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Analysis of a Caputo HIV and Malaria Co-Infection Epidemic Model

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Abstract In this paper, we investigate a fractional-order compartmental HIV and Malaria co-infection epidemic model using the Caputo derivative. The existence and uniqueness of the solution to the proposed fractional-order model were investigated using fixed point theorem techniques. To demonstrate that the proposed fractional-order model is both mathematically and epidemiologically well-posed, we compute the model's positivity and boundedness, which is an important feature in epidemiology. Finally, we analyze the dynamic behavior of each of the state variables using a recent and powerful computational technique known as the fractional Euler method.

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1. Introduction

HIV (Human immunodeficiency virus) is the virus as the name describe that destroy CD4 cell in the body which lead to a condition called AIDS (Acquired immunodeficiency Syndrome) (stage 3 HIV) which weaken the immune system of the body. As of the end of 2019, over 38 million people were infected with HIV. The destruction of the CD4 cell can lead to the presence of opportunistic infection caused by other viruses, bacteria or fungi [1, 26]. This virus is only transmitted between human beings, and these infections cause several health problems and even lead to the death of a person [20].

Malaria is among the biggest disease treating humanity, which is spread through the bite of infected female Anopheles mosquito cause by Plasmodium parasites. Malaria symptoms appear in a non-immune person after 10-15 days of an infectious mosquito bite, which include fever, headache, and chills. Two decades of research have shown that HIV-related immunosuppression is correlated with increased malaria infection, burden, and treatment failure, and with complicated malaria, irrespective of immune status, as in families living in the same household with HIV-infected individual, the higher rates of malaria infection on the HIV-positive individual is higher than any family member without an HIV-infection [25]. The author in [18] developed a mathematical model on HIV and malaria co-infection which tries to identify the application of the control strategy on the co-infection. Similarly, [32] develop a mathematical model which concludes that the best strategy reduce malaria and HIV co-infection is the combine malaria prevention and ARV treatment in a population. Also, [32] conclude that the implementation of multiple interventions such as co-trimoxazole prophylaxis, ITNs, and ARV may reduce the effect of malaria in HIV-infected patients.

The fractional calculus is one of the most tantivy frim area of analysis which turns over derivatives and integrals of real and complex orders and thus it generalizes the traditional calculus. This calculus has been capturing the curiosity of a big number of researchers by the marvelous results they obtained when they replaced integer-order derivatives with fractional ones for the sake of a better understanding of the real-world phenomena [14, 19, 21, 23, 24]. The most well-known and used fractional derivatives are the Riemann-Liouville and Caputo fractional derivatives. These two derivatives are defined for different spaces but they coincide in some cases. When one considers Riemann-Liouville initial value problems the initial conditions contain fractional derivatives given at the initial point. These conditions have no loud and clear physical interpretation. Nevertheless, when a Caputo initial value problem is being considered, the initial conditions are nothing but derivatives of integer orders described at the initial point and similar to the traditional initial conditions [27]. Amidst the qualitative properties of the solutions of fractional differential systems which need handling, the issue of the existence and uniqueness of these solutions are of the most significant. There is a big number of articles analyzing the existence and uniqueness of a variety of differential equations involving Caputo or Caputo-like fractional derivatives using the techniques of fixed point theorems, we reefer's to [4–7, 11–13] and the references cited therein.

A huge number of classical and fractional-order mathematical models have been developed, analyzed, and applied which motivate our understanding and predictive ability

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about a variety of infectious diseases qualitatively and quantitatively, see for example, [2, 8–10, 15–17, 28–30] and the references cited therein. Recently, Idris et al. [3] investigated the dynamics of a combined HIV-COVID-19 co-infection model in frame of Atangana-Baleanu Caputo derivative. The authors derived the existence and uniqueness of solution via Schaefer and Banach fixed point theorems. A fractional-order COVID-19 model to investigate the effects of preventive measures, future outbreaks, and potential control strategies were proposed by Amjad et al. [31].

In this paper, we present a Caputo fractional-order model to investigate the complex behavior of an HIV/Malaria epidemic. Existence and uniqueness of solutions were establish using the techniques of fixed point theorems. Consequently, the results of the fractional-order derivative are simulated and analyzed in all compartments. Several simulations are performed to obtain a comprehensive understanding of the mechanisms of the proposed model.

2. Preliminaries Concepts

In this section, we will review some basic definitions and fundamental concepts of the Caputo fractional operator, which will be used later in the theoretical study of the fractional-order HIV and Malaria co-infection epidemic model.

Definition 2.1. [27] Suppose $r \in \mathbb{R}_+$. The operator I_a^r, defined on $L^1[a,b]$ given by

$$I_a^r f(t) = \frac{1}{\Gamma(r)} \int_a^t \frac{f(t)}{(t-x)^{1-r}} dx, \tag{2.1}$$

for $a \leq t \leq b$, is referred to the fractional integral of order r in the sense of Riemann-Liouville where $\Gamma(\cdot)$ denotes the gamma function defined by

$$\Gamma(u) = \int_0^\infty x^{u-1} e^{-x} dx, u \in \mathbb{C}/\{0, 1, 2, \ldots\}.$$

Definition 2.2. [27] Let $n-1 < r \leq n$ and $f \in C^n[a,b],\ n \in \mathbb{N}$. The operator

$${}^C D_a^r f(t) = \frac{1}{\Gamma(n-r)} \int_a^t \frac{1}{(t-x)^{1-n+r}} \frac{d^n}{dt^n} f(x) dx, \quad t > a, \tag{2.2}$$

is referred to the fractional derivative of order r in the sense of Caputo. Note that if $r \to n$ then ${}^C D_a^r f(t) = \frac{d^n}{dt^n} f(t)$.

Lemma 2.3. *[27] Let $f : [a,b] \to \mathbb{R}$ be a continuous function for any $z \in C^1[a,b]$. Then, $z(t)$ is a solution of the following Caputo fractional-order differential equation:*

$$\begin{cases} {}^C D_a^r z(t) = f(t),\ t \in [a,b],\ 0 < r \leq 1, \\ z(a) = z_a, \end{cases}$$

if and only if $z(t)$ satisfies the integral equation of the form:

$$z(t) = z_a - \frac{1}{\Gamma(r)} \int_a^t \frac{f(t)}{(t-x)^{1-r}} dx.$$

3. Fractional-order HIV and Malaria co-infection model

In this section, we consider a mathematical compartmental model based on HIV and malaria co-infection which shows the dynamics of malaria on HIV negative and positive human population and the vector population. Specifically, the population is divided into two, Human population and vector population. In the Human population $N(t)$ at time t is subdivided into six (5) compartments, namely: $S(t)$ the susceptible population, which are assumed not have a double infection at a time; $I_m(t)$ population infected with malaria; I_h population infected with HIV; I_{hm} population infected with HIV and malaria simultaneously and A population infected with AID. The vector population $N(t)$ is subdivided into two (2) compartment, namely: S_v susceptible vector population and I_v infected vector population.

The integer-order model as proposed in [18], consists of seven systems of the nonlinear first-order differential equation. Hence, the proposed fractional-order model in the setting of Caputo fractional derivatives is of the form:

$$\begin{aligned}
{}^{C}D^{r}_{0,t}S(t) &= \Lambda - \beta_m S I_v - \beta_h S I_h + \alpha_1 I_m - \delta S,\\
{}^{C}D^{r}_{0,t}I_m(t) &= \beta_m S I_v - \alpha_1 I_m - \phi_1\beta_h I_m I_h - \delta I_m,\\
{}^{C}D^{r}_{0,t}I_h(t) &= \beta_h S I_h + \alpha_2 I_{hm} - \phi_2\beta_m I_h I_v - \gamma_1 I_h - \delta I_h,\\
{}^{C}D^{r}_{0,t}I_{hm}(t) &= \phi_1\beta_h I_m I_h + \phi_2\beta_m I_h I_v - \alpha_2 I_{hm} - (\delta+\mu_d) I_{hm} - \gamma_2 I_{hm},\\
{}^{C}D^{r}_{0,t}A(t) &= \gamma_1 I_h + \gamma_2 I_{hm} - (\delta+\mu_a) A,\\
{}^{C}D^{r}_{0,t}S_v(t) &= \Lambda_v - \beta_v S_v(I_m + I_{hm}) - \delta_v S_v,\\
{}^{C}D^{r}_{0,t}I_v(t) &= \beta_v S_v(I_m + I_{hm}) - \delta_v I_v.
\end{aligned} \tag{3.1}$$

with the initial conditions

$$S(0)\geq 0,\ I_m(0)\geq 0,\ I_h(0)\geq 0,\ I_{hm}(0)\geq 0,\ A(0)\geq 0, S_v(0)\geq 0 \text{ and } I_v(0)\geq 0, \tag{3.2}$$

in which the descriptions of state variables as well as the parameters associated with the model is given respectively, in Tables 1 and 2.

It is paramount to note that the fractional-order models are important and useful in understanding the dynamics of real-word phenomena than the classical models due to hereditary properties and description of memory. Also, the integer-order derivative does not explore the dynamics between two different points.

Table 1. State variables

Compartment	Description
S	Susceptible people
I_m	Infected people with malaria
I_h	Infected people with HIV
I_{hm}	Infected people with malaria and HIV
A	Infected people with AIDS
S_v	Susceptible vector population
I_v	Infected vector population

TABLE 2. Parameters with their description.

Parameters	Description
Λ	Host population recruitment rate
Λ_v	Vector population recruitment rate Q
β_h	HIV infection rate
β_m	Host infection rate
β_v	Vector infection rate
α_1	Malaria recovery rate
α_2	Malaria recovery rate among HIV-malaria
ϕ_1	Rate of Progression from I_m to I_{hm}
ϕ_2	Rate of Progression from I_h to I_{hm}
γ_1	Rate of Progression from I_h to A
γ_2	Rate of Progression from I_{hm} to A
δ	Host natural death rate
δ_v	Vector natural death rate
μ_a	AIDS disease induced death rate
μ_d	HIV-malaria disease induced death rate

4. Qualitative Analysis of the proposed fractional-order Model

This section is devoted to investigating the existence and uniqueness of the proposed fractional-order epidemic model (3.1) via the techniques of fixed point theorems. Besides, the positivity and boundedness of the model which shows the model is both mathematically and epidemiologically well-posed are presented.

4.1. Existence and Uniqueness Results

Utilizing the concepts of fixed point theorems, this subsection is devoted to studying the existence and uniqueness of solutions to the fractional-order HIV and Malaria epidemic co-infection model (3.1). The analysis is based on the followings facts, that is for a given Banach space $\mathcal{S}$ with $t \in [0,T]$, $S, I_m, I_h, I_{hm}, A, S_v, I_v \in C(G,\mathcal{S}) \cap L^1_{loc}(G,\mathcal{S})$ and $G = \{(t,z) : t \in [0,T], z \in \mathcal{B}(0,\kappa)\}$ for some $T, \kappa > 0$.

Theorem 4.1. *Let $\mathcal{S}$ be a Banach space and $\mathcal{B}$ be a closed, convex and bounded subset of $\mathcal{S}$. If $\Phi : \mathcal{B} \to \mathcal{B}$ is a condensing map, then Φ has a fixed point in $\mathcal{B}$.*

Note that the given initial value problem on the cylinder $G = \{(t,z) \in \mathbb{R} \times \mathcal{S} : t \in [0,T], z \in \mathcal{B}(0,\kappa)\}$ for some $T, \kappa > 0$ and there exits $q \in (0,r)$, $S^1, S^2, L, L_1 \in L_{\frac{1}{q}}([0,T],\mathbb{R}_+)$ and the functions $S^1, I^1_m, I^1_h, I^1_{hm}, A^1, S^1_v, I^1_v, S^2, I^2_m, I^2_h, I^2_{hm}, A^2, S^2_v, I^2_v \in C(\mathbb{R},S) \cap L^1_{loc}(\mathbb{R},\mathcal{S})$ such that $S = S^1 + S^2$, $I_m = I^1_m + I^2_m$, $I_h = I^1_h + I^2_h$, $I_{hm} = I^1_{hm} + I^2_{hm}$, $A = A^1 + A^1$, $S_v = S^1_v + S^2_v$, $I_v = I^1_v + I^1_v$ and the given hypotheses holds:
(H_1). $S^1, I^1_m, I^1_h, I^1_{hm}, A^1, S^1_v, I^1_v$ are bounded and Lipschitz.
(H_2). $S^2, I^2_m, I^2_h, I^2_{hm}, A^2, S^2_v, I^2_v$ compact and bounded.
(H_3). $|\mathbb{R}(t,z) - \mathbb{R}(t,y)| \leq L_1 \|z - y\|$, for all $(t,z),(t,y) \in \mathbb{R}$.
Applying the fractional operator I^r_0 to both sides of system (3.1) and using Lemma 2.3, the following Lemma can be stated.

Lemma 4.2. *The proposed fractional-order epidemic model under consideration subject to a single initial condition is equivalent with the following system of Volterra integral*

equations:

$$\begin{aligned}
S(t) &= S(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} S^1(s, S(s))ds + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} S^2(s, S(s))ds \\
I_m(t) &= I_m(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} I_m^1(s, I_m(s))ds + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} I_m^2(s, I_m(s))ds \\
I_h(t) &= I_h(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} I_h^1(s, I_h(s))ds + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} I_h^2(s, I_h(s))ds \\
I_{hm}(t) &= I_{hm}(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} I_{hm}^1(s, I_{hm}(s))ds + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} I_{hm}^2(s, I_{hm}(s))ds \\
A(t) &= I_m(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} A^1(s, A(s))ds + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} A^2(s, A(s))ds \\
S_v(t) &= S_v(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} S_v^1(s, S_v(s))ds + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} S_v^2(s, S_v(s))ds \\
I_v(t) &= I_v(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} I_v^1(s, I_v(s))ds + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} I_v^2(s, I_v(s))ds
\end{aligned} \tag{4.1}$$

Based on the above system of integral equations (4.1), we state and prove the existence of the solutions.

Theorem 4.3. *The initial value problem under consideration possess at least one in the interval* $[0, T]$ *if the condition*

$$\Delta = \frac{\alpha \|L\|_{\frac{1}{q}} T^{\lambda}}{\Gamma(r)} < 1, \text{ where } \lambda = (r-q), \ \alpha = \left(\frac{1-q}{r-q}\right)^{1-q},$$

holds.

Proof. Taking κ such that $\kappa \geq |S(0)| + \Gamma(r)^{-1}\alpha \left(\|M_1\|_{\frac{1}{q}} + \|M_2\|_{\frac{1}{q}}\right) T^r$ and suppose $\mathcal{B}_\kappa = \{z : \|z\| \leq \kappa\}$ be the closed ball in $BC([0, T], \mathcal{S})$ with $\sup \|\cdot\|$. In view of system (4.1), we obtain that $S : \mathcal{B}_\kappa \to BC([0, T], \mathcal{S}),\ z \to S^1 z + S^2 z$ with the followings:

$$\begin{aligned}
S^1(t) &= S(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} S^1(s, z(s))ds, \\
S^1(t) &= \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} S^2(s, z(s))ds,
\end{aligned}$$

as a solution of the proposed model (3.1). It is now to be proved in the following steps that $S(t)$ is a condensing and thus the existence of a fixed point for $S(t)$ holds from Theorem 4.1.

Step 1. We show that $S(\mathcal{B}_\kappa) \subset \mathcal{B}_\kappa$. Now, for any $z \in \mathcal{B}_\kappa$, yields

$$\begin{aligned}
\|S(t)\| &\leq |S(0)| + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} S(s, z(s)) ds \\
&\leq |S(0)| + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} S^1(s, z(s)) ds + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} S^2(s, z(s)) ds \\
&\leq |S(0)| + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} M_1(s) ds + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} M_2(s) ds \\
&\leq |S(0)| + \frac{1}{\Gamma(r)} \left(\int_0^t (t-s)^{\frac{r-1}{1-q}} ds \right)^{1-q} \left[\left(\int_0^t M_1^{\frac{1}{q}}(s) ds \right)^q + \left(\int_0^t M_2^{\frac{1}{q}}(s) ds \right)^q \right] \\
&\leq |S(0)| + \frac{\alpha \left(\|M_1\|_{\frac{1}{q}} + \|M_2\|_{\frac{1}{q}} \right)}{\Gamma(r)} T^r \\
&\leq \kappa,
\end{aligned} \tag{4.2}$$

and thus $S(\mathcal{B}_\kappa) \subset \mathcal{B}_\kappa$.
Step 2. We show that S^1 is a contraction. Let $z, y \in \mathcal{B}_r$, then

$$\begin{aligned}
|S^1 z(t) - S^1 y(t)| &\leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} L(s) |z(s) - y(s)| ds \\
&\leq \frac{1}{\Gamma(r)} \left(\int_0^t (t-s)^{\frac{r-1}{1-q}} ds \right)^{1-q} \left(\int_0^t L^{\frac{1}{q}}(s) ds \right)^q \|z - y\| \\
&\leq \frac{\alpha \|L\|_{\frac{1}{q}} T^\lambda}{\Gamma(r)} \|z - y\| \\
&\leq \Delta \|z - y\|,
\end{aligned} \tag{4.3}$$

thus $S^1(t)$ is a contraction.
Step 3. We show that $S^2(t)$ is compact. For $0 \leq t_1 \leq t_2 \leq T$, we obtain

$$\begin{aligned}
|S^2 z(t_1) - S^2 y(t_2)| &\leq \frac{1}{\Gamma(r)} \left| \int_0^{t_2} (t_2 - s)^{r-1} S^2(s, z(s)) ds - \int_0^{t_1} (t_1 - s)^{r-1} S^2(s, z(s)) ds \right| \\
&\leq \frac{1}{\Gamma(r)} \left| \int_0^{t_1} (t_2 - s)^{r-1} S^2(s, z(s)) ds + \int_{t_1}^{t_2} (t_2 - s)^{r-1} S^2(s, z(s)) ds \right. \\
&\quad \left. - \int_0^{t_1} (t_1 - s)^{r-1} S^2(s, z(s)) ds \right| \\
&\leq \frac{1}{\Gamma(r)} \int_0^{t_1} \left((t_1 - s)^{r-1} - (t_2 - s)^{r-1} \right) M_2(s) ds \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2 - s)^{r-1} M_2(s) ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(r)}\left(\int_0^t M_2^{\frac{1}{q}}(s)ds\right)^q\left[\left(\int_0^{t_1}\left((t_1-s)^{r-1}-(t_2-s)^{r-1}\right)^{\frac{r-1}{1-q}}\right)^{1-q}\right.\\
&\quad\left.+\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{r-1}{1-q}}ds\right)^{1-q}\right]\\
&\leq \frac{a}{\Gamma(r)}\|M_2\|_{\frac{1}{q}}\left[\left((t_2-t_1)^{\frac{r-1}{1-q}}\right)^{1-q}+(t_2-t_1)^{r-q}\right]\\
&\leq \frac{2a\|M_2\|_{\frac{1}{q}}}{\Gamma(r)}(t_2-t_1)^{r-q}\\
&\longrightarrow 0,\ \text{as } t_2\to t_1.
\end{aligned}
\tag{4.4}
$$

Therefore, as a consequence of Arzela-Ascoli principle [15], together with steps $1-3$, implies that S^2 is compact. Since S^1 is a contraction and S^2 is compact therefore completely continuous as shown in [33] that the map $S = S_1 + S_2$ is condensing on $\mathcal{B}_\kappa$, and in view of systems (4.1) above shows the existence of a fixed point of S. Repeating the same above procedure for the remaining state variables $I_m,\ I_h,\ I_{hm},\ A,\ S_v$ and I_v the results follows. ∎

Theorem 4.4. *Suppose that assumption (H_3) holds. Then there exists a unique solutions of the initial value problems under consideration on the interval $[0,T]$ provided that*

$$\Delta = \frac{\alpha\|L\|_{\frac{1}{q}}T^\lambda}{\Gamma(r)} < 1.$$

Proof. Consider the operator defined by

$$F[S(t)] = S(0) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}S(s,S(s))ds. \tag{4.5}$$

Thus, for $S, S_1 \in \mathcal{B}_r$ gives

$$
\begin{aligned}
|F[S(t)] - F[S_1(t)]| &\leq \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}L_1(s)|S(s)-S_2(s)|ds\\
&\leq \frac{1}{\Gamma(r)}\left(\int_0^t (t-s)^{\frac{r-1}{1-q}}ds\right)^{1-q}\left(\int_0^t L_1^{\frac{1}{q}}(s)ds\right)^q\|S-S_1\|\\
&\leq \frac{\alpha\|L_1\|_{\frac{1}{Q}}T^\lambda}{\Gamma(r)}\|S-S_1\|\\
&\leq \Delta\|S-S_1\|.
\end{aligned}
\tag{4.6}
$$

Hence, we conclude the existence of a unique solution for the proposed fractional-order model (3.1). ∎

4.2. Positivity and boundedness of solution

Positivity and boundedness of solutions are important features of epidemiological models. We show that for all $t > 0$, all state variables are nonnegative, implying that any

trajectory that begins with a positive initial condition will remain positive for all $t > 0$. From systems (3.1), we have

$$\begin{aligned}
{}^{C}D^{r}_{0,t}S(t)|_{S=0} &= \Lambda + \alpha_1 I_m \geq 0,\\
{}^{C}D^{r}_{0,t}I_m(t)|_{I_m=0} &= \beta_m S I_v \geq 0,\\
{}^{C}D^{r}_{0,t}I_h(t)|_{I_h=0} &= \alpha_2 I_{hm} \geq 0,\\
{}^{C}D^{r}_{0,t}I_{hm}(t)|_{I_{hm}=0} &= \phi_1\beta_h I_m I_h + \phi_2\beta_m I_h I_v \geq 0,\\
{}^{C}D^{r}_{0,t}A(t)|_{A=0} &= \gamma_1 I_h + \gamma_2 I_{hm} \geq 0,\\
{}^{C}D^{r}_{0,t}S_v(t)|_{S_v=0} &= \Lambda_v \geq 0,\\
{}^{C}D^{r}_{0,t}I_v(t)|_{I_v=0} &= \beta_v S_v(I_m + I_{hm}) \geq 0,
\end{aligned} \tag{4.7}$$

on each hyperplane bounding the non-negative octant. Moreover, Let $N(t) = S(t) + I_m(t) + I_h(t) + I_{hm}(t) + A(t)$ and $N_v(t) = S_v(t) + I_v(t)$ be the total number of human and vector population respectively. Then, adding the first five equations of (3.1) yields

$${}^{C}D^{r}_{0,t}N(t) = \Lambda - \delta N - \mu_a A - \mu_d I_{hm} \leq \Lambda - \delta N, \tag{4.8}$$

then one has

$$N(t) \leq \left(N(0) - \frac{\Lambda}{\delta}\right) E_r(-\delta t^r) + \frac{\Lambda}{\delta}.$$

Similarly,

$$N_v(t) \leq \left(N_v(0) - \frac{\Lambda_v}{\delta_v}\right) E_r(-\delta_v t^r) + \frac{\Lambda_v}{\delta_v}.$$

Hence, the biological feasible region for the fractional-order model (3.1) is

$$\Omega = \Omega_h \times \Omega_v \subset \mathbb{R}^5_+ \times \mathbb{R}^2_+ \tag{4.9}$$

where

$$\Omega_h = \left\{(S(t), I_m(t), I_h(t), I_{hm}(t), A(t))\mathbb{R}^5_+ : 0 \leq N(t) \leq \frac{\Lambda}{\delta}\right\}$$

and

$$\Omega_v = \left\{(S_v(t), I_v(t))\mathbb{R}^2_+ : 0 \leq N_v(t) \leq \frac{\Lambda_v}{\delta_v}\right\}.$$

Therefore, the region Ω is positively invariant so that no solution path moves beyond the boundary of Ω. Thus proposed fractional-order model (3.1) is both mathematically and epidemiologically well-posed.

5. Numerical simulations and discussion

To get insight into the solutions trajectories, both the classical and fractional-order models require a numerical scheme. To this purpose, we employ a recent and effective numerical scheme recently introduced in [22]. The numerical scheme used in the present

analysis is given by

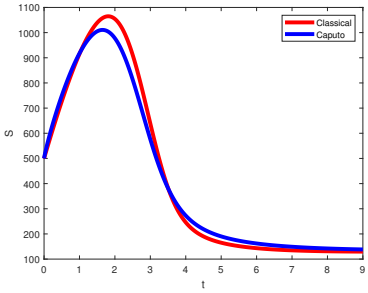
$$
\begin{aligned}
{}^{C}S_{n+1} &= S(0) + \frac{(\Delta t)^r}{\Gamma(r+1)} \sum_{k=0}^{n} \Big((k+1)^r - k^r \Big) \\
&\quad \times \Big(\Lambda - S_{n-k}(\beta_m I_v^{n-k} - \beta_h I_h^{n-k} - \delta) + \alpha_1 I_m^{n-k} \Big), \\
{}^{C}I_m^{n+1} &= I_m(0) + \frac{(\Delta t)^r}{\Gamma(r+1)} \sum_{k=0}^{n} \Big((k+1)^r - k^r \Big) \\
&\quad \times \Big(\beta_m S_{n-k} I_v^{n-k} - \alpha_1 I_m^{n-k} - \phi_1 \beta_h I_m^{n-k} I_h^{n-k} - \delta I_m^{n-k} \Big), \\
{}^{C}I_h^{n+1} &= I_h(0) + \frac{(\Delta t)^r}{\Gamma(r+1)} \sum_{k=0}^{n} \Big((k+1)^r - k^r \Big) \\
&\quad \times \Big(\beta_h S_{n-k} I_h^{n-k} + \alpha_2 I_{hm}^{n-k} - \phi_2 \beta_m I_h^{n-k} I_v^{n-k} - \gamma_1 I_h^{n-k} - \delta I_h^{n-k} \Big), \\
{}^{C}I_{hm}^{n+1} &= I_{hm}(0) + \frac{(\Delta t)^r}{\Gamma(r+1)} \sum_{k=0}^{n} \Big((k+1)^r - k^r \Big) \\
&\quad \times \Big(\phi_1 \beta_h I_m^{n-k} I_h^{n-k} + \phi_2 \beta_m I_h^{n-k} I_v^{n-k} - \alpha_2 I_{hm}^{n-k} - (\delta + \mu_d) I_{hm}^{n-k} - \gamma_2 I_{hm}^{n-k} \Big), \\
{}^{C}A_{n+1} &= A(0) + \frac{(\Delta t)^r}{\Gamma(r+1)} \sum_{k=0}^{n} \Big((k+1)^r - k^r \Big) \Big(\gamma_1 I_h^{n-k} + \gamma_2 I_{hm}^{n-k} - (\delta + \mu_a) A_{n-k} \Big), \\
{}^{C}S_v^{n+1} &= S_v(0) + \frac{(\Delta t)^r}{\Gamma(r+1)} \sum_{k=0}^{n} \Big((k+1)^r - k^r \Big) \\
&\quad \times \Big(\Lambda_v - \beta_v S_v^{n-k} (I_m^{n-k} + I_{hm}^{n-k}) - \delta_v S_v^{n-k} \Big), \\
{}^{C}I_v^{n+1} &= I_v(0) + \frac{(\Delta t)^r}{\Gamma(r+1)} \sum_{k=0}^{n} \Big((k+1)^r - k^r \Big) \Big(\beta_v S_v^{n-k} (I_m^{n-k} + I_{hm}^{n-k}) - \delta_v S_v^{n-k} \Big).
\end{aligned}
\tag{5.1}
$$

Besides, the aforementioned numerical scheme is convergent, conditionally stable, and possesses error bounds. These features of a numerical method guarantee its safe use. Table 3, provides the values of the running parameters used during the numerical simulations of the proposed fractional-order model.

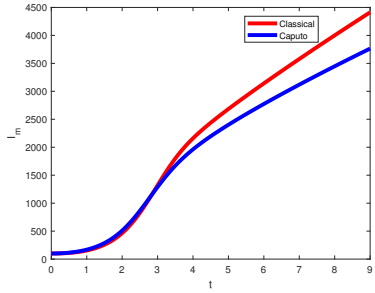
TABLE 3. Parameters values [18].

Parameters	Value
Λ	500/year
Λ_v	5000/year
β_h	0.00031/year
β_m	0.00045/year
β_v	0.00035/year
α_1	0.2/year
α_2	0.2/year
ϕ_1	1.1/year
ϕ_2	1.15/year
γ_1	0.01/year
γ_2	0.05/year
δ	0.02/year
δ_v	0.1429/year
μ_a	0.02/year
μ_d	0.03/year

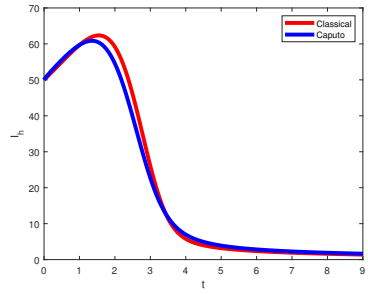
Whereas the initial conditions are given by $S(0) = 500,\ I_m(0) = 100,\ I_h(0) = 50,\ I_{hm}(0) = 30,\ A(0) = 50,\ S_v(0) = 5000,\ I_v(0) = 100$. Using the parameters values in Table 3, Figure 1, explore the dynamics trajectories of each compartments for the classical and fractional-order model which shows the correlation and accuracy of the fractional-order model.
In Figure 2, it can be noticed that the fractional-order r is varying for values $1, 0.88$ and 0.66. One can easily see the robust nature of the Caputo derivative than the classical integer-order derivative of the model. Besides, decreasing values of fractional-order r leads to a decreasing in each of the compartments.



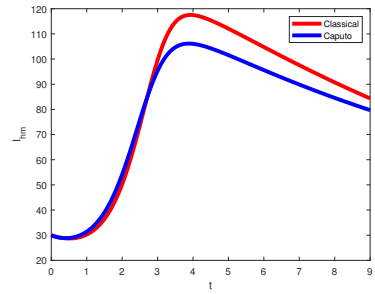
(A) Susceptible individuals



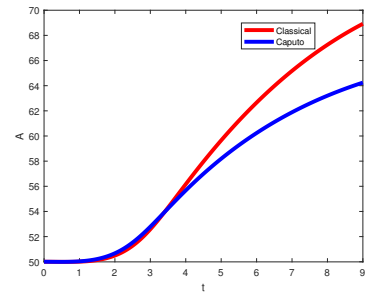
(B) Infected individuals with Malaria



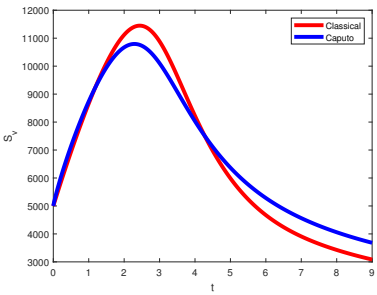
(C) Infected individuals with HIV



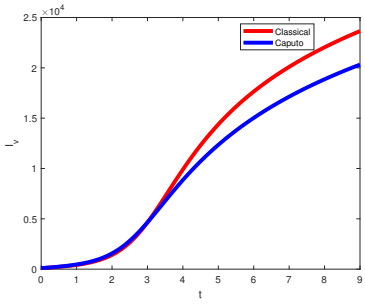
(D) Infected individuals with Malaria and HIV



(E) Infected individuals with AIDS

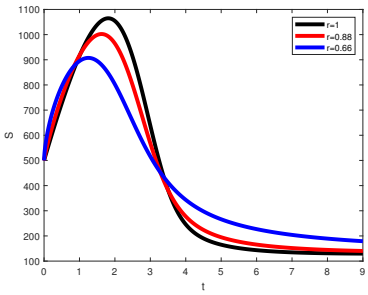


(F) Susceptible vector

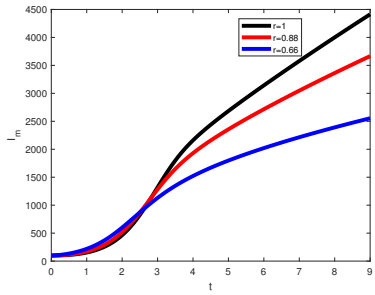


(G) Infected vector

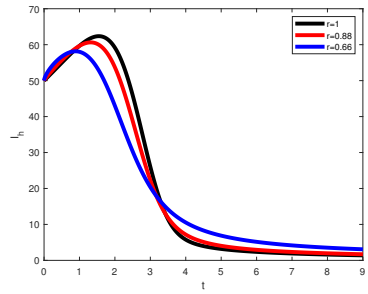
FIGURE 1. Dynamics behavior of each state variables for the classical and Caputo version of the model.



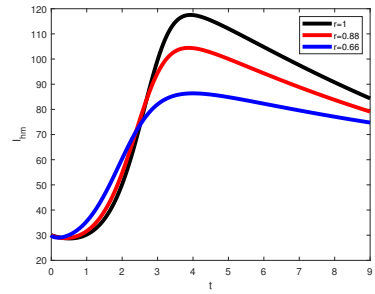
(A) Susceptible individuals



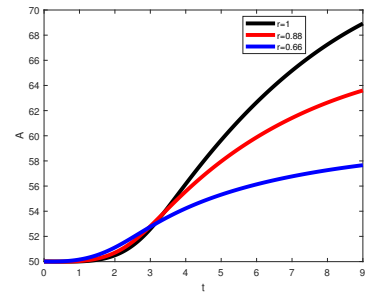
(B) Infected individuals with Malaria



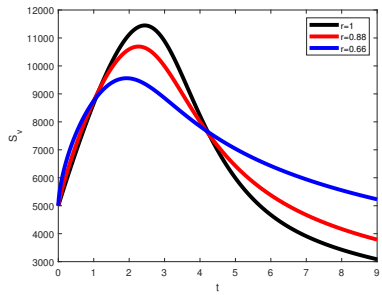
(C) Infected individuals with HIV



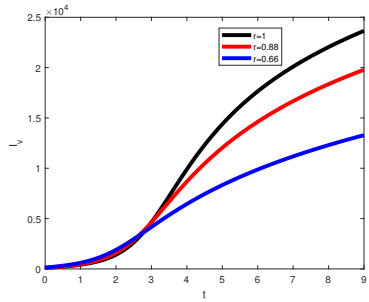
(D) Infected individuals with AIDS



(E) Infected individuals with AIDS



(F) Susceptible vector



(G) Infected vector

FIGURE 2. Dynamics behavior of each state variables with different fractional-order

6. Conclusions

The Caputo fractional-order derivative is employed to investigate the dynamic behavior of a combined HIV and Malaria epidemic co-infection model. The present analysis shows the existence and uniqueness of solutions of the proposed fractional-order epidemic model by utilizing the techniques of fixed point theorems. Moreover, to shows the model is mathematically and epidemiologically well-posed, positivity and boundedness of solutions are presented. To explore the dynamic behavior between two different points, the Caputo fractional-order epidemic model was solved via an effective numerical scheme.

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