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The Enlarged Banach Contraction Mappings with Application to Implicit Functional Integral Equations

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Abstract Our main goal is to investigate the fixed point results for enlarged Banach contraction mappings inspired by weak contraction mappings in metric spaces. We give some illustrative examples for showing the usability of our results. As applications, we analyze the existence of solutions for implicit functional integral equations using our theoretical fixed point results.

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1. Introduction and Preliminaries

In 1922, Banach [1] proved one of the famous results in mathematics named the Banach contraction principle, which is used to establish the existence and uniqueness of a solution for an integral equation. This principle says that if (X, d) is a complete metric space and $T : X \to X$ is a self mapping satisfying

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1.1}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$, then there exists a unique $x \in X$ such that $Tx = x$, that is, T has a unique fixed point. Since then, because of its simpleness, efficacy, and fruitfulness, it has become very famous and a fundamental tool for solving existing several problems in many branches of pure and applied mathematics.

Motivated by the Banach contraction principle, many mathematicians invented new interesting fixed point results for several kinds of linear/nonlinear mappings in various spaces. Fixed point results for Kannan contraction mappings and Chatterjea contraction mappings in [7] and [3], respectively, are classical results in this trend. These two results are shown below.

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Theorem 1.1 ([7]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the Kannan contractive condition, that is,*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point.

Theorem 1.2 ([3]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the Chatterjea contractive condition, that is,*

$$d(Tx, Ty) \leq \lambda[d(x, Ty) + d(y, Tx)] \tag{1.3}$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point.

In 1972, the idea of three famous contractive conditions (1.1), (1.2) and (1.3) are combined to only one generalization named Zamfirescu contractive condition which is appeared in the following definition.

Definition 1.3 ([8]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called a Zamfirescu contraction mapping if there are constants $\alpha \in [0, 1)$ and $\beta, \gamma \in [0, 1/2)$ such that for each $x, y \in X$, one of the following conditions holds:

(Z_1) $d(Tx, Ty) \leq \alpha d(x, y)$;
(Z_2) $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$;
(Z_3) $d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]$.

It is easy to see that for a self mapping T on a metric space (X, d), T is a Zamfirescu contraction mapping if and if and only if there is a constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq k \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \tag{1.4}$$

for all $x, y \in X$. The following theorem is a fixed point result for Zamfirescu contraction mappings.

Theorem 1.4 ([8]). *Let (X, d) be a complete metric space and $T : X \to X$ be a Zamfirescu contraction mapping. Then T has a unique fixed point.*

Inspired by the extensive literature devoted to all contraction mappings aforementioned, in 2004, Berinde [2] introduced the concept of a new contractive condition as follows:

Definition 1.5 ([2]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called a weak contraction mapping if there are constants $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \tag{1.5}$$

for all $x, y \in X$.

It is obvious that each of the contractive conditions of Banach [1], Kannan [7], Chatterjea [3] and Zamfirescu [8] implies (1.5). Moreover, the class of weak contraction mappings covers the half class of Ciric contraction mappings in [4], which is one of the most general class of several contraction mappings. The following theorem is the main fixed point result for weak contraction mappings in [2].

Theorem 1.6 ([2])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a weak contraction mapping. Then T has a unique fixed point. Moreover, if there are constants $\delta_1 \in [0,1)$ and $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \delta_1 d(x,y) + L_1 d(x,Tx) \tag{1.6}$$

for all $x, y \in X$. Then the fixed point of T is unique.

Theorem 1.6 is an extension of the Banach contraction principle, Theorem 1.1, Theorem 1.2, Theorem 1.4 and many fixed point results. Over recent years, the idea of weak contraction mappins is attracted many mathematicians to invent the new contractions.

On the other hand, the extension of Banach contraction principle through the rational expression of the contractive condition is first investigated by Dass and Gupta [5]. Afterwards, many authors proved several fixed point results through contractive condition of rational type in metric spaces. For instance, Jaggi [6] introduced the following rational expression.

Definition 1.7 ([6])**.** Let (X,d) be a complete metric space. A mapping $T : X \to X$ is called a rational type contraction mapping if it is continuous and there are constants $\alpha, \beta \in [0,1)$ such that $\alpha + \beta < 1$ and

$$d(Tx,Ty) \leq \frac{\alpha d(x,Tx)d(y,Ty)}{d\,(x,y)} + \beta d\,(x,y) \tag{1.7}$$

for all $x, y \in X$ with $x \neq y$.

Jaggi [6] also gave the following fixed point result for rational type contraction mappings in the above definition.

Theorem 1.8 ([6])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a rational type contraction mapping. If T^p is continuous for some $p \in \mathbb{N}$, then T has a unique fixed point.*

This paper aims to introduce many new types of contraction mapping named enlarged Banach contraction mappings. These contraction mappings are the new rational expression inspired by weak contraction mappings of Berinde [2]. Many fixed point results for enlarged Banach contraction mappings are analyzed in metric spaces. Moreover, we give some illustrative examples to show the usability of our results. Based on the interesting impact of the theory of implicit functional integral equation, our theoretical fixed point results are applied to claim the existence of solutions for implicit integral equations

2. Enlarged Banach Contraction Mappings

Several new enlarged Banach contraction mappings are first introduced in this section. Moreover, fixed point results for such enlarged Banach contraction mappings are investigated. We begin with the definition of the first enlarged Banach contraction mapping.

Definition 2.1. Let (X,d) be a metric space. A mapping $T : X \to X$ is called an enlarged Banach contraction mapping type 1 if there are constants $\delta \in [0,1)$ and $L, \overline{L} \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + \frac{Ld(x,Ty)d(y,Tx)}{\overline{L}\,[d(x,y) - d(x,Tx)] + 1} \tag{2.1}$$

for all $x, y \in X$.

Remark 2.2. It is easy to see that each Banach contraction mapping with the Banach contractive constant $k \in [0, 1)$ is an enlarged Banach contraction mapping type 1 with constants $\delta = k$ and $L = \overline{L} = 0$.

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ be an enlarged Banach contraction mapping type 1. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to a fixed point of T.*

Proof. Let $x_0 \in X$. Define the Picard sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1} \tag{2.2}$$

for all $n \in \mathbb{N}$. From (2.1), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \delta d(x_{n-1}, x_n) + \frac{Ld(x_{n-1}, Tx_n)\, d(x_n, Tx_{n-1})}{\overline{L}[d(x_{n-1}, x_n) - d(x_{n-1}, Tx_{n-1})] + 1} \\ &= \delta d(x_{n-1}, x_n) + \frac{Ld(x_{n-1}, x_{n+1})\, d(x_n, x_n)}{\overline{L}[d(x_{n-1}, x_n) - d(x_{n-1}, x_n)] + 1} \\ &= \delta d(x_{n-1}, x_n) \\ &\;\vdots \\ &\leq \delta^n d(x_0, x_1). \end{aligned}$$

For each $m, n \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq \delta^n d(x_0, x_1) + \delta^{n+1} d(x_0, x_1) + \ldots + \delta^{m-1} d(x_0, x_1) \\ &= \left(\delta^n + \delta^{n+1} + \ldots + \delta^{m-1}\right) d(x_0, x_1) \\ &\leq \left(\delta^n + \delta^{n+1} + \ldots \right) d(x_0, x_1) \\ &= \frac{\delta^n}{1-\delta} d(x_0, x_1). \end{aligned} \tag{2.3}$$

By taking the limit as $m, n \to \infty$ in (2.3), we have $d(x_n, x_m) \to 0$ and so $\{x_n\}$ is a Cauchy sequence in X. By using the completeness of X, we obtain $x_n \to z$ as $n \to \infty$ for some $z \in X$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, Tz) &= d(Tx_n, Tz) \\ &\leq \delta d(x_n, z) + \frac{Ld(x_n, Tz)\, d(z, Tx_n)}{\overline{L}[d(x_n, z) - d(x_n, Tx_n)] + 1} \\ &= \delta d(x_n, z) + \frac{Ld(x_n, Tz)\, d(z, x_{n+1})}{\overline{L}[d(x_n, z) - d(x_n, x_{n+1})] + 1}. \end{aligned} \tag{2.4}$$

Taking the limit as $n \to \infty$ in (2.4), we have $d(z, Tz) = 0$ and so $z = Tz$. This completes the proof. ∎

Example 2.4. Let $X = [0, 3]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = \begin{cases} 2 & \text{if } x \in [0, 3) \\ \frac{3}{5} & \text{if } x = 3. \end{cases}$$

First, we will show that the Berinde fixed point theorem in [2] can not be applied in this example whenever L in Definition 1.5 is greater than or equal to 1. To show this, we will claim that T is not a weak contraction mapping with $L \geq 1$. Indeed, for $x = \frac{13}{5}$ and $y = 3$, we get and

$$\begin{aligned} d(Tx, Ty) &= \left| 2 - \frac{3}{5} \right| \\ &= \frac{7}{5} \\ &= \frac{2}{5} + 1 \\ &= \left| \frac{13}{5} - 3 \right| + |3 - 2| \\ &> \delta \left| \frac{13}{5} - 3 \right| + L\,|3 - 2| \\ &= \delta d(x, y) + L d(y, Tx) \end{aligned}$$

for all $\delta \in [0, 1)$ and $L \geq 1$. It yields that T is not a weak contraction mapping. Therefore, the Berinde fixed point theorem in [2] is not useful is this situation.

Second, we will show that the Banach contraction mapping principle can not be applied in this example. To show this, we will claim that T is not a Banach contraction mapping. Indeed, for $x = \frac{8}{5}$ and $y = 3$, we get

$$d(Tx, Ty) = \left| 2 - \frac{3}{5} \right| = \frac{7}{5} > k\left(\frac{7}{5}\right) = kd(x, y)$$

for all $k \in [0, 1)$. It yields that T is not a Banach contraction mappping. Therefore, the Banach contraction mapping principle is not useful is this situation.

Next, we will show that Theorem 2.3 can guarantee the existence of a fixed point of T. To apply Theorem 2.3, we will show that T is an enlarged Banach contraction mapping type 1 with constants $\delta \in \left[\frac{2}{3}, 1\right)$, $\overline{L} = 0.1$ and $L = 1$. Suppose that $x, y \in X$. We will divide our showing into 4 cases.

Case 1: If $x, y \in [0, 3)$, then

$$d(Tx, Ty) = 0 \leq \delta d(x, y) + \frac{Ld(x, Ty)d(y, Tx)}{\overline{L}\,[d(x, y) - d(x, Tx)] + 1}.$$

Case 2: If $x = y = 3$, then

$$d(Tx, Ty) = 0 \leq \delta d(x, y) + \frac{Ld(x, Ty)d(y, Tx)}{\overline{L}\,[d(x, y) - d(x, Tx)] + 1}.$$

Case 3: If $x \in [0,3)$ and $y = 3$, then

$$\begin{aligned} d\left(Tx,Ty\right) &= \left|2 - \frac{3}{5}\right| \\ &= \frac{7}{5} \\ &\leq \frac{2}{3}\left|x-3\right| + \frac{\left|x-\frac{3}{5}\right|\left|3-2\right|}{0.1\left(\left|x-3\right|-\left|x-2\right|\right)+1} \\ &= \delta d(x,y) + \frac{Ld(x,Ty)d(y,Tx)}{\overline{L}\left[d(x,y)-d(x,Tx)\right]+1}. \end{aligned}$$

Figure 1 shows the validity of the above inequality. In this figure, the solid line is the value of $d\left(Tx,Ty\right) = \frac{7}{5}$ and the dash line is the graph of a function

$$[0,3) \ni x \mapsto \frac{2}{3}\left|x-3\right| + \frac{\left|x-\frac{3}{5}\right|\left|3-2\right|}{0.1\left(\left|x-3\right|-\left|x-2\right|\right)+1}. \tag{2.5}$$

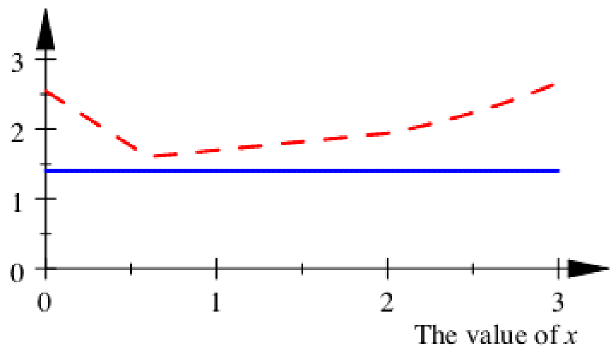

FIGURE 1. Showing the validity for Case 3 in Example 2.4

Case 4: If $x = 3$ and $y \in [0,3)$, then

$$\begin{aligned} d\left(Tx,Ty\right) &= \left|\frac{3}{5} - 2\right| \\ &= \frac{7}{5} \\ &\leq \frac{2}{3}\left|3-y\right| + \frac{\left|3-2\right|\left|y-\frac{3}{5}\right|}{0.1\left(\left|3-y\right|-\left|3-\frac{3}{5}\right|\right)+1} \\ &= \delta d(x,y) + \frac{Ld(x,Ty)d(y,Tx)}{\overline{L}\left[d(x,y)-d(x,Tx)\right]+1}. \end{aligned}$$

Figure 2 shows the validity of the above inequality. In this figure, the solid line is the value of $d\left(Tx,Ty\right)=\dfrac{7}{5}$ and the dash line is the graph of a function

$$[0,3)\ni y\mapsto \frac{2}{3}\left|3-y\right|+\frac{\left|3-2\right|\left|y-\frac{3}{5}\right|}{0.1\left(\left|3-y\right|-\left|3-\frac{3}{5}\right|\right)+1}. \tag{2.6}$$

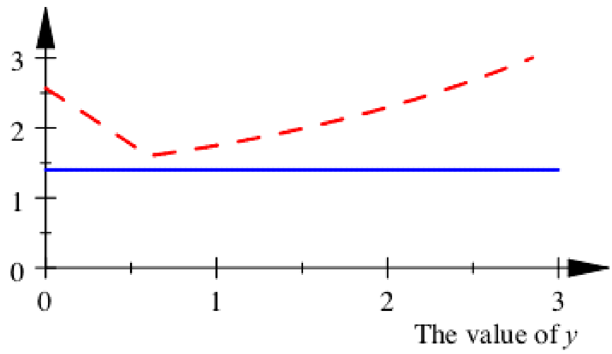

FIGURE 2. Showing the validity for Case 4 in Example 2.4

From all cases, we obtain the inquality (2.1) holds for all $x,y\in X$. This means that T is an enlarged Banach contraction mapping type 1. Therefore, all hypotheses of Theorem 2.3 hold and so the existence of a fixed point of T can be obtained from Theorem 2.3.

It points out that the uniqueness of the fixed point for an enlarged Banach contraction mapping type 1 can not be applied from the inequality (2.1). However, this situation does not occur for an enlarged Banach contraction mapping in the following definition.

Definition 2.5. Let (X,d) be a metric space and $T:X\to X$ be a mapping.

(1) T is called an enlarged Banach contraction mapping type 2 if there are constants $\delta\in[0,1)$ and $L,\overline{L}\geq 0$ such that

$$d(Tx,Ty)\leq \delta d(x,y)+\frac{Ld(y,Tx)d(y,Ty)}{\overline{L}\left[d(x,y)-d(x,Tx)\right]+1} \tag{2.7}$$

for all $x,y\in X$.

(2) T is called an enlarged Banach contraction mapping type 3 if there are constants $\delta\in[0,1)$ and $L,\overline{L}\geq 0$ such that

$$d(Tx,Ty)\leq \delta d(x,y)+\frac{Ld(x,Tx)d(y,Tx)}{\overline{L}\left[d(x,y)-d(x,Tx)\right]+1} \tag{2.8}$$

for all $x,y\in X$.

Similarly with Remark 2.2, each Banach contraction mapping with the Banach contractive constant $k\in[0,1)$ is an enlarged Banach contraction mapping type i with constants $\delta=k$ and $L=\overline{L}=0$ for all $i\in\{2,3\}$.

Theorem 2.6. *Let (X,d) be a complete metric space and $T : X \to X$ be an enlarged Banach contraction mapping type i for some $i \in \{2,3\}$. Then T has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to a fixed point of T.*

Proof. Since the proof for $i = 3$ is similar to the proof for $i = 2$, we omit the proof for $i = 3$ in this article. Now, we suppose that T is an enlarged Banach contraction mapping type 2. Let $x_0 \in X$. Define the Picard sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1} \tag{2.9}$$

for all $n \in \mathbb{N}$. From (2.7), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \delta d(x_{n-1}, x_n) + \frac{Ld(x_n, Tx_{n-1})\, d(x_n, Tx_n)}{\overline{L}\left[d(x_{n-1}, x_n) - d(x_{n-1}, Tx_{n-1})\right] + 1} \\ &= \delta d(x_{n-1}, x_n) + \frac{Ld(x_n, x_n)\, d(x_n, x_{n+1})}{\overline{L}\left[d(x_{n-1}, x_n) - d(x_{n-1}, x_n)\right] + 1} \\ &= \delta d(x_{n-1}, x_n) \\ &\vdots \\ &\leq \delta^n d(x_0, x_1). \end{aligned}$$

For each $m, n \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq \delta^n d(x_0, x_1) + \delta^{n+1} d(x_0, x_1) + \dots + \delta^{m-1} d(x_0, x_1) \\ &= \left(\delta^n + \delta^{n+1} + \dots + \delta^{m-1}\right) d(x_0, x_1) \\ &\leq \left(\delta^n + \delta^{n+1} + \dots\ \right) d(x_0, x_1) \\ &= \frac{\delta^n}{1-\delta} d(x_0, x_1). \end{aligned} \tag{2.10}$$

By taking the limit as $m, n \to \infty$ in (2.10), we have $d(x_n, x_m) \to 0$ and so $\{x_n\}$ is a Cauchy sequence in X. By using the completeness of X, we obtain $x_n \to z$ as $n \to \infty$ for some $z \in X$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, Tz) &= d(Tx_n, Tz) \\ &\leq \delta d(x_n, z) + \frac{Ld(z, Tx_n)\, d(z, Tz)}{\overline{L}\left[d(x_n, z) - d(x_n, Tx_n)\right] + 1} \\ &= \delta d(x_n, z) + \frac{Ld(z, x_{n+1})\, d(z, Tz)}{\overline{L}\left[d(x_n, z) - d(x_n, x_{n+1})\right] + 1}. \end{aligned} \tag{2.11}$$

Taking the limit as $n \to \infty$ in (2.11), we have $d(z, Tz) = 0$ and so $z = Tz$.
Finally, we may assume that w is another fixed point of T. Then

$$d(z,w) = d(Tz, Tw) \leq \delta d(z,w) + \frac{Ld(w,Tz)d(w,Tw)}{\overline{L}\left[d(z,w) - d(z,Tz)\right] + 1} = \delta d(z,w).$$

It yields that $d(z,w) = 0$ and so $z = w$. Therefore, T has a unique fixed point. This completes the proof. ∎

Next, we give the definition of the remaining enlarged Banach contraction mappings as follows.

Definition 2.7. Let (X, d) be a metric space and $T : X \to X$ be a mapping.

(1) T is called an enlarged Banach contraction mapping type 4 if there are constants $\delta \in [0, 1)$ and $L, L', \overline{L} \geq 0$ such that $L \leq L'$ and

$$d(Tx, Ty) \leq Ld(x, y) + \frac{\delta d(x, Tx) - L'd(x, y)}{\overline{L}\,[d(x, y) - d(x, Tx)] + 1} \tag{2.12}$$

for all $x, y \in X$.

(2) T is called an enlarged Banach contraction mapping type 5 if there are constants $\delta \in [0, 1)$ and $L, L', \overline{L} \geq 0$ such that $L \leq L'$ and

$$d(Tx, Ty) \leq Ld(x, Ty) + \frac{\delta d(x, y) - L'd(x, Ty)}{\overline{L}\,[d(x, y) - d(x, Tx)] + 1} \tag{2.13}$$

for all $x, y \in X$.

Remark 2.8. If T is a Banach contraction mapping with the Banach contractive constant $k \in [0, 1)$, then the following assertions hold:

(1) T is an enlarged Banach contraction mapping type 4 with constants $\delta = 0$, $\overline{L} = 0$ and $L, L' \geq 0$ such that $L - L' = k$;

(2) T is an enlarged Banach contraction mapping type 5 with constants $\delta = k$, $\overline{L} = 0$ and $L, L' \geq 0$ such that $L = L'$.

Theorem 2.9. *Let (X, d) be a complete metric space and $T : X \to X$ be an enlarged Banach contraction mapping type i for some $i \in \{4, 5\}$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to a fixed point of T.*

Proof. This theorem can be proved by using a similar argumentation as in the proof of Theorem 2.3. ∎

Example 2.10. Let $X = [0, 1]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{1}{14} & \text{if } x \in [0, 1) \\[2ex] \dfrac{1}{7} & \text{if } x = 1. \end{cases}$$

We will show that T is an enlarged Banach contraction mapping type 4 with constants $L, L', \overline{L} = 1$ and $\delta = \frac{1}{2}$. Suppose that $x, y \in X$. We will divide our showing into 4 cases.

Case 1: If $x, y \in [0, 1)$, then

$$d(Tx, Ty) = 0 \leq Ld(x, y) + \frac{\delta d(x, Tx) - L'd(x, y)}{\overline{L}\,[d(x, y) - d(x, Tx)] + 1}.$$

Case 2: If $x = y = 1$, then

$$d(Tx, Ty) = 0 \leq Ld(x, y) + \frac{\delta d(x, Tx) - L'd(x, y)}{\overline{L}\,[d(x, y) - d(x, Tx)] + 1}.$$

Case 3: If $x \in [0,1)$ and $y = 1$, then

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{1}{14} - \frac{1}{7}\right| \\ &= \frac{1}{14} \\ &\leq |x-1| + \frac{\frac{1}{2}\left|x - \frac{1}{14}\right| - |x-1|}{\left(|x-1| - \left|x - \frac{1}{14}\right|\right) + 1} \\ &= Ld(x,y) + \frac{\delta d(x,Tx) - L'd(x,y)}{\overline{L}\left[d(x,y) - d(x,Tx)\right] + 1}. \end{aligned}$$

Figure 3 shows the validity of the above inequality. In this figure, the solid line is the value of $d(Tx,Ty) = \frac{1}{14}$ and the dash line is the graph of a function

$$[0,1) \ni x \mapsto |x-1| + \frac{\frac{1}{2}\left|x - \frac{1}{14}\right| - |x-1|}{\left(|x-1| - \left|x - \frac{1}{14}\right|\right) + 1}. \tag{2.14}$$

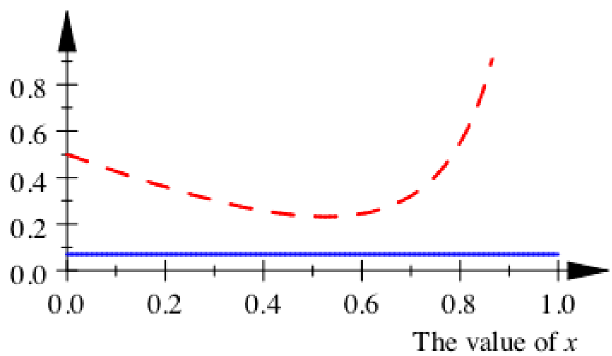

FIGURE 3. Showing the validity for Case 3 in Example 2.10

Case 4: If $x = 1$ and $y \in [0,1)$, then

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{1}{7} - \frac{1}{14}\right| \\ &= \frac{1}{14} \\ &\leq |1-y| + \frac{\frac{1}{2}\left|1 - \frac{1}{7}\right| - |1-y|}{\left(|1-y| - \left|1 - \frac{1}{7}\right|\right) + 1} \\ &= Ld(x,y) + \frac{\delta d(x,Tx) - L'd(x,y)}{\overline{L}\left[d(x,y) - d(x,Tx)\right] + 1}. \end{aligned}$$

Figure 4 shows the validity of the above inequality. In this figure, the solid line is the value of $d(Tx,Ty) = \frac{1}{14}$ and the dash line is the graph of a function

$$[0,1) \ni y \mapsto |1-y| + \frac{\frac{1}{2}\left|1 - \frac{1}{7}\right| - |1-y|}{\left(|1-y| - \left|1 - \frac{1}{7}\right|\right) + 1}. \tag{2.15}$$

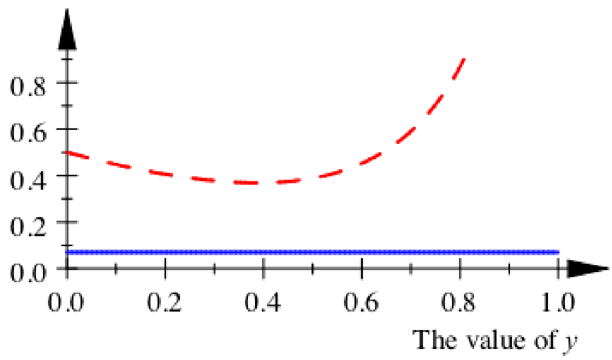

FIGURE 4. Showing the validity for Case 4 in Example 2.10

From all cases, we obtain the inquality (2.12) holds for all $x, y \in X$. This means that T is an enlarged Banach contraction mapping type 4. Therefore, all hypotheses of Theorem 2.9 hold and so the existence of a fixed point of T can be applied from Theorem 2.9.

3. SOLUTIONS FOR IMPLICIT FUNCTIONAL INTEGRAL EQUATIONS

Motivated by the impact of the existence of solutions for a problem in implicit function theory, we use the theoretical fixed point results for enlarged Banach contraction mappings obtained from the previous section to investigate an existence theorem for solutions of the implicit functional integral equation (3.1) in the next theorem.

Theorem 3.1. *Let $X = C[0,1]$ be a set of all continuous real-value functions on $[0,1]$ and d be a Chebyshev distance on X. Consider the mapping $T : X \to X$, formulating from the nonlinear implicit functional integral equation, which is defined for each $x \in X$ by*

$$(Tx)(t) = \delta x(t) + \int_0^t f(t, \tau, x(\tau), (Tx)(\tau))\, d\tau \tag{3.1}$$

for all $t \in [0,1]$, where $\delta \in (0,1)$ and $f : [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ is a given continuous mapping. Suppose that the following conditions hold:

(1) T is continuous;
(2) $T(B) \subseteq B$, where $B := \{z \in X : 0 \leq z(t) \leq 1\}$;
(3) there are $\delta \in [0,1)$ and $L, \overline{L} \geq 0$ such that for each $x, y \in B$, we have

$$\int_0^t |f(t, \tau, x(\tau), (Tx)(\tau)) - f(t, \tau, y(\tau), (Ty)(\tau))|\, d\tau \leq \frac{Ld(x,Ty)d(y,Tx)}{\overline{L}\,[d(x,y) - d(x,Tx)] + 1}$$

for all $t \in [0,1]$.

Then T has a fixed point.

Proof. It is easy to see that B is closed in X. Then (B,d) is complete. From (2), we know that B is T-invariant. Hence, we must to apply Theorem 2.3 with the restriction of T on B. Now, we will show that $T|_B$ is an enlarged Banach contraction mapping type 1. For each $x, y \in B$ and $t \in [0,1]$, we have

$$\begin{aligned} &|(Tx)(t) - (Ty)(t)| \\ = & \left|\delta(x(t) - y(t)) + \int_0^t f(t,\tau,x(\tau),(Tx)(\tau))\,d\tau - \int_0^t f(t,\tau,y(\tau),(Ty)(\tau))\,d\tau\right| \\ \leq & \ \delta|x(t) - y(t)| + \left|\int_0^t f(t,\tau,x(\tau),(Tx)(\tau))\,d\tau - \int_0^t f(t,\tau,y(\tau),(Ty)(\tau))\,d\tau\right| \\ \leq & \ \delta|x(t) - y(t)| + \int_0^t |f(t,\tau,x(\tau),(Tx)(\tau)) - f(t,\tau,y(\tau),(Ty)(\tau))|\,d\tau \\ \leq & \ \delta|x(t) - y(t)| + \frac{Ld(x,Ty)d(y,Tx)}{\overline{L}\left[d(x,y) - d(x,Tx)\right] + 1} \end{aligned}$$

Taking the maximum on the right hand side for all $t \in [0,1]$, we obtain

$$|(Tx)(t) - (Ty)(t)| \leq \delta d(x,y) + \frac{Ld(x,Ty)d(y,Tx)}{\overline{L}\left[d(x,y) - d(x,Tx)\right] + 1}$$

for all $t \in [0,1]$. This implies that

$$d(Tx,Ty) \leq \delta d(x,y) + \frac{Ld(x,Ty)d(y,Tx)}{\overline{L}\left[d(x,y) - d(x,Tx)\right] + 1}$$

for all $x, y \in X$. Now, all conditions of Theorem 2.3 are satisfied. Therefore, T has a fixed point. ∎

Remark 3.2. We can use Theorem 2.6 or Theorem 2.9 with a similar technique as in the proof of Theorem 3.1 for making the new existence result of the implicit functional integral equation (3.1).

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