Thai Journal of Mathematics
Volume 19 Number 3 (2021)
Pages 865–884

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Local Well-posedness of Nonlinear Time–fractional Diffusion Equation

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Abstract We study the local well-posedness of the following time-fractional nonlinear diffusion equation

$$\begin{cases} {}^C D_{0,t}^{\alpha,\lambda} u - \Delta u = |u|^{p-1} u, & (x,t) \in \mathbb{R}^n \times (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $0 < \alpha < 1$, $\lambda \geq 0$, $T < \infty$, $p > 1$, $u_0 \in C_0(\mathbb{R}^n)$ and ${}^C D_{0,t}^{\alpha,\lambda}$ denotes Caputo tempered fractional derivative of order α. The local existence and uniqueness results are obtained from heat kernel and fixed point theorem. Then, we extend the solution to establish a maximal mild solution. Moreover, we provide estimate for continuous dependence on initial condition.

MSC: 26A33; 35A01; 35B30

Keywords: tempered fractional calculus; existence and uniqueness; diffusion equation

Submission date: 26.04.2021 / Acceptance date: 20.07.2021

1. Introduction

Fractional calculus is a rapidly developing field of research, at the interface between many phenomena and differential equations such as chemistry, physics, biology, engineering, epidemiology, etc (see [1–3]). The research on fractional calculus becomes a focus area of study due to the fact that some dynamical models can be described more accurately with fractional derivatives than the ones with integer-order derivatives. In recent years, there have been many studies on fractional differential equations in various aspects such as existence and uniqueness of solution [4–9], stability of solutions [10, 11], numerical solutions [12, 13] and optimal control problem [14, 15].

One of the main research focuses on fractional calculus is the theory of fractional nonlinear diffusion equations. Nonlinear diffusion equations is an essential class of parabolic partial differential equations, derive from a variety of diffusion phenomena which appear

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Published by The Mathematical Association of Thailand.

extensively in nature. Many research suggested mathematical models of physical problems in many fields and scientific phenomena, such as heat transfer, fluid mechanics, plasma physics, plasma waves, thermoelasticity, biochemistry and dynamics of biological groups [16–20]. These models are determined by replacing the first-order time derivative from the classical diffusion equation with a fractional derivative of order α and $0 < \alpha < 1$. So, appreciable attention has been attracted to the time fractional diffusion equation. Recently, researchers have studied fractional calculus for nonlinear diffusion equations based on fixed point theorem [21–26] and in terms of fractional evolution equation [27–30].

In practical applications, there are different types of fractional operators such as Riemann-Liouville, Caputo, Riesz, Hilfer, etc (see [1, 31–34]). Fractional calculus involves the operation of convolution with a power law. In particular, changing the memory kernel of the fractional operator by multiplying with an exponential factor leads to the notion of tempered fractional calculus [35]. Tempered fractional calculus is applied for describing the transition between normal and anomalous diffusions (or the anomalous diffusion in finite time or bounded physical space). Thus, it is natural to find in its focused area of study, for example, tempered fractional diffusion equations, tempered fractional Brownian motion. Moreover, it can be accepted as the generalization of fractional calculus because it can be reduced to the classical of Riemann-Liouville and Caputo fractional calculus.

In 2015, Zhang and Sun [25] investigated the following time-fractional diffusion equation with the Caputo fractional derivative:

$$\begin{cases} {}^C D_{0,t}^{\alpha} u - \Delta u = |u|^{p-1} u, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $p > 1$ and ${}^C D_{0,t}^{\alpha}$ is Caputo fractional derivative. They obtained the Fujita critical exponent of (1.1) that if $1 < p < 1 + \frac{2}{n}$, then any nontrivial positive solution of (1.1) blows up in finite time, while if $p \geq 1 + \frac{2}{n}$ and the initial value is sufficiently small, problem (1.1) exists a global solution. This motivates us to study time-fractional diffusion equation with a more general fractional derivative.

This paper is concerned with the local well-posedness of the following Cauchy problems for nonlinear diffusion equations:

$${}^C D_{0,t}^{\alpha,\lambda} u - \Delta u = |u|^{p-1} u, \quad (x,t) \in \mathbb{R}^n \times (0,T], \tag{1.2}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $0 < \alpha < 1$, $\lambda \geq 0, p > 1$, $0 < T < \infty$, ${}^C D_{0,t}^{\alpha,\lambda}$ denotes Caputo tempered fractional derivative and

$$u_0 \in C_0(\mathbb{R}^n) = \left\{ u \in C(\mathbb{R}^n) | \lim_{|x|\to\infty} u(x) = 0 \right\}.$$

The aim of this paper is to investigate a well-posed problem, that is, a solution exists, the solution is unique and the solution depends continuously upon the given data. Firstly, we derive a mild solution for the problem (1.2)-(1.3) by using Laplace transform and Fourier transform in terms of the heat kernel. Then, the local existence and uniqueness of mild solution for the problem (1.2)-(1.3) is investigated by means of fixed point theorem. Moreover, we extend the result to prove the maximal existence of solutions. The continuous dependence of the solution is studied in the end.

This paper is organized as follows. In Section 2, we introduce some notions, definitions, preliminaries that will be useful. Next, in Section 3, we derive the mild solution for problem (1.2)-(1.3). Then, we prove local existence and uniqueness of solutions to the problem (1.2)-(1.3) and the maximal existence of solutions will also be investigated in Section 4. In Section 5, the continuous dependence of solutions on initial data will be obtained. Finally, we give an example to illustrate the main results in the last section.

2. Preliminaries

We provide some preliminary details, results and definitions of fractional calculus in this section which are important throughout this paper.

Definition 2.1. The Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(s) = \int_0^\infty e^{-\tau} \tau^{s-1} d\tau.$$

Definition 2.2 ([36])**.** If $\varphi \in L^1(\mathbb{R}^n)$, we define its Fourier transform

$$\mathcal{F}[\varphi(x)] = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) dx =: \hat{\varphi}(\xi), \qquad \xi \in \mathbb{R}^n$$

and its inverse Fourier transform

$$\mathcal{F}^{-1}[\varphi(\xi)] = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \varphi(\xi) d\xi, \qquad x \in \mathbb{R}^n.$$

Definition 2.3 ([36])**.** The Laplace transform of a function f is defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t) dt, \qquad t > 0,$$

where $f(t)$ is a given function.

Here, we are have identified $u(t) = u(\cdot, t)$.

Definition 2.4 (Riemann-Liouville tempered fractional integral, [37])**.** Let $u(t)$ be piecewise continuous on (a, b), $u(t) \in L^1([a, b])$ and α, λ be parameters with $\alpha > 0$ and $\lambda \geq 0$. The Riemann-Liouville tempered fractional integral operator $I_{a,t}^{\alpha,\lambda}$ of order α is defined by

$$\begin{aligned} I_{a,t}^{\alpha,\lambda} u(t) &:= e^{-\lambda t} I_{a,t}^{\alpha} \left(e^{\lambda t} u(t)\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} u(s) ds, \qquad a \leq t \leq b, \end{aligned}$$

where $I_{a,t}^{\alpha}$ denotes the Riemann-Liouville fractional integral

$$I_{a,t}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds.$$

Definition 2.5 (Riemann-Liouville tempered fractional derivative [37])**.** For $n \in \mathbb{N}$, $n - 1 < \alpha < n$ and $\lambda \geq 0$. The Riemann-Liouville tempered fractional derivative operator $D_{a,t}^{\alpha,\lambda}$ of order α is defined by

$$\begin{aligned} {}^{RL}D_{a,t}^{\alpha,\lambda} u(t) &:= e^{-\lambda t} \left({}^{RL}D_{a,t}^{\alpha} \left(e^{\alpha t} u(t)\right)\right) \\ &= \frac{e^{-\lambda t}}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} e^{\lambda s} u(s) ds, \qquad a \leq t \leq b, \end{aligned}$$

where ${}^{RL}D^{\alpha}_{a,t}$ denotes the Riemann-Liouville fractional derivative

$$ {}^{RL}D^{\alpha}_{a,t}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}u(s)ds. $$

Definition 2.6 (Caputo tempered fractional derivative, [37]). For $n \in \mathbb{N}$, $n-1<\alpha<n$ and $\lambda \geq 0$. The Caputo tempered fractional derivative operator ${}^{C}D^{\alpha,\lambda}_{a,t}$ of order α is defined by

$$ \begin{aligned} {}^{C}D^{\alpha,\lambda}_{a,t}u(t) &:= e^{-\lambda t}\left({}^{C}D^{\alpha}_{a,t}\left(e^{\lambda t}u(t)\right)\right) \\ &= \frac{e^{-\lambda t}}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\frac{d^n e^{\lambda s}u(s)}{ds^n}ds, \quad a \leq t \leq b, \end{aligned} $$

where ${}^{C}D^{\alpha}_{a,t}$ denotes the Caputo fractional derivative

$$ {}^{C}D^{\alpha}_{a,t}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\frac{d^n u(s)}{ds^n}ds. $$

Lemma 2.7 ([37]). *Let $u(t) \in AC^n[a,b]$ and $n-1<\alpha<n$. Then the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the composite properties*

$$ I^{\alpha,\lambda}_{a,t}\left[{}^{C}D^{\alpha,\lambda}_{a,t}u(t)\right] = u(t) - \sum_{k=0}^{n-1} e^{-\lambda t}\frac{(t-a)^k}{k!}\left[\left.\frac{d^k\left(e^{\lambda t}u(t)\right)}{dt^k}\right|_{t=a}\right] $$

and

$$ {}^{C}D^{\alpha,\lambda}_{a,t}\left[I^{\alpha,\lambda}_{a,t}u(t)\right] = u(t) \qquad \text{if } \alpha \in (0,1). $$

Lemma 2.8 ([37]). *The Laplace transform of the Riemann-Liouville tempered fractional integral is given by*

$$ \mathscr{L}\left\{I^{\alpha,\lambda}_{0,t}f(t)\right\} = (s+\lambda)^{-\alpha}\mathscr{L}\{f(t)\}. $$

Definition 2.9 ([31]). The Mittag-Leffler function is defined by

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \qquad z \in \mathbb{C}, \alpha > 0. $$

The two-parameter Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \qquad z \in \mathbb{C}, \alpha>0, \beta>0. $$

One can see $E_{\alpha}(z) = E_{\alpha,1}(z)$ and $E_1(z) = E_{1,1}(z) = e^z$ from the above equation.

Lemma 2.10 ([31]). *The Laplace transform of two parameter Mittag-Leffler function is*

$$ \mathscr{L}\left\{t^{\beta-1}E_{\alpha,\beta}\left(-at^{\alpha}\right)\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}+a}, \qquad \mathrm{Re}(s) > |a|^{\frac{1}{\alpha}}, $$

where t and s are the variables in the time domain and Laplace domain, respectively.

Lemma 2.11 ([38]). *Let $\alpha > 0$ and v be a nonnegative locally integrable function on $[0,T]$ for some $T \leq \infty$. Assume that*

(1) $h(t) \geq 0$ is a nondecreasing continuous function on $[0,T]$,

(2) there exists $M > 0$ such that $h(t) \le M$, and
(3) $u(t)$ is a nonnegative and locally integrable on $0 \le t < T$.

If

$$u(t) \le v(t) + h(t) \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

then

$$u(t) \le v(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(h(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} v(s) \right] ds, \quad 0 \le t \le T.$$

Moreover, if $v(t)$ is nondecreasing function on $[0,T]$ then

$$u(t) \le v(t) E_\alpha \left(h(t)\Gamma(\alpha) t^\alpha \right).$$

Definition 2.12 ([1, 39])**.** The Wright type function is given by

$$\phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)} = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-z)^k \Gamma(\alpha(k+1)) \sin(\pi(k+1)\alpha)}{k!}$$

for $0 < \alpha < 1$ and $z \in \mathbb{C}$.

Proposition 2.13 ([1, 39])**.** *The wright type function ϕ_α is an entire function and has the following properties:*

(1) $\phi_\alpha(\theta) \ge 0$ for $\theta \ge 0$ and $\int_0^\infty \phi_\alpha(\theta) d\theta = 1$;
(2) $\int_0^\infty \phi_\alpha(\theta)\theta^r d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r > -1$;
(3) $\int_0^\infty \phi_\alpha(\theta) e^{-z\theta} d\theta = E_\alpha(-z), \quad z \in \mathbb{C}$;
(4) $\alpha \int_0^\infty \theta \phi_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,\alpha}(-z), \quad z \in \mathbb{C}$.

From the problem (1.2)-(1.3), we denote $A = \Delta$ and it generates a semigroup $\{T(t)\}_{t\ge0}$ on $C_0(\mathbb{R}^n)$ with domain $D(A) = \{u \in C_0(\mathbb{R}^n) | \Delta u \in C_0(\mathbb{R}^n)\}$. Then $\{T(t)\}_{t\ge0}$ is an analytic contractive semigroup $C_0(\mathbb{R}^n)$ and, for $t > 0, x \in \mathbb{R}^n$ we have

$$T(t)u = \int_{\mathbb{R}^n} \mathcal{G}(x-y,t) u(y,t) dy,$$

where $\mathcal{G}(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}$ and $T(t)$ is a contractive semigroup on $L^p(\mathbb{R}^n)$ for $p \ge 1$.

Lemma 2.14 (Lemma 3.5.8, [40])**.** *For $t > 0$, we have*

$$\|T(t)u_0\|_{L^p(\mathbb{R}^n)} \le (4\pi t)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_{L^q(\mathbb{R}^n)}$$

for $u_0 \in L^q(\mathbb{R}^n)$ and $1 \le q \le p \le \infty$.

3. Representation formula of mild solutions

Lemma 3.1. *Let $u_0 \in C_0(\mathbb{R}^n)$ and $f \in L^1((0,T], C_0(\mathbb{R}^n))$. The solution of the following problem:*

$$\begin{cases} {}^C D_{0,t}^{\alpha,\lambda} u(x,t) - \Delta u(x,t) = f(x,t), & (x,t) \in \mathbb{R}^n \times (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $0 < \alpha < 1$ *and* $\lambda \geq 0$ *satisfies the following integral equation*

$$u(x,t) = e^{-\lambda t} S_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) f(x,s) ds,$$

where

$$S_\alpha(t) u_0 = \int_0^\infty \phi_\alpha(\theta) \left(\int_{\mathbb{R}^n} \mathcal{G}(x-y, t^\alpha \theta) u_0(y) dy \right) d\theta$$

and

$$T_\alpha(t) u_0 = \int_0^\infty \alpha\theta \left(\int_{\mathbb{R}^n} \phi_\alpha(\theta) \mathcal{G}(x-y, t^\alpha \theta) u_0(y) dy \right) d\theta \qquad \text{for } t \geq 0.$$

Proof. Taking the Fourier transform with respect to x into Equation (3.1) , we obtain

$${}^C D_{0,t}^{\alpha,\lambda} \hat{U}(\xi,t) + |\xi|^2 \hat{U}(\xi,t) = \hat{F}(\xi,t) \quad \text{and} \quad \hat{U}(\xi,0) = \hat{U}_0(\xi).$$

Applying Lemma 2.7, we obtain

$$\hat{U}(\xi,t) = e^{-\lambda t} \hat{U}_0(\xi) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\lambda(t-\tau)} \left(-|\xi|^2 \hat{U}(\xi,\tau) + \hat{F}(\xi,\tau) \right) d\tau. \tag{3.2}$$

Taking Laplace transform with respect to t into both side of (3.2), we obtain

$$\begin{aligned}
\mathscr{L}\left\{\hat{U}(\xi,t)\right\} &= \mathscr{L}\left\{e^{-\lambda t}\hat{U}_0(\xi)\right\} \\
&\quad + \frac{1}{(s+\lambda)^\alpha} \mathscr{L}\left\{\left(-|\xi|^2 \hat{U}(\xi,t) + \hat{F}(\xi,t)\right)\right\} \\
\left(1 + (s+\lambda)^{-\alpha} |\xi|^2\right) \mathscr{L}\left\{\hat{U}(\xi,t)\right\} &= \frac{1}{s+\lambda} \hat{U}_0(\xi) + \frac{1}{(s+\lambda)^\alpha} \mathscr{L}\left\{\hat{F}(\xi,t)\right\} \\
\left((s+\lambda)^{-\alpha} + |\xi|^2\right) \mathscr{L}\left\{\hat{U}(\xi,t)\right\} &= (s+\lambda)^{\alpha-1} \hat{U}_0(\xi) + \mathscr{L}\left\{\hat{F}(\xi,t)\right\} \\
\mathscr{L}\left\{\hat{U}(\xi,t)\right\} &= \left((s+\lambda)^\alpha + |\xi|^2\right)^{-1} \\
&\quad \times \left[(s+\lambda)^{\alpha-1} \hat{U}_0(\xi) + \mathscr{L}\left\{\hat{F}(\xi,t)\right\}\right].
\end{aligned}$$

Applying the inverse Laplace transform to the both sides of the above equation, we obtain

$$\begin{aligned}
\hat{U}(\xi,t) &= \mathscr{L}^{-1}\left\{\left((s+\lambda)^\alpha + |\xi|^2\right)^{-1} (s+\lambda)^{\alpha-1}\right\} \hat{U}_0(\xi) \\
&\quad + \mathscr{L}^{-1}\left\{\left((s+\lambda)^\alpha + |\xi|^2\right)^{-1}\right\} * \mathscr{L}^{-1}\left\{\mathscr{L}\left\{\hat{F}(\xi,t)\right\}\right\} \\
&= e^{-\lambda t} E_\alpha(-|\xi|^2 t^\alpha) \hat{U}_0(\xi) + e^{-\lambda t} t^{\alpha-1} E_{\alpha,\alpha}\left(-|\xi|^2 t^\alpha\right) * \hat{F}(\xi,t) \\
&= e^{-\lambda t} E_\alpha(-|\xi|^2 t^\alpha) \hat{U}_0(\xi) + \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} E_{\alpha,\alpha}\left(-|\xi|^2 (t-s)^\alpha\right) \hat{F}(\xi,s) ds \\
&= e^{-\lambda t} \int_0^\infty \phi_\alpha(\theta) e^{-|\xi|^2 t^\alpha \theta} d\theta \hat{U}_0(\xi) \\
&\quad + \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \left(\alpha \int_0^\infty \theta \phi_\alpha(\theta) e^{\left(-|\xi|^2 (t-s)^\alpha\right)} d\theta \right) \hat{F}(\xi,s) ds.
\end{aligned}$$

Since

$$\mathcal{F}^{-1}\left[e^{-|\xi|^2 t}\right] = \frac{1}{(2t)^{n/2}} e^{-|x|^2/(4t)},$$

the inverse Fourier transform of the above equation can be represented as

$$
\begin{aligned}
&u(x,t)\\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\mathcal{F}^{-1}\left[e^{-\lambda t}\int_0^\infty \phi_\alpha(\theta)e^{-|\xi|^2 t^\alpha\theta}d\theta\right] * u_0(x)\\
&\quad + \int_0^t \frac{1}{(2\pi)^{\frac{n}{2}}}\mathcal{F}^{-1}\left[(t-s)^{\alpha-1}e^{-\lambda(t-s)}\left(\alpha\int_0^\infty \theta\phi_\alpha(\theta)e^{\left(-|\xi|^2(t-s)^\alpha\right)}d\theta\right)\right] * f(x,s)ds\\
&= e^{-\lambda t}\int_0^\infty \phi_\alpha(\theta)\left\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-|\xi|^2 t^\alpha\theta}e^{ix\xi}d\xi\right\}d\theta * u_0(x)\\
&\quad + \int_0^t (t-s)^{\alpha-1}e^{-\lambda(t-s)}\\
&\qquad \times\left(\alpha\int_0^\infty \theta\phi_\alpha(\theta)\left\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\left(-|\xi|^2(t-s)^\alpha\right)}e^{ix\xi}d\xi\right\}d\theta\right) * f(x,s)ds\\
&= e^{-\lambda t}\int_0^\infty \phi_\alpha(\theta)\mathcal{G}(x,t^\alpha\theta)d\theta * u_0(x)\\
&\quad + \int_0^t (t-s)^{\alpha-1}e^{-\lambda(t-s)}\left(\alpha\int_0^\infty \theta\phi_\alpha(\theta)\mathcal{G}(x,(t-s)^\alpha\theta)d\theta\right) * f(x,s)ds,
\end{aligned}
$$

where $\mathcal{G}(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{\frac{-|x|^2}{4t}}$ is the heat kernel.

Therefore, the solution of Equation (3.1) can be represented by

$$
\begin{aligned}
&u(x,t)\\
&= e^{-\lambda t}\int_0^\infty \phi_\alpha(\theta)\left(\int_{\mathbb{R}^n}\mathcal{G}(x-y,t^\alpha\theta)u_0(y)dy\right)d\theta\\
&\quad + \int_0^t (t-s)^{\alpha-1}e^{-\lambda(t-s)}\left(\int_0^\infty \alpha\theta\left(\int_{\mathbb{R}^n}\phi_\alpha(\theta)\mathcal{G}(x-y,(t-s)^\alpha\theta)f(y,s)dy\right)d\theta\right)ds.
\end{aligned}
$$

■

Lemma 3.2 ([25]). *The operator $S_\alpha(t)$ for $t>0$ has the following properties:*

(1) If $u_0 \geq 0$ and $u_0 \neq 0$, then

$$S_\alpha(t)u_0 > 0 \qquad \text{and} \qquad \|S_\alpha(t)u_0\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}.$$

(2) If $1 \leq q \leq p \leq \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} < \frac{2}{n}$, then

$$\|S_\alpha(t)u_0\|_{L^p(\mathbb{R}^n)} \leq (4\pi t^\alpha)^{-\frac{n}{2r}}\frac{\Gamma\left(1-\frac{n}{2r}\right)}{\Gamma\left(1-\frac{n}{2r}\alpha\right)}\|u_0\|_{L^q(\mathbb{R}^n)}.$$

Lemma 3.3 ([25]). *The operator $T_\alpha(t)$ for $t>0$ has the following properties:*

(1) If $u_0 \geq 0$ and $u_0 \neq 0$, then

$$T_\alpha(t)u_0 > 0 \qquad \text{and} \qquad \|T_\alpha(t)u_0\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}.$$

(2) If $1 \leq q \leq p \leq \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} < \frac{4}{n}$, then

$$\|T_\alpha(t)u_0\|_{L^p(\mathbb{R}^n)} \leq \alpha(4\pi t^\alpha)^{-\frac{n}{2r}}\frac{\Gamma\left(2-\frac{n}{2r}\right)}{\Gamma\left(1+\alpha-\alpha\frac{n}{2r}\right)}\|u_0\|_{L^q(\mathbb{R}^n)}.$$

Lemma 3.4. *Assume that* $f \in L^p([0,T];C_0(\mathbb{R}^n))$, $p>1$, $\alpha p>1$ *and*

$$w(t)=\int_0^t (t-s)^{\alpha-1}e^{-\lambda(t-s)}T_\alpha(t-s)f(s)ds.$$

Then $w \in C\left([0,T];C_0(\mathbb{R}^n)\right)$.

Proof. The proof follows a similar argument to the proof of Lemma 2.4 in [25]. Let $X=C_0\left(\mathbb{R}^n\right)$. We have that for every $h>0$ and $t \le t+h \le T$,

$$\begin{aligned}
&w(t+h)-w(t)\\
&=\int_0^{t+h}(t+h-s)^{\alpha-1}e^{-\lambda(t+h-s)}T_\alpha(t+h-s)f(s)ds\\
&\qquad-\int_0^t (t-s)^{\alpha-1}e^{-\lambda(t-s)}T_\alpha(t-s)f(s)ds\\
&=\alpha\int_0^{t+h}\int_0^\infty \theta\phi_\alpha(\theta)(t+h-s)^{\alpha-1}e^{-\lambda(t+h-s)}T\left((t+h-s)^\alpha\theta\right)f(s)d\theta ds\\
&\qquad-\alpha\int_0^t\int_0^\infty \theta\phi_\alpha(\theta)(t-s)^{\alpha-1}e^{-\lambda(t-s)}T\left((t-s)^\alpha\theta\right)f(s)d\theta ds\\
&=\alpha\int_t^{t+h}\int_0^\infty \theta\phi_\alpha(\theta)(t+h-s)^{\alpha-1}e^{-\lambda(t+h-s)}T\left((t+h-s)^\alpha\theta\right)f(s)d\theta ds\\
&\qquad+\alpha\int_0^t\int_0^\infty \theta\phi_\alpha(\theta)\Big[(t+h-s)^{\alpha-1}e^{-\lambda(t+h-s)}T\left((t+h-s)^\alpha\theta\right)\\
&\qquad\quad-(t-s)^{\alpha-1}e^{-\lambda(t-s)}T\left((t-s)^\alpha\theta\right)\Big]f(s)d\theta ds\\
&=: I_1+I_2.
\end{aligned}$$

We estimate I_1 by Hölder inequality and Proposition 2.13 to obtain

$$\begin{aligned}
\|I_1\|_X &\le \alpha\int_t^{t+h}\int_0^\infty \theta\phi_\alpha(\theta)(t+h-s)^{\alpha-1}e^{-\lambda(t+h-s)}\|f(s)\|_X d\theta ds\\
&=\frac{1}{\Gamma(\alpha)}\int_t^{t+h}\left(\int_0^\infty \phi_\alpha(\theta)d\theta\right)(t+h-s)^{\alpha-1}e^{-\lambda(t+h-s)}\|f(s)\|_X ds\\
&\le \frac{1}{\Gamma(\alpha)}\|f\|_{L^p((0,T),X)}\left(\int_t^{t+h}\left[(t+h-s)^{\alpha-1}e^{-\lambda(t+h-s)}\right]^{\frac{p}{p-1}}ds\right)^{\frac{p-1}{p}}\\
&\le \frac{e^{-\lambda(t+h)}}{\Gamma(\alpha)}\|f\|_{L^p((0,T),X)}\left(\int_t^{t+h}\left[(t+h-s)^{\alpha-1}e^{\lambda s}\right]^{\frac{p}{p-1}}ds\right)^{\frac{p-1}{p}}\\
&\le \frac{1}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\|f\|_{L^p((0,T),X)}h^{\frac{p\alpha-1}{p}}.
\end{aligned}\tag{3.3}$$

To estimate I_2, consider for $0<s<t$

$$\begin{aligned}
&\left\|(t+h-s)^{\alpha-1}e^{-\lambda(t+h-s)}T\left((t+h-s)^\alpha\theta\right)f(s)-(t-s)^{\alpha-1}e^{-\lambda(t-s)}T\left((t-s)^\alpha\theta\right)f(s)\right\|_X\\
&\qquad\le 2(t-s)^{\alpha-1}\|f(s)\|_X
\end{aligned}$$

and there exists constant $C > 0$ such that

$$\begin{aligned}
&\left\| \left[(t+h-s)^{\alpha-1} e^{-\lambda(t+h-s)} T\left((t+h-s)^{\alpha}\theta\right) - (t-s)^{\alpha-1} e^{-\lambda(t-s)} T\left((t-s)^{\alpha}\theta\right) \right] f(s) \right\|_X \\
&\le \left| (t+h-s)^{\alpha-1} e^{-\lambda(t+h-s)} - (t-s)^{\alpha-1} e^{-\lambda(t-s)} \right| \left\| T\left((t+h-s)^{\alpha}\theta\right) f(s) \right\|_X \\
&\qquad + (t-s)^{\alpha-1} e^{-\lambda(t-s)} \left\| \left(T\left((t+h-s)^{\alpha}\theta\right) - T\left((t-s)^{\alpha}\theta\right)\right) f(s) \right\|_X \\
&\le C(t-s)^{\alpha-2} h \|f(s)\|_X.
\end{aligned}$$

It follows

$$\begin{aligned}
\|I_2\|_X &\le C \int_0^t \int_0^\infty \alpha\theta\phi_\alpha(\theta) \min\left\{ \frac{1}{(t-s)^{1-\alpha}}, \frac{h}{(t-s)^{2-\alpha}} \right\} d\theta \|f(s)\|_X ds \\
&\le \frac{C}{\Gamma(\alpha)} \left(\int_0^t \left(\min\left\{ \frac{1}{(t-s)^{1-\alpha}}, \frac{h}{(t-s)^{2-\alpha}} \right\} \right)^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \|f\|_{L^p((0,T),X)}.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^t & \left(\min\left\{ \frac{1}{(t-s)^{1-\alpha}}, \frac{h}{(t-s)^{2-\alpha}} \right\} \right)^{\frac{p}{p-1}} ds \\
&= \int_0^t \left(\min\left\{ \frac{1}{s^{1-\alpha}}, \frac{h}{s^{2-\alpha}} \right\} \right)^{\frac{p}{p-1}} ds \\
&\le \int_0^\infty \left(\min\left\{ \frac{1}{s^{1-\alpha}}, \frac{h}{s^{2-\alpha}} \right\} \right)^{\frac{p}{p-1}} ds \\
&= \int_0^h s^{p(\alpha-1)/(p-1)} ds + \int_h^\infty h^{\frac{p}{p-1}} s^{\frac{p(\alpha-2)}{p-1}} ds \\
&= \frac{p(p-1)}{(p\alpha-1)(p+1-p\alpha)} h^{\frac{p\alpha-1}{p-1}},
\end{aligned}$$

we obtain

$$\|I_2\|_X \le C \|f\|_{L^p((0,T),X)} h^{\frac{p\alpha-1}{p}}. \tag{3.4}$$

By the estimation (3.3) and (3.4), we conclude

$$\|w(t+h) - w(t)\|_X \le \left(\frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha-1} \right)^{\frac{p-1}{p}} h^{\frac{p\alpha-1}{p}} + C h^{\frac{p\alpha-1}{p-1}} \right) \|f\|_{L^p((0,T),X)}.$$

It follows that

$$\|w(t+h) - w(t)\|_X \to 0 \quad \text{as } h \to 0$$

which implies that w is continuous. ∎

4. Local existence and uniqueness Results

For this section, we will prove the local existence and uniqueness of mild solution for problem (1.2)-(1.3). Firstly, we introduce the definition of mild solution of (1.2)-(1.3).

Definition 4.1. Let $u_0 \in C_0(\mathbb{R}^n)$ and $T > 0$. A function $u \in C([0,T]; C_0(\mathbb{R}^n))$ is called a mild solution of the problem (1.2)-(1.3) if satisfies the following integral equation

$$u(t) = e^{-\lambda t} S_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) |u|^{p-1} u(s) ds.$$

Theorem 4.2. *Let $u_0 \in C_0(\mathbb{R}^n)$. Then, there exists $T > 0$ such that the problem (1.2)-(1.3) has a unique mild solution u in $C([0,T]; C_0(\mathbb{R}^n))$ provided that satisfy*

$$\frac{2^p T^\alpha e^{\lambda T}}{\alpha \Gamma(\alpha)} \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^n)} \leq 1 \qquad \text{and} \qquad \frac{4^{p-1} p T^\alpha e^{\lambda T}}{\alpha \Gamma(\alpha)} \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^n)} < 1. \tag{4.1}$$

Proof. For given $T > 0$ and $u_0 \in C_0(\mathbb{R}^n)$, let

$$E_T = \{u | u \in C([0,T]; C_0(\mathbb{R}^n)), \|u\|_{L^\infty((0,T); L^\infty(\mathbb{R}^n))} \leq 2\|u_0\|_{L^\infty(\mathbb{R}^n)}\}$$

and

$$d(u,v) = \max_{t\in[0,T]} \|u(t) - v(t)\|_{L^\infty(\mathbb{R}^n)} \qquad \text{for } u, v \in E_T.$$

Since $C([0,T]; C_0(\mathbb{R}^n))$ is a Banach space, (E_T, d) is a complete metric space. We define the operator as

$$(\mathcal{A}u)(t) = e^{-\lambda t} S_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) \left|u^{p-1}\right| u(s) ds, \quad u \in E_T.$$

Then, $\mathcal{A}u \in C([0,T]; C_0(\mathbb{R}^n))$ by Lemma 3.4. First, we will show that $\mathcal{A}$ is a self map on E_T. Let $u \in E_T$. Then for any $t \in [0,T]$ by Lemma 3.2 and 3.3, we have

$$\begin{aligned}
&\|(\mathcal{A}u)(t)\|_{L^\infty(\mathbb{R}^n)} \\
&\leq e^{-\lambda t} \|S_\alpha(t) u_0\|_{L^\infty(\mathbb{R}^n)} + \left\| \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) \left|u(s)^{p-1}\right| u(s) ds \right\|_{L^\infty(\mathbb{R}^n)} \\
&\leq e^{-\lambda t} \|u_0\|_{L^\infty(\mathbb{R}^n)} + \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} |T_\alpha(t-s)| \, |u(s)|^{p-1} u(s)_{L^\infty(\mathbb{R}^n)} ds \\
&\leq e^{-\lambda t} \|u_0\|_{L^\infty(\mathbb{R}^n)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \|u(s)\|^p_{L^\infty(\mathbb{R}^n)} ds \\
&\leq e^{-\lambda t} \|u_0\|_{L^\infty(\mathbb{R}^n)} + \frac{2^p \|u_0\|^p_{L^\infty(\mathbb{R}^n)} e^{-\lambda t}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\lambda s} ds \\
&\leq \|u_0\|_{L^\infty(\mathbb{R}^n)} + \frac{2^p \|u_0\|^p_{L^\infty(\mathbb{R}^n)} e^{\lambda T}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&= \|u_0\|_{L^\infty(\mathbb{R}^n)} + \frac{2^p \|u_0\|^p_{L^\infty(\mathbb{R}^n)} e^{\lambda T}}{\Gamma(\alpha)} \left(\frac{t^\alpha}{\alpha}\right) \\
&\leq \|u_0\|_{L^\infty(\mathbb{R}^n)} + \frac{2^p T^\alpha e^{\lambda T}}{\alpha \Gamma(\alpha)} \|u_0\|^p_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

It follows that

$$\|(\mathcal{A}u)(t)\|_{L^\infty(\mathbb{R}^n)} \leq 2\|u_0\|_{L^\infty(\mathbb{R}^n)}.$$

Now, we show that $\mathcal{A}$ is a contraction map. Let $u, v \in E_T$. Then, for any $t \in [0, T]$ we have

$$
\begin{aligned}
&\|(\mathcal{A}u)(t) - (\mathcal{A}v)(t)\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \left\| T_\alpha(t-s)|u(s)|^{p-1}u(s) - T_\alpha(t-s)|v(s)|^{p-1}v(s) \right\|_{L^\infty(\mathbb{R}^n)} ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \left\| |u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s) \right\|_{L^\infty(\mathbb{R}^n)} ds \\
&\leq \frac{p}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \left(\|u(s)\|^{p-1}_{L^\infty(\mathbb{R}^n)} + \|v(s)\|^{p-1}_{L^\infty(\mathbb{R}^n)} \right) \|u(s) - v(s)\|_{L^\infty(\mathbb{R}^n)} ds \\
&\leq \frac{4^{p-1} p \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^n)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} \|u(s) - v(s)\|_{L^\infty(\mathbb{R}^n)} ds \\
&\leq \frac{4^{p-1} p \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^n)}}{\Gamma(\alpha)} \|u(s) - v(s)\|_{L^\infty((0,T);L^\infty(\mathbb{R}^n))} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} ds \\
&= \frac{4^{p-1} p \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^n)} e^{-\lambda t}}{\Gamma(\alpha)} \|u(s) - v(s)\|_{L^\infty((0,T);L^\infty(\mathbb{R}^n))} \int_0^t (t-s)^{\alpha-1} e^{\lambda s} ds \\
&= \frac{4^{p-1} p \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^n)} t^\alpha e^{\lambda T}}{\alpha\Gamma(\alpha)} \|u(s) - v(s)\|_{L^\infty((0,T);L^\infty(\mathbb{R}^n))} \\
&\leq \frac{4^{p-1} p \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^n)} T^\alpha e^{\lambda T}}{\alpha\Gamma(\alpha)} \|u(s) - v(s)\|_{L^\infty((0,T);L^\infty(\mathbb{R}^n))} \\
&< \|u(s) - v(s)\|_{L^\infty((0,T);L^\infty(\mathbb{R}^n))}.
\end{aligned}
$$

Therefore $\mathcal{A}$ is a contraction map on E_T . As a consequence of the Banach fixed point theorem, there exists a unique fixed point $u^* \in E_T$ such that $u^* = \mathcal{A}u^*$. Therefore, u^* is the unique mild to problems (1.2)-(1.3) on $[0, T]$. ■

Theorem 4.3. *Let $u_0 \in C_0(\mathbb{R}^n)$. Then, there exists a maximal time $T_{max} = T(u_0) > 0$ such that the problem (1.2)-(1.3) has a unique mild solution u in $C([0, T]; C_0(\mathbb{R}^n))$ and either $T_{max} = +\infty$ or $T_{max} < +\infty$ and $\|u\|_{L^\infty((0,t);C_0(\mathbb{R}^n))} \to \infty$ as $t \to T_{max}$.*

Proof. We notice that a mild solution u of the problem (1.2)-(1.3) defined on $[0,\ T]$ can be extended to a larger interval $[0,\ T+h]$ with $h > 0$. Let $v(t) = u(t+T)$ be a mild solution of

$$
\begin{cases}
{}^C D^{\alpha,\lambda}_{0,t} v - \Delta v = |v|^{p-1} v, & t \in (0, h], \\
v(0) = u(T).
\end{cases}
$$

Therefore, repeating the methods of steps in Theorem 4.2, we can prove that there exists a maximal interval $[0, T_{\max})$ such that the mild solution u of the problem (1.2)-(1.3), where

$$T_{\max} = \sup\{T > 0 | \text{there exists a mild solution } u \text{ of (1.2)-(1.3) in } C([0, T]; C_0(\mathbb{R}^n))\}.$$

Assume that $T_{\max} < +\infty$ and there exists $M > 0$ such that

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq M, \qquad \text{for } t \in [0, T_{\max}).$$

Now, we will claim that $\lim_{t \to T_{\max-}} u(t)$ exists in $C_0(\mathbb{R}^n)$. For any $0 < t_1 < t_2 < T_{\max}$, we have

$$
\begin{aligned}
&\|u(t_2) - u(t_1)\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \le \left\|e^{-\lambda t_2} S_\alpha(t_2)u_0 - e^{-\lambda t_1} S_\alpha(t_1)u_0\right\|_{L^\infty(\mathbb{R}^n)} \\
&\qquad + \left\| \int_0^{t_2} (t_2 - s)^{\alpha-1} e^{-\lambda(t_2-s)} T_\alpha(t_2 - s)|u(s)|^{p-1}u(s)ds \right. \\
&\qquad\quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} e^{-\lambda(t_1-s)} T_\alpha(t_1 - s)|u(s)|^{p-1}u(s)ds \right\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \le \left\|e^{-\lambda t_2} S_\alpha(t_2)u_0 - e^{-\lambda t_1} S_\alpha(t_1)u_0\right\|_{L^\infty(\mathbb{R}^n)} \\
&\qquad + \left\| \int_0^{t_1} (t_2 - s)^{\alpha-1} e^{-\lambda(t_2-s)} T_\alpha(t_2 - s)\,|u(s)|^{p-1}\,u(s)ds \right. \\
&\qquad\qquad + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} e^{-\lambda(t_2-s)} T_\alpha(t_2 - s)|u(s)|^{p-1}u(s)ds \\
&\qquad\qquad - \int_0^{t_1} (t_1 - s)^{\alpha-1} e^{-\lambda(t_1-s)} T_\alpha(t_1 - s)|u(s)|^{p-1}u(s)ds \\
&\qquad\qquad + \int_0^{t_1} (t_1 - s)^{\alpha-1} e^{-\lambda(t_2-s)} T_\alpha(t_2 - s)|u(s)|^{p-1}u(s)ds \\
&\qquad\qquad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} e^{-\lambda(t_2-s)} T_\alpha(t_2 - s)|u(s)|^{p-1}u(s)ds \right\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \le \left\|e^{-\lambda t_2} S_\alpha(t_2)u_0 - e^{-\lambda t_1} S_\alpha(t_1)u_0\right\|_{L^\infty(\mathbb{R}^n)} \\
&\qquad + \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] e^{-\lambda(t_2-s)} T_\alpha(t_2 - s)|u(s)|^{p-1}u(s)ds \right\|_{L^\infty(\mathbb{R}^n)} \\
&\qquad + \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} e^{-\lambda(t_2-s)} T_\alpha(t_2 - s)|u(s)|^{p-1}u(s)ds \right\|_{L^\infty(\mathbb{R}^n)} \\
&\qquad + \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} e^{\lambda s} [e^{-\lambda t_2} T_\alpha(t_2 - s) - e^{-\lambda t_1} T_\alpha(t_1 - s)]|u(s)|^{p-1}u(s)ds \right\|_{L^\infty(\mathbb{R}^n)} \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

It can be shown that $I_1 \to 0$ as $t_1 \to t_2$ and we obtain that

$$I_2 \le \frac{M^p e^{\lambda T_{\max}}}{\Gamma(\alpha+1)} [t_2^\alpha - t_1^\alpha - (t_2 - t_1)^\alpha]$$

and

$$I_3 \le \frac{M^p e^{\lambda T_{\max}}}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha$$

and hence $I_2 \to 0$ and $I_3 \to 0$ as $t_2 \to t_1$.

For $t_1 = 0$ and $0 < t_2 \leq T$, it is easy to see that $I_4 = 0$. Then, for any $\varepsilon \in (0, t_1)$, we have

$$\begin{aligned} I_4 \leq & \left\| \int_0^{t_1-\varepsilon} (t_1-s)^{\alpha-1} e^{\lambda s} [e^{-\lambda t_2} T_\alpha(t_2-s) - e^{-\lambda t_1} T_\alpha(t_1-s)] |u(s)|^{p-1} u(s) ds \right\|_{L^\infty(\mathbb{R}^n)} \\ & + \left\| \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\alpha-1} e^{\lambda s} [e^{-\lambda t_2} T_\alpha(t_2-s) - e^{-\lambda t_1} T_\alpha(t_1-s)] |u(s)|^{p-1} u(s) ds \right\|_{L^\infty(\mathbb{R}^n)} \\ \leq & \sup_{0 \leq s < t_1-\varepsilon} \left\| e^{-\lambda(t_2-s)} T_\alpha(t_2-s) - e^{-\lambda(t_1-s)} T_\alpha(t_1-s) \right\|_{L^\infty(\mathbb{R}^n)} \times \frac{M^p \varepsilon^\alpha}{\alpha} + \frac{2 M^p \varepsilon^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

It follows that $I_4 \to 0$ as $t_2 \to t_1$ and $\varepsilon \to 0$. This implies that

$$\|u(t_2) - u(t_1)\|_{L^\infty(\mathbb{R}^n)} \to 0 \qquad \text{as } t_2 \to t_1.$$

Hence $u_{T_{\max}} := \lim\limits_{t \to T_{\max}-} u(t)$ exists in $C_0(\mathbb{R}^n)$. We define $u(T_{\max}) = u_{T_{\max}}$. Then, $u \in C([0, T_{\max}], C_0(\mathbb{R}^n))$ and then, by Lemma 3.4,

$$\int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) |u|^{p-1} u(s) ds \in C\left([0, T_{\max}], C_0\left(\mathbb{R}^N\right)\right).$$

For $h > 0$ and $\delta > 0$, we let

$$E_{h,\delta} = \{u \in C([T_{\max}, T_{\max}+h]; C_0(\mathbb{R}^n)) \mid u(T_{\max}) = u_{T_{\max}}, d(u, u_{T_{\max}}) \leq \delta\},$$

where

$$d(u, v) = \max_{t \in [T_{\max}, T_{\max}+h]} \|u(t) - v(t)\|_{L^\infty(\mathbb{R}^n)}$$

for $u, v \in E_{h,\delta}$. Since $C([T_{\max}, T_{\max}+h]; C_0(\mathbb{R}^n))$ is a Banach space, $(E_{h,\delta}, d)$ is a complete metric space. We define the operator $\mathcal{K}$ on $E_{h,\delta}$

$$\begin{aligned} (\mathcal{K}v)(t) = & e^{-\lambda t} S_\alpha(t) u_0 + \int_0^{T_{\max}} (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) |u|^{p-1} u(s) ds \\ & + \int_{T_{\max}}^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) |v|^{p-1} v(s) ds, \quad v \in E_{h,\delta}. \end{aligned}$$

It is clearly that $\mathcal{K}v \in C([T_{\max}, T_{\max}+h]; C_0(\mathbb{R}^n))$ and $(\mathcal{K}v)(T_{\max}) = u_{T_{\max}}$. Let $v \in E_{h,\delta}$. Then, for any $t \in [T_{\max}, T_{\max}+h]$

$$\begin{aligned} & \|(\mathcal{K}v)(t) - u_{T_{\max}}\|_{L^\infty(\mathbb{R}^n)} \\ & \leq \left\| e^{-\lambda t} S_\alpha(t) u_0 - e^{-\lambda T_{\max}} S_\alpha(T_{\max}) u_0 \right\|_{L^\infty(\mathbb{R}^n)} \\ & \quad + \left\| \int_0^{T_{\max}} (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) |u|^{p-1} u(s) ds \right. \\ & \qquad \left. - \int_0^{T_{\max}} (T_{\max}-s)^{\alpha-1} e^{-\lambda(T_{\max}-s)} T_\alpha(T_{\max}-s) |u|^{p-1} u(s) ds \right\|_{L^\infty(\mathbb{R}^n)} \\ & \quad + \left\| \int_{T_{\max}}^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) |v|^{p-1} v(s) ds \right\|_{L^\infty(\mathbb{R}^n)} \\ & =: J_1 + J_2 + J_3. \end{aligned}$$

Taking $h>0$ small enough so that $J_1 \leq \frac{\delta}{3}, J_2 \leq \frac{\delta}{3}$ and

$$
\begin{aligned}
J_3 \leq & \left\| \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) \Big(\left| v^{p-1} \right| v(s) - |u_{T_{\max}}|^{p-1} u_{T_{\max}} \Big) ds \right\|_{L^\infty(\mathbb{R}^n)} \\
& + \left\| \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(t-s)} T_\alpha(t-s) \, |u_{T_{\max}}|^{p-1} u_{T_{\max}} \right\|_{L^\infty(\mathbb{R}^n)} \\
\leq & \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(T_{\max}-s)} \left\| T_\alpha(t-s) \Big(\left| v^{p-1} \right| v(s) - |u_{T_{\max}}|^{p-1} u_{T_{\max}} \Big) \right\|_{L^\infty(\mathbb{R}^n)} ds \\
& + \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(T_{\max}-s)} \left\| T_\alpha(t-s) |u_{T_{\max}}|^{p-1} u_{T_{\max}} \right\|_{L^\infty(\mathbb{R}^n)} ds \\
\leq & \frac{p}{\Gamma(\alpha)} \left(\|v\|^{p-1}_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} + \|u_{T_{\max}}\|^{p-1}_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} \right) \\
& \quad \times \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(T_{\max}-s)} \|v(s) - u_{T_{\max}}\|_{L^\infty(\mathbb{R}^n)} ds \\
& \quad + \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(T_{\max}-s)} \left\| T_\alpha(t-s) |u_{T_{\max}}|^{p-1} u_{T_{\max}} \right\|_{L^\infty(\mathbb{R}^n)} ds \\
\leq & \frac{p}{\Gamma(\alpha)} \left(\|v\|^{p-1}_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} + \|u_{T_{\max}}\|^{p-1}_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} \right) \\
& \quad \times \delta \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(T_{\max}-s)} ds \\
& + \frac{\|u_{T_{\max}}\|^{p}_{L^\infty(\mathbb{R}^n)}}{\Gamma(\alpha)} \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(T_{\max}-s)} ds \\
\leq & \frac{p}{\Gamma(\alpha)} \left(\|v\|_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} + \|u_{T_{\max}}\|_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} \right)^{p-1} \\
& \quad \times \frac{\delta}{\alpha} e^{\lambda(t-T_{\max})} (t-T_{\max})^\alpha + \frac{\|u_{T_{\max}}\|^{p}_{L^\infty(\mathbb{R}^n)}}{\Gamma(\alpha+1)} e^{\lambda(t-T_{\max})} (t-T_{\max})^\alpha \\
\leq & \frac{\delta}{3}
\end{aligned}
$$

for $t \in [T_{\max}, T_{\max}+h]$. Then, we obtain

$$
\|(\mathcal{K}v)(t) - u_{T_{\max}}\|_{L^\infty(\mathbb{R}^n)} \leq \delta, \qquad t \in [T_{\max}, T_{\max}+h].
$$

Next, we show that $\mathcal{K}$ is a contraction map on $E_{h,\delta}$ for h small enough. Let $w, v \in E_{h,\delta}$. Then, for any $t \in [T_{\max}, T_{\max}+h]$ we have

$$
\begin{aligned}
& \|(\mathcal{K}w)(t) - (\mathcal{K}v)(t)\|_{L^\infty(\mathbb{R}^n)} \\
& \quad \leq \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{-\lambda(t-s)} \left\| T_\alpha(t-s) \left(|w|^{p-1} w(s) - |v|^{p-1} v(s) \right) \right\|_{L^\infty(\mathbb{R}^n)} ds \\
& \quad \leq \|w-v\|_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} \\
& \qquad \times p \left(\|w\|^{p-1}_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} + \|v\|^{p-1}_{L^\infty((T_{\max}, T_{\max}+h); L^\infty(\mathbb{R}^n))} \right) \\
& \qquad \times \frac{e^{-\lambda T_{\max}}}{\Gamma(\alpha)} \int_{T_{\max}}^{t} (t-s)^{\alpha-1} e^{\lambda s} ds
\end{aligned}
$$

$$\leq \|w-v\|_{L^\infty((T_{\max},T_{\max}+h);L^\infty(\mathbb{R}^n))}$$
$$\times\, p\left(\|w\|_{L^\infty((T_{\max},T_{\max}+h);L^\infty(\mathbb{R}^n))}+\|v\|_{L^\infty((T_{\max},T_{\max}+h);L^\infty(\mathbb{R}^n))}\right)^{p-1}$$
$$\times\,\frac{e^{\lambda(t-T_{\max})}}{\Gamma(\alpha+1)}(t-T_{\max})^\alpha$$
$$\leq 2^{p-1}p\Big(\delta+\|u_{T_{\max}}\|_{L^\infty(\mathbb{R}^n)}\Big)^{p-1}\frac{e^{\lambda(t-T_{\max})}}{\Gamma(\alpha+1)}(t-T_{\max})^\alpha$$
$$\times\,\|w-v\|_{L^\infty((T_{\max},T_{\max}+h);L^\infty(\mathbb{R}^n))}$$
$$\leq 2^{p-1}p\Big(\delta+\|u_{T_{\max}}\|_{L^\infty(\mathbb{R}^n)}\Big)^{p-1}\frac{e^{\lambda h}}{\Gamma(\alpha+1)}h^\alpha\|w-v\|_{L^\infty((T_{\max},T_{\max}+h);L^\infty(\mathbb{R}^n))}.$$

Choosing h small enough such that

$$2^{p-1}p\Big(\delta+\|u_{T_{\max}}\|_{L^\infty(\mathbb{R}^n)}\Big)^{p-1}\frac{e^{\lambda h}}{\Gamma(\alpha+1)}h^\alpha\leq\frac{1}{2}.$$

Then, it follows that $\mathcal{K}$ is a contraction map on $E_{h,\delta}$. By the Banach fixed point theorem, there exists a unique fixed point $v^*\in E_{h,\delta}$ such that $(\mathcal{K}v^*)(t)=v^*(t)$. Since $v\,(T_{\max})=\mathcal{K}v\,(T_{\max})=u_{T_{\max}}$, we can define

$$\tilde{u}(t)=\begin{cases} u(t), & t\in[0,T_{\max}) \\ v(t), & t\in[T_{\max},T_{\max}+h]\,, \end{cases}$$

then $\tilde{u}(t)\in C\left(([0,T_{\max}+h],C_0(\mathbb{R}^n)\right)$ and

$$\tilde{u}(t)=e^{-\lambda t}S_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}e^{-\lambda(t-s)}T_\alpha(t-s)|\tilde{u}|^{p-1}\tilde{u}(s)ds.$$

Thus, $\tilde{u}(t)$ is a mild solution of (1.2)-(1.3), which contradicts with the definition of $T_{\max}$.
■

5. Continuous dependence on initial conditions

Now, we give a characterization of continuous dependence on initial conditions of the problem (1.2)–(1.3).

Theorem 5.1. *Let u and w be solutions to the following problem:*

$$\begin{cases} {}^C D_{0,t}^{\alpha,\lambda}u-\Delta u= |u|^{p-1}u, & (x,t)\in\mathbb{R}^n\times(0,T_{\max}] \\ u(x,0)=u_0(x), & x\in\mathbb{R}^n \end{cases}$$

and

$$\begin{cases} {}^C D_{0,t}^{\alpha,\lambda}w-\Delta w= |w|^{p-1}w, & (x,t)\in\mathbb{R}^n\times(0,\tilde{T}_{\max}] \\ w(x,0)=w_0(x), & x\in\mathbb{R}^n \end{cases}$$

for some $T_{\max}<\infty$ and $\tilde{T}_{\max}<\infty$ satisfying the condition (4.1). Then, for all $t\in[0,T]$, we have

$$\|u(t)-w(t)\|_{L^\infty(\mathbb{R}^n)}\leq e^{-\lambda t}\|u_0-w_0\|_{L^\infty(\mathbb{R}^n)}E_\alpha\left(p\left[2\left(\|u_0\|_{L^\infty(\mathbb{R}^n)}+\|w_0\|_{L^\infty(\mathbb{R}^n)}\right)\right]^{p-1}t^\alpha\right)$$

for some $0<T\leq\min\{T_{\max},\tilde{T}_{\max}\}$.

Proof. By Lemma 3.1, we have

$$u(t)=e^{-\lambda t}S_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}e^{-\lambda(t-s)}T_\alpha(t-s)|u|^{p-1}u(s)ds,\quad \text{for } t\in[0,T_{\max}]$$

and

$$w(t)=e^{-\lambda t}S_\alpha(t)w_0+\int_0^t(t-s)^{\alpha-1}e^{-\lambda(t-s)}T_\alpha(t-s)|w|^{p-1}w(s)ds\quad \text{for } t\in[0,\tilde{T}_{\max}].$$

By the proof of Theorem 4.2, we can find $0<T\le\min\{T_{\max},\tilde{T}_{\max}\}$ such that

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)}\le 2\|u_0\|_{L^\infty(\mathbb{R}^n)}\quad\text{and}\quad \|w(t)\|_{L^\infty(\mathbb{R}^n)}\le 2\|w_0\|_{L^\infty(\mathbb{R}^n)}$$

for all $t\in[0,T]$.

Setting $x(t):=\|u(t)-w(t)\|_{L^\infty(\mathbb{R}^n)}$. Then, we obtain

$$\begin{aligned}
x(t)&\le e^{-\lambda t}\|S_\alpha(t)\left(u_0-w_0\right)\|_{L^\infty(\mathbb{R}^n)}\\
&\quad+\int_0^t(t-s)^{\alpha-1}e^{-\lambda(t-s)}\|T_\alpha(t-s)|u|^{p-1}u(s)-T_\alpha(t-s)|w|^{p-1}w(s)\|_{L^\infty(\mathbb{R}^n)}ds\\
&\le e^{-\lambda t}\|u_0-w_0\|_{L^\infty(\mathbb{R}^n)}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}e^{-\lambda(t-s)}\||u|^{p-1}u(s)-|w|^{p-1}w(s)\|_{L^\infty(\mathbb{R}^n)}ds\\
&\le e^{-\lambda t}\|u_0-w_0\|_{L^\infty(\mathbb{R}^n)}\\
&\quad+\frac{2^{p-1}p}{\Gamma(\alpha)}\left(\|u_0\|_{L^\infty(\mathbb{R}^n)}+\|w_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1}\\
&\qquad\qquad\times\int_0^t(t-s)^{\alpha-1}e^{-\lambda(t-s)}\left\|u(s)-w(s)\right\|_{L^\infty(\mathbb{R}^n)}ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&e^{\lambda t}x(t)\\
&\quad\le\|u_0-w_0\|_{L^\infty(\mathbb{R}^n)}\\
&\qquad+\frac{2^{p-1}p}{\Gamma(\alpha)}\left(\|u_0\|_{L^\infty(\mathbb{R}^n)}+\|w_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1}\\
&\qquad\qquad\times\int_0^t(t-s)^{\alpha-1}e^{\lambda s}\left\|u(s)-w(s)\right\|_{L^\infty(\mathbb{R}^n)}ds\\
&\quad=x(0)+\frac{2^{p-1}p}{\Gamma(\alpha)}\left(\|u_0\|_{L^\infty(\mathbb{R}^n)}+\|w_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1}\int_0^t(t-s)^{\alpha-1}e^{\lambda s}\left\|x(s)\right\|_{L^\infty(\mathbb{R}^n)}ds.
\end{aligned}$$

Applying Lemma 2.11, we obtain

$$e^{\lambda t}x(t)\le x(0)E_\alpha\left(\frac{2^{p-1}p}{\Gamma(\alpha)}\left(\|u_0\|_{L^\infty(\mathbb{R}^n)}+\|w_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1}\Gamma(\alpha)t^\alpha\right)$$

and hence

$$x(t)\le e^{-\lambda t}x(0)E_\alpha\left(2^{p-1}p\left(\|u_0\|_{L^\infty(\mathbb{R}^n)}+\|w_0\|_{L^\infty(\mathbb{R}^n)}\right)^{p-1}t^\alpha\right).$$

∎

6. An example

In this section, we present an example of the time-fractional nonlinear diffusion equation. Let u and w be solution of the following problem:

$$\begin{cases} {}^{C}D_{0,t}^{\frac{1}{6},1}u - \Delta u = |u|^2 u, & (x,t) \in \mathbb{R} \times (0,1], \\ u(x,0) = \frac{1}{8}e^{-x^2-2}, & x \in \mathbb{R} \end{cases} \tag{6.1}$$

and

$$\begin{cases} {}^{C}D_{0,t}^{\frac{1}{6},1}w - \Delta w = |w|^2 w, & (x,t) \in \mathbb{R} \times (0,\frac{3}{2}], \\ w(x,0) = \frac{1}{20}\sin^2(x), & x \in \mathbb{R}, \end{cases} \tag{6.2}$$

respectively. Here, we let $\alpha = \frac{1}{6}$, $\lambda = 1$, $p = 3$, $T = 1$, $\tilde{T} = \frac{3}{2}$, $u_0(x) = \frac{1}{8}e^{-x^2-2}$ and $w_0(x) = \frac{1}{20}\sin^2(x)$.

Then, we calculate the condition (4.1) to obtain

$$\frac{2^p T^\alpha e^{\lambda T}}{\alpha\Gamma(\alpha)} \|u_0\|_{L^\infty(\mathbb{R})}^{p-1} = \frac{8e}{\Gamma\left(\frac{7}{6}\right)}\left(\frac{1}{8e^2}\sup_{x\in\mathbb{R}} e^{-x^2}\right)^2 = \frac{1}{8e^3\Gamma\left(\frac{7}{6}\right)} \approx 0.00671 \le 1,$$

$$\frac{2^p \tilde{T}^\alpha e^{\lambda \tilde{T}}}{\alpha\Gamma(\alpha)} \|w_0\|_{L^\infty(\mathbb{R})}^{p-1} = \frac{8\left(\frac{3}{2}\right)^{\frac{1}{6}} e^{\frac{3}{2}}}{\Gamma\left(\frac{7}{6}\right)}\left(\frac{1}{20}\sup_{x\in\mathbb{R}} \sin^2(x)\right)^2 = 0.10337 \le 1,$$

$$\frac{4^{p-1} p T^\alpha e^{\lambda T}}{\alpha\Gamma(\alpha)} \|u_0\|_{L^\infty(\mathbb{R})}^{p-1} = \frac{48e}{\Gamma\left(\frac{7}{6}\right)}\left(\frac{1}{8e^2}\sup_{x\in\mathbb{R}} e^{-x^2}\right)^2 = \frac{3}{4e^3\Gamma\left(\frac{7}{6}\right)} \approx 0.04025 < 1$$

and

$$\frac{4^{p-1} p \tilde{T}^\alpha e^{\lambda \tilde{T}}}{\alpha\Gamma(\alpha)} \|w_0\|_{L^\infty(\mathbb{R})}^{p-1} = \frac{48\left(\frac{3}{2}\right)^{\frac{1}{6}} e^{\frac{3}{2}}}{\Gamma\left(\frac{7}{6}\right)}\left(\frac{1}{20}\sup_{x\in\mathbb{R}} \sin^2(x)\right)^2 \approx 0.62024 < 1.$$

By Theorem 4.2, we obtain that the problem (6.1) and (6.2) have the unique mild solutions u and w, respectively. Moreover, the mild solution u of the problem (6.1) is continuously dependent on the initial data (6.1) with

$$\|u(t) - w(t)\| \le e^{-t}\left(\frac{1}{8}e^{-2} + \frac{1}{20}\right) E_\alpha\left(3\left(\frac{1}{4e^2} + \frac{1}{10}\right)^2 t^\alpha\right) \qquad \text{for all } t \in [0,T],$$

with $0 < T \le \min\{T, \tilde{T}\}$.

7. Conclusion

In this paper, we investigated local existence uniqueness of mild solution which is basic concepts of knowledge. These results can be extended to maximal existence result. Furthermore, the continuous dependence on initial conditions are also proved. Other aspects in qualitative analysis of solutions of diffusion equations such as blow-up, global existence and regularity could be further investigated under various fractional derivatives and conditions.

Acknowledgements

The authors would like to thank referees for their suggestions. Moreover, the first author is grateful to the Science Achievement Scholarship of Thailand for financial support throughout my study.

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