle \leq 0\}.$$

In this article, we concentrate on the Tseng's extragradient method [31] that uses only one projection for each iteration:

$$\begin{cases} x_0 \in \mathcal{C}, \\ y_n = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(x_n)], \\ x_{n+1} = y_n + \lambda\big[\mathcal{G}(x_n) - \mathcal{G}(y_n)\big]. \end{cases} \tag{1.3}$$

where $0 < \lambda < \frac{1}{L}$.

The main objective of this research is to study quasimonotone variational inequalities in infinite dimensional Hilbert spaces. To show that the iterative sequence generated by Tseng's extragradient method for the solution of quasimonotone variational inequalities converges weakly to a solution. The proposed method is based on the projection method described in [31]. each iteration, the method only requires solving one projection on the feasible set. If some suitable conditions are imposed on the control parameters, the iterative sequences generated by our methods converge weakly to some solution to the problem. We also present examples to explain the computational performance of the new method.

The paper is organized in the following manner. In Sect. 2, some preliminary results were presented. Sect. 3 provides a new algorithm and its convergence study. Finally, Sect. 4 presents some numerical results to point out the practical efficiency of the proposed method.

2. Preliminaries

For all $x, y \in \mathcal{H}$, we have

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

A *metric projection* $P_{\mathcal{C}}(x)$ of $x \in \mathcal{H}$ is defined by

$$P_{\mathcal{C}}(x) = \arg\min\{\|x-y\| : y \in \mathcal{C}\}.$$

Lemma 2.1. *Assume $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ be a metric projection from $\mathcal{H}$ onto $\mathcal{C}$. Then*

(i) *Let $x \in \mathcal{C}$ and $y \in \mathcal{H}$, we have*

$$\|x - P_{\mathcal{C}}(y)\|^2 + \|P_{\mathcal{C}}(y) - y\|^2 \leq \|x-y\|^2;$$

(ii) *$z = P_{\mathcal{C}}(x)$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \ \forall y \in \mathcal{C};$$

(iii) *For $y \in \mathcal{C}$ and $x \in \mathcal{H}$*

$$\|x - P_{\mathcal{C}}(x)\| \leq \|x - y\|.$$

Lemma 2.2. [2] *For any $x, y \in \mathcal{H}$ and $\ell \in \mathbb{R}$. Then*

(i)

$$\|\ell x + (1-\ell)y\|^2 = \ell\|x\|^2 + (1-\ell)\|y\|^2 - \ell(1-\ell)\|x-y\|^2;$$

(ii)

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Lemma 2.3. [21] *Let $\mathcal{C}$ be a nonempty set of $\mathcal{H}$ and $\{x_n\}$ be a sequence in $\mathcal{H}$ such that*

(*i*) *for every $x \in \mathcal{C}$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*

(*ii*) *each sequentially weak cluster point of $\{x_n\}$ within $\mathcal{C}$.*

Then, $\{x_n\}$ converges weakly to a point in $\mathcal{C}$.

Lemma 2.4. [28] *Let $\mathcal{G} : \mathcal{C} \to \mathcal{H}$ be a pseudomonotone and continuous operator. Then, x^* is a solution of the problem (VIP) if and only if x^* is a solution of the following problem:*

$$\textit{Find } x \in \mathcal{C} \textit{ such that } \langle \mathcal{G}(y), y - x\rangle \geq 0, \ \forall y \in \mathcal{C}.$$

3. Main Results

In this section, we present an iterative algorithm for solving quasimonotone variational inequalities that is based on Tseng's extragradient method that does not require either knowledge of the Lipschitz constant of the operator or additional projection.

Algorithm 1

Step 0: Choose $x_0 \in \mathcal{C}$ and $0 < \lambda < \frac{1}{L}$.
Step 1: Compute

$$y_n = P_{\mathcal{C}}(x_n - \lambda \mathcal{G}(x_n)).$$

If $x_n = y_n$, then STOP and y_n is a solution. Otherwise, go to **Step 2**.
Step 2: Compute

$$x_{n+1} = y_n + \lambda\big[\mathcal{G}(x_n) - \mathcal{G}(y_n)\big].$$

Set $n = n + 1$ and go back to **Step 1**.

In order to prove the weak convergence, it is considered that the following conditions have been satisfied:

($\mathcal{G}1$) The solution set of problem (VIP) is denoted by Ω is nonempty;
($\mathcal{G}2$) An operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is said to quasimonotone if

$$\big\langle \mathcal{G}(x), y - x \big\rangle > 0 \Longrightarrow \big\langle \mathcal{G}(y), y - x \big\rangle \geq 0, \ \forall\, x, y \in \mathcal{C}; \tag{QM}$$

($\mathcal{G}3$) An operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is *Lipschitz continuous* with constant $L > 0$ such that

$$\|\mathcal{G}(x) - \mathcal{G}(y)\| \leq L\|x - y\|, \ \forall\, x, y \in \mathcal{C}; \tag{LC}$$

($\mathcal{G}4$) An operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is *sequentially weakly continuous* if $\{\mathcal{G}(x_n)\}$ converges weakly to $\mathcal{G}(x)$ for every sequence $\{x_n\}$ converges weakly to x.

Lemma 3.1. *Suppose that $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ satisfies the conditions ($\mathcal{G}1$)-($\mathcal{G}4$) and sequence $\{x_n\}$ generated by Algorithm 1. Then, we have*

$$\|x_{n+1} - x^*\|^2 \leq \big\|x_n - x^*\big\|^2 - \big(1 - \lambda^2 L^2\big)\big\|x_n - y_n\big\|^2.$$

Proof. Since $x^* \in \Omega$, we have

$$\begin{aligned}
&\big\|x_{n+1} - x^*\big\|^2 \\
&= \big\|y_n + \lambda[\mathcal{G}(x_n) - \mathcal{G}(y_n)] - x^*\big\|^2 \\
&= \big\|y_n - x^*\big\|^2 + \lambda^2\big\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\big\|^2 + 2\lambda\langle y_n - x^*, \mathcal{G}(x_n) - \mathcal{G}(y_n)\rangle \\
&= \big\|y_n + x_n - x_n - x^*\big\|^2 + \lambda^2\big\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\big\|^2 + 2\lambda\langle y_n - x^*, \mathcal{G}(x_n) - \mathcal{G}(y_n)\rangle \\
&= \big\|y_n - x_n\big\|^2 + \big\|x_n - x^*\big\|^2 + 2\langle y_n - x_n, x_n - x^*\rangle \\
&\quad + \lambda^2\big\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\big\|^2 + 2\lambda\langle y_n - x^*, \mathcal{G}(x_n) - \mathcal{G}(y_n)\rangle \\
&= \big\|x_n - x^*\big\|^2 + \big\|y_n - x_n\big\|^2 + 2\langle y_n - x_n, y_n - x^*\rangle + 2\langle y_n - x_n, x_n - y_n\rangle \\
&\quad + \lambda^2\big\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\big\|^2 + 2\lambda\langle y_n - x^*, \mathcal{G}(x_n) - \mathcal{G}(y_n)\rangle.
\end{aligned} \tag{3.1}$$

It is given that

$$y_n = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(x_n)]$$

and further it implies that

$$\langle x_n - \lambda \mathcal{G}(x_n) - y_n, y - y_n \rangle \leq 0, \ \forall\, y \in \mathcal{C}. \tag{3.2}$$

Thus, we have

$$\langle x_n - y_n, x^* - y_n \rangle \leq \lambda \langle \mathcal{G}(x_n), x^* - y_n \rangle. \tag{3.3}$$

Combining expressions (3.1) and (3.3), we have

$$\begin{aligned}
&\big\|x_{n+1} - x^*\big\|^2 \\
&\leq \big\|x_n - x^*\big\|^2 + \big\|y_n - x_n\big\|^2 + 2\lambda\langle \mathcal{G}(x_n), x^* - y_n\rangle - 2\langle x_n - y_n, x_n - y_n\rangle \\
&\quad + \lambda^2 \big\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\big\|^2 - 2\lambda\langle \mathcal{G}(x_n) - \mathcal{G}(y_n), x^* - y_n\rangle \\
&= \big\|x_n - x^*\big\|^2 - \big\|x_n - y_n\big\|^2 + \lambda^2\big\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\big\|^2 - 2\lambda\langle \mathcal{G}(y_n), y_n - x^*\rangle.
\end{aligned} \tag{3.4}$$

It is given that x^* is the solution of the problem (VIP) implies that

$$\langle \mathcal{G}(x^*), y - x^* \rangle \geq 0, \ \forall\, y \in \mathcal{C}.$$

It implies that

$$\langle \mathcal{G}(y), y - x^* \rangle \geq 0, \ \forall\, y \in \mathcal{C}.$$

Substituting $y = y_n \in \mathcal{C}$, we have

$$\langle \mathcal{G}(y_n), y_n - x^* \rangle \geq 0. \tag{3.5}$$

From expressions (3.4) and (3.5), we obtain

$$\begin{aligned}
\big\|x_{n+1} - x^*\big\|^2 &\leq \big\|x_n - x^*\big\|^2 - \big\|x_n - y_n\big\|^2 + \lambda^2 L^2 \big\|x_n - y_n\big\|^2 \\
&= \big\|x_n - x^*\big\|^2 - \big(1 - \lambda^2 L^2\big)\big\|x_n - y_n\big\|^2.
\end{aligned} \tag{3.6}$$

∎

Theorem 3.2. *Assume that an operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ satisfies the conditions $(\mathcal{G}1)$–$(\mathcal{G}4)$. Then, the sequence $\{x_n\}$ generated by the Algorithm 1 converges weakly to $x^* \in \Omega$.*

Proof. By using Lemma 3.1, we have

$$\big\|x_{n+1} - x^*\big\|^2 \leq \big\|x_n - x^*\big\|^2 - \big(1 - \lambda^2 L^2\big)\big\|x_n - y_n\big\|^2. \tag{3.7}$$

Since $0 < \lambda < \frac{1}{L}$, thus we obtain

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{3.8}$$

Thus, the expression (3.8) implies that

$$\lim_{n\to\infty} \|x_n - x^*\| = l, \text{ for some finite } l \geq 0. \tag{3.9}$$

From expression (3.7), we have

$$\big(1 - \lambda^2 L^2\big)\big\|x_n - y_n\big\|^2 \leq \big\|x_n - x^*\big\|^2 - \big\|x_{n+1} - x^*\big\|^2. \tag{3.10}$$

Due to the existence of $\lim_{n\to\infty} \|x_n - x^*\| = l$, we deduce that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.11}$$

Thus, we have

$$\lim_{n\to\infty} \|y_n - x^*\| = l. \tag{3.12}$$

It follows that

$$\|x_{n+1} - y_n\| = \|y_n + \lambda[\mathcal{G}(x_n) - \mathcal{G}(y_n)] - y_n\| \le \lambda L\|x_n - y_n\|.$$

The above expression implies that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.13}$$

It follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| \le \lim_{n\to\infty} \|x_n - y_n\| + \lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.14}$$

This implies that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Now, we show that the sequence $\{x_n\}$ converges weakly to $x^* \in \Omega$. Indeed, since $\{x_n\}$ is bounded, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Next, we prove that $\hat{x} \in \Omega$. Indeed, we have

$$y_{n_k} = P_{\mathcal{C}}[x_{n_k} - \lambda_{n_k}\mathcal{G}(x_{n_k})]$$

that is equivalent to

$$\langle x_{n_k} - \lambda_{n_k}\mathcal{G}(x_{n_k}) - y_{n_k}, y - y_{n_k}\rangle \le 0,\ \forall\, y \in \mathcal{C}. \tag{3.15}$$

The inequality mentioned above implies that

$$\langle x_{n_k} - y_{n_k}, y - y_{n_k}\rangle \le \lambda_{n_k}\langle \mathcal{G}(x_{n_k}), y - y_{n_k}\rangle,\ \forall\, y \in \mathcal{C}. \tag{3.16}$$

Thus, we obtain

$$\frac{1}{\lambda_{n_k}}\langle x_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle \mathcal{G}(x_{n_k}), y_{n_k} - x_{n_k}\rangle \le \langle \mathcal{G}(x_{n_k}), y - x_{n_k}\rangle,\ \forall\, y \in \mathcal{C}. \tag{3.17}$$

Since $\min\left\{\frac{\mu}{L}, \lambda_1\right\} \le \lambda \le \lambda_1$ and $\{x_{n_k}\}$ is a bounded sequence. By the use of $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$ and $k \to \infty$ in (3.17), we obtain

$$\liminf_{k\to\infty} \langle \mathcal{G}(x_{n_k}), y - x_{n_k}\rangle \ge 0,\ \forall\, y \in \mathcal{C}. \tag{3.18}$$

Moreover, we have

$$\begin{aligned}&\langle \mathcal{G}(y_{n_k}), y - y_{n_k}\rangle \\ &= \langle \mathcal{G}(y_{n_k}) - \mathcal{G}(x_{n_k}), y - x_{n_k}\rangle + \langle \mathcal{G}(x_{n_k}), y - x_{n_k}\rangle + \langle \mathcal{G}(y_{n_k}), x_{n_k} - y_{n_k}\rangle.\end{aligned} \tag{3.19}$$

Since $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$ and $\mathcal{G}$ is L-Lipschitz continuity on $\mathcal{H}$ implies that

$$\lim_{k\to\infty} \|\mathcal{G}(x_{n_k}) - \mathcal{G}(y_{n_k})\| = 0, \tag{3.20}$$

which together with expressions (3.19) and (3.20), we obtain

$$\liminf_{k\to\infty} \langle \mathcal{G}(y_{n_k}), y - y_{n_k}\rangle \ge 0,\ \forall\, y \in \mathcal{C}. \tag{3.21}$$

To prove further, let us take a positive sequence $\{\epsilon_k\}$ that is convergent to zero and decreasing. For each $\{\epsilon_k\}$ we denote by m_k the smallest positive integer such that

$$\langle \mathcal{G}(x_{n_i}), y - x_{n_i}\rangle + \epsilon_k > 0,\ \forall\, i \ge m_k, \tag{3.22}$$

where the existence of m_k follows from (3.21). Since sequence $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{m_k\}$ is increasing.

Case I: If there exists a subsequence $\{x_{n_{m_{k_j}}}\}$ subsequence of $\{x_{n_{m_k}}\}$ such that $\mathcal{G}(x_{n_{m_{k_j}}}) = 0$ $(\forall j)$. Let $j \to \infty$, we obtain

$$\langle \mathcal{G}(\hat{x}), y - \hat{x}\rangle = \lim_{j\to\infty} \langle \mathcal{G}(x_{n_{m_{k_j}}}), y - \hat{x}\rangle = 0. \tag{3.23}$$

Thus, $\hat{x} \in \mathcal{C}$ and imply that $\hat{x} \in \Omega$.

Case II: If there exits $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \geq N_0$, $\mathcal{G}(x_{n_{m_k}}) \neq 0$. Consider that

$$\Upsilon_{n_{m_k}} = \frac{\mathcal{G}(x_{n_{m_k}})}{\|\mathcal{G}(x_{n_{m_k}})\|^2}, \ \forall n_{m_k} \geq N_0. \tag{3.24}$$

Due to the above definition, we obtain

$$\langle \mathcal{G}(x_{n_{m_k}}), \Upsilon_{n_{m_k}} \rangle = 1, \ \forall n_{m_k} \geq N_0. \tag{3.25}$$

Moreover, expressions (3.22) and (3.25), for all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{G}(x_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - x_{n_{m_k}} \rangle > 0. \tag{3.26}$$

Since $\mathcal{G}$ is quasimonotone, then

$$\langle \mathcal{G}(y + \epsilon_k \Upsilon_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - x_{n_{m_k}} \rangle > 0. \tag{3.27}$$

For all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{G}(y), y - x_{n_{m_k}} \rangle \geq \langle \mathcal{G}(y) - \mathcal{G}(y + \epsilon_k \Upsilon_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - x_{n_{m_k}} \rangle - \epsilon_k \langle \mathcal{G}(y), \Upsilon_{n_{m_k}} \rangle. \tag{3.28}$$

Due to $\{x_{n_k}\}$ weakly converges to $\hat{x} \in \mathcal{C}$ through $\mathcal{G}$ is sequentially weakly continuous on the set $\mathcal{C}$, we get $\{\mathcal{G}(x_{n_k})\}$ weakly converges to $\mathcal{G}(\hat{x})$. Suppose that $\mathcal{G}(\hat{x}) \neq 0$, we have

$$\|\mathcal{G}(\hat{x})\| \leq \liminf_{k \to \infty} \|\mathcal{G}(x_{n_k})\|. \tag{3.29}$$

Since $\{x_{n_{m_k}}\} \subset \{x_{n_k}\}$ and $\lim_{k\to\infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k\to\infty} \|\epsilon_k \Upsilon_{n_{m_k}}\| = \lim_{k\to\infty} \frac{\epsilon_k}{\|\mathcal{G}(x_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{G}(\hat{x})\|} = 0. \tag{3.30}$$

Next, consider $k \to \infty$ in (3.28), we obtain

$$\langle \mathcal{G}(y), y - \hat{x} \rangle \geq 0, \ \forall y \in \mathcal{C}. \tag{3.31}$$

Thus, we infer that $\hat{x} \in \Omega$. Therefore, we proved that:

(1) for every $x^* \in \Omega$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists;
(2) every sequential weak cluster point of the sequence $\{x_n\}$ is in Ω.

By Lemma 2.3, the sequence $\{x_n\}$ converges weakly to $x^* \in \Omega$. ∎

Next, we introduce a variant of Algorithm 1 in which the constant step size λ is chosen adaptively and thus produced a sequence λ_n that does not require the knowledge of the Lipschitz-type constant L.

Lemma 3.3. *The sequence $\{\lambda_n\}$ generated by (3.32) is decreasing monotonically and converges to $\lambda > 0$.*

Proof. It is given that $\mathcal{G}$ is Lipschitz-continuous with constant $L > 0$. Let $\mathcal{G}(x_n) \neq \mathcal{G}(y_n)$ such that

$$\begin{aligned} \frac{\mu\|x_n - y_n\|}{\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\|} &\geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} \\ &\geq \frac{\mu}{L}. \end{aligned} \tag{3.33}$$

The above expression implies that $\lim_{n\to\infty} \lambda_n = \lambda$. ∎

Algorithm 2

Step 0: Choose $x_0 \in \mathcal{C}$, $\mu \in (0,1)$ and $\lambda_0 > 0$.
Step 1: Compute

$$y_n = P_{\mathcal{C}}(x_n - \lambda_n \mathcal{G}(x_n)).$$

If $x_n = y_n$, then STOP and y_n is a solution. Otherwise, go to **Step 2**.
Step 2: Compute

$$x_{n+1} = y_n + \lambda_n\big[\mathcal{G}(x_n) - \mathcal{G}(y_n)\big].$$

Set $n = n+1$ and go back to **Step 1**.
Step 3: Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|x_n - y_n\|}{\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\|}\right\} & \text{if} \quad \mathcal{G}(x_n) - \mathcal{G}(y_n) \neq 0, \\ \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.32}$$

Lemma 3.4. *Suppose that $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ satisfies conditions ($\mathcal{G}1$)-($\mathcal{G}4$) and sequence x_n generated by Algorithm 1. Then, we have*

$$\|x_{n+1} - x^*\|^2 \leq \big\|x_n - x^*\big\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\big\|x_n - y_n\big\|^2.$$

4. Numerical Illustrations

The computational results of the proposed schemes are described in this section, in contrast to some related work in the literature and also in the analysis of how variations in control parameters affect the numerical effectiveness of the proposed algorithms. All computations are done in MATLAB R2018b and run on HP i 5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Consider that $\mathcal{H} = l_2$ is a real Hilbert space with sequences of real numbers satisfying the following condition

$$\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2 + \cdots < +\infty. \tag{4.1}$$

Assume that $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\mathcal{G}(x) = (5 - \|x\|)x, \ \forall\, x \in \mathcal{C},$$

where $\mathcal{C} = \{x \in \mathcal{H} : \|x\| \leq 3\}$. It is easy to see that $\mathcal{G}$ is weakly sequentially continuous on $\mathcal{H}$ and $\Omega = \{0\}$. For any $x, y \in \mathcal{C}$, we have

$$\begin{aligned} \big\|\mathcal{G}(x) - \mathcal{G}(y)\big\| &= \big\|(5 - \|x\|)x - (5 - \|y\|)y\big\| \\ &= \big\|5(x-y) - \|x\|(x-y) - (\|x\| - \|y\|)y\big\| \\ &\leq 5\|x-y\| + \|x\|\|x-y\| + \big|\|x\| - \|y\|\big|\|y\| \\ &\leq 5\|x-y\| + 3\|x-y\| + 3\|x-y\| \\ &= 11\|x-y\|. \end{aligned} \tag{4.2}$$

Hence $\mathcal{G}$ is L-Lipschitz continuous with $L = 11$. For any $x, y \in \mathcal{C}$ let $\big\langle \mathcal{G}(x), y - x\big\rangle > 0$ such that

$$(5 - \|x\|)\big\langle x, y - x\big\rangle > 0.$$

Since $\|x\| \leq 3$ implies that

$$\langle x, y-x \rangle > 0.$$

Thus, we have

$$\begin{aligned} \langle \mathcal{G}(y), y-x \rangle &= (5-\|y\|)\langle y, y-x \rangle \\ &\geq (5-\|y\|)\langle y, y-x \rangle - (5-\|y\|)\langle x, y-x \rangle \\ &\geq 2\|x-y\|^2 \geq 0. \end{aligned} \tag{4.3}$$

Thus, we shown that $\mathcal{G}$ is quasimonotone on $\mathcal{C}$. A projection on the set $\mathcal{C}$ is computed explicitly as follows:

$$P_{\mathcal{C}}(x) = \begin{cases} x \quad \text{if} \quad \|x\| \leq 3, \\ \frac{3x}{\|x\|}, \quad \text{otherwise.} \end{cases}$$

The control conditions have been taken as follows: (1) (Algorithm 1): $\lambda_0 = \frac{5}{11}$; (2) (Algorithm 2): $\lambda_0 = \frac{5}{11}, \mu = 0.44$.

TABLE 1. Numerical results values for Example 4.1.

	Number of Iterations		Elapsed time in seconds	
x_0	**Algorithm 1**	**Algorithm 2**	**Algorithm 1**	**Algorithm 2**
$(3, 3, \cdots, 3_{1000}, 0, 0, \cdots)$	47	53	5.563843	7.145738
$(1, 2, \cdots, 1000, 0, 0, \cdots)$	58	59	6.563731	8.352731
$(5, 5, \cdots, 5_{10000}, 0, 0, \cdots)$	63	68	6.474532	9.563846
$(100, 100, \cdots, 100_{10000}, 0, 0, \cdots)$	84	79	11.56463	12.56383

CONCLUSION

In this study, we considered a weak convergence result for the variational inequalities problem involving quasimonotone and the Lipschitz continuous operator, but the Lipschitz constant is unknown. We modify the extragradient method with a natural step size rule. The weak convergence result is proved without any provision for additional projections.

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