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Strong Convergence of a Parallel Extragradient-Like Algorithm Involving Pseudo-Monotone Mappings for Solving Common Variational Inequality Problems

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Abstract The purpose of this paper is to introduce a new parallel extragradient-like algorithm for solving common variational inequality problems with pseudo-monotone and Lipschitz continuous mappings in a Hilbert space. The iterative algorithm combines inertial technique and hybrid extragradient ideas with the Armijo-like step size rule. Strong convergence of the algorithm is obtained under some suitable conditions.

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1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty, convex, and closed subsets of H. Recell that the nearest point (or metric) projection from H onto C is defined by

$$Proj_C^H y := \{x \in C : \|x - y\| = dist_C(y)\},\ \forall y \in H, \tag{1.1}$$

where $dist_C(y) := inf_{x\in C}\|x - y\|$. The nearest point projection operator plays a crucial role innonsmooth analysis, convex programming, and fixed point problems of nonexpansive-like mappings.

Let $F : C \to H$ be a mapping, recall that the following classical variational inequality problem (VIP) is to find $z \in C$ such that

$$\langle F(z), x - z\rangle \geq 0 \text{ for all } x \in C. \tag{1.2}$$

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From now on, the solution set of the variational inequality problem by $VI(C, F)$. One knows that x solves problem $VI(C, F)$ if and only if x is a fixed point of the mapping $Proj_C^H(I - \mu F)$, that is,

$$x = Proj_C^H(x - \mu Fx), \quad \mu > 0. \tag{1.3}$$

The VIP(1.2), which serves as a powerful and efficient mathematical model, unifies many classical concepts in convex programming, nonsmooth analysis, variational analysis, and functional analysis, such as complementarity problems, systems of nonlinear equations, and equilibrium problems under a general setting. In recent years, much attention has been given to developing efficient and implementable numerical solution methods for solving variational inequality problems and related convex optimization problems, see [1, 3–5] and the references therein.

Korpelevich [6] proposed the extragradient method in 1976 for solving saddle point problems, and it has since been extended to the VIP. It was proved that, in a finite dimensional space, the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in H, \\ y_n &= Proj_C^H(x_n - \alpha F x_n), \\ x_{n+1} &= Proj_C^H(x_n - \alpha F y_n), n \geq 1, \end{aligned} \tag{1.4}$$

where $F : C \to H$ is a monotone and L-Lipschitz continuous mapping, and $\alpha\ \epsilon(0, \frac{1}{L})$, converges weakly to a solution of variational problem (1.2); see [6] and the references therein.

Recently, the extragradient method has been extensively investigated and extended to the framework of infinite dimensional spaces, and a number of efficient modifications were obtained; see [7–11] and the references therein. In particular, an alternative modification to the extragradient method is the following algorithm proposed by Tseng [12].

$$\begin{aligned} x_1 &\in H, \\ y_n &= Proj_C^H(x_n - \alpha F x_n), \\ x_{n+1} &= Proj_C^H(y_n - \alpha(F y_n - F x_n)), n \geq 1, \end{aligned} \tag{1.5}$$

where F is a monotone and L-Lipchitz continuous mapping, X is some convex and closed set, and α is a constant in $(0, \frac{1}{L})$. In real-world applications, such as physics [13], economics [17], image recovery [18], and control theory [19], many results or modelings are investigated in the setting of infinite dimension spaces. Strong convergence, that is, convergence in the norm, is much more useful and desirable in such problems than weak convergence, that is, convergence in the weak topology, because strong convergence of iterative sequences translates into tangible properties, for example, the energy $\|x_n - \overline{x}\|$of the error between the solution $\overline{x}$and the iterate x_n becomes arbitary. The natural question that arises is how to construct an efficient algorithm which guarantees the strong convergence, that is, norm convergence, in the setting of infinite dimensional spaces with no compact conditions on involved operators. To obtain an answer to this question, Nadezhkina and Takahashi [20] introduced the following hybrid extragradient method

$$\begin{aligned} x_1 &\in H, \\ y_n &= Proj_C^H(x_n - \alpha F x_n), \\ z_n &= Proj_C^H(x_n - \alpha F y_n), \\ C_n &= \{\omega \in C : \|z_n - \omega\| \le \|x_n - \omega\|\}, \\ Q_n &= \{\omega \in C : \langle x_n - \omega, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= Proj_{C_n \bigcap Q_n}^H(x_0), n \ge 1, \end{aligned} \tag{1.6}$$

where $F : C \to H$ is a monotone and L-Lipschitz continuous mapping with $\alpha \epsilon \left(0, \frac{1}{L}\right)$. They proved that the sequence $\{x_n\}$ generated by (1.9) converges in norm to $Proj_{VI(C,F)}^H(x_0)$.

To implement the extragradient algorithm, one needs to calculate projections two times at each iteration. This may affect the efficiency of the algorithm from viewpoint of computation. With some choices on set C_n, we show that it is possible to throw out one projection at each iteration in (1.4) or (1.9).

Recently, Censor, Gibali, and Reich [**?**] and Malitsky and Semenov [9] investigated hybrid extragradient algorithms with the strong restrictions that the potential mapping is both monotone and Lipschitz continuous. However, it is not very clear if convergence is still available if monotonicity is replaced by pseudo-monotonicity. We, in this paper, consider the pseudo-monotonicity in our convergence analysis, which is one of the highlights of this paper. Based on the methods and techniques discussed in this manuscript, many convex optimization problems could be extended to pseudo-convex optimization problems. Very recently, Liu and Qin [33] introduced new extragradient-like algorithm for solving a variational inequality problem with a pseudo-monotone and Lipschitz continuous mapping in a Hilbert space. This algorithm was generated as follow:

$$\begin{aligned} x_0, x_1 &\in H, \\ y_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ z_n &= Proj_C^H(x_n - \mu_n F y_n), \\ & \quad \textit{where } \mu_n \textit{ is chosen to be the largest } \mu \in \left\{\varrho, \varrho\sigma, \varrho\sigma^2, \ldots\right\} \textit{ such that} \\ & \quad \mu\|F z_n - F y_n\| \le \nu\|z_n - y_n\| \\ w_n &= z_n - \mu_n(F z_n - F y_n), \\ C_n &= \{p \in H : \|x_n - p\|^2 - \left(1 - 2\nu^2\right)\|x_n - z_n\|^2 + 2\nu^2\alpha_n^2\|x_{n-1} - x_n\|^2 \\ & \quad \ge \|w_n - p\|^2\}, \\ Q_n &= \{p \in H : \langle x_n - p, x_0 - x_n\rangle \ge 0\}, n \ge 1, \\ x_{n+1} &= Proj_{C_n \bigcap Q_n}^H(x_0), n \ge 1, \end{aligned} \tag{1.7}$$

On the other hand, the inertial extrapolation, which was first proposed by Polyak [21] as an acceleration process, has been employed to solve various convex minimization problems recently. It is based on the heavy ball method of the two-order time dynamical system. Inertial type methods involve two iterative steps and the second iterative step is obtained with the aid of previous two iterates. They can be viewed as an efficient technique to deal with various iterative algorithms, in particular, the projection-based algorithms; see [14–16, 22–25, 27, 28, 28].

Recently, Gibali [29] introduced a self-adaptive method by adopting Armijo-like searches and obtained a convergence result in a finite dimensional space. This methods uses variable and nonmonotone stepsizes avoiding using the Lipschitz continuity; however, it uses two values of operator F at each iteration even with fixed steps.

In this research we focus on the common variational inequality problems (CVIP) is to find $z \epsilon C$ such that

$$\langle F_i(z), x - z \rangle \geqslant 0, \text{ for all } x \epsilon C \text{ and } i = 1, 2, 3, ..., N. \tag{1.8}$$

In 2012, Censor et al. [30] presented the algorithm for solving the CVIP here,finite elements are computed in parallel of each iterations. The closed convex subset $C_n^1, C_n^2, ..., C_n^N$ are constructed getting x_{n+1} which is projected onto the intersection of these closed convex subset. This algorithm generated by $x_1 \epsilon H$ and compute

$$\begin{aligned}
y_n^i &= P_{K_i}(x_n - \lambda_n^i A_i(x_n)), \\
z_n^i &= P_{K_i}(x_n - \lambda_n^i A_i(y_n^i)), \\
C_n^i &= \{z \epsilon H : \langle x_n - z_n^i, z - x_n - \gamma_n^i \left(z_n^i - x_n\right) \rangle \leq 0\}, \\
C_n &= \bigcap_{i=1}^{N} C_n^i \ , \\
W_n &= \{z \epsilon H : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \\
x_{n+1} &= P_{C_n \cap W_n} x_1,
\end{aligned} \tag{1.9}$$

where K_i is a nonempty closed and convex subset of H.

Recently, some authors have constructed different fast iterative algorithms with the aim of inertial extrapolation, such as inertial proximal algorithms, inertial forward-backward splitting algorithms, inertial Mann algorithms, and inertial subgradient extragradient algorithms. Among first-order methods, there has always been some trade-off between methods with variable stepsizes and ones with fixed stepsizes. Methods requiring fixed stepsizes necessitate knowledge of the Lipschitz constant of monotone mappings.One can estimate the Lipschitz constant from the above; however, this estimation is often quite conservative. As a result, methods with fixed stepsizes typically use tiny steps.Methods with variables stepsizes execute some search procedures with the goal of locating an appropriate stepsize in each case iteration. They are more flexible and often allow us to use a larger step than what is predicted by the Lipschitz constant. At this moment, there are some adaptivity techniques for variational inequality problems.

In this paper, inspired and motivated by the methods, a parallel extragradient-like algorithm is proposed for solving common variational inequalities problems involving pseudomonotone mapping in real Hilbert spaces. It is important to note that our proposed scheme is effective. In particular, our algorithm can solve common pseudomonotone variational inequalities. The proof of strong convergence of the proposed algorithm is proved without knowing the Lipschitz constant of the operator F. The proposed algorithm could be seen as a modification of the methods that have appeared. A strong convergence theorem is proved under mild conditions.

2. Preliminaries

Now let us recall some related definitions here. A mapping $F : H \to H$ is said to be:

(i) L-lipschitz continuous if and only if there exists a positive constant $L > 0$ such that

$$\|F\upsilon' - F\upsilon\| \leq L\|\upsilon' - \upsilon\|, \quad \text{for all } \upsilon', \upsilon \in H; \tag{2.1}$$

(ii) Monotone if and only if

$$\langle \upsilon' - \upsilon, F\upsilon' - F\upsilon \rangle \geq 0, \quad \text{for all } \upsilon', \upsilon \in H; \tag{2.2}$$

(iii) Pseudo-monotone if and only if

$$\langle F\upsilon', \upsilon - \upsilon' \rangle \geq 0 \Rightarrow \langle F\upsilon, \upsilon - \upsilon' \rangle \geq 0, \quad \text{for all} \upsilon', \upsilon \in H; \tag{2.3}$$

(iv) Sequentially weakly continuous if and only if, for each sequence $\{x_n\}, \{x_n\}$ converges weakly to x implies that $F(x_n)$ converges weakly to $F(x_n)$.

In order to prove our main result, we also need the following lemmas. The first one is known as the KK property.

Lemma 2.1. [31] *Let H be a Hilbert space. Let $\{x_n\}$ be a vector sequence in H. If $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$, then $x_n \rightharpoonup x$.*

Lemma 2.2. [32] *Let H be a Hilbert space and let $F : C \to H$ be a pseudomonotone and continuous mapping. Then, $\widehat{x}$ is a solution of the $VI(C, F)$ if and only if $\langle F(x), x - \widehat{x} \rangle \geq 0, \forall x \in C$.*

3. Main Results

In this section, we do several computational experiments in support of the convergence of the proposed algorithm and compare our algorithm with some existing algorithms in literatures. First,we introduce two algorithm, which solve our proposed problems the following assumptions will be used through the rest of this paper.

(a) Mapping $F_i : H \to H$ is pseudo-monoton, L-Lipschitz continuous, and sequentially weakly continuous on bounded sets.

(b) Solution set $\bigcap_{i=1}^{N} VI(C, F_i) \neq \emptyset$.

Algorithm 3.1. (Parallel inertial hybrid extragradient algorithm with the Armijo-like step).

Initialization: Let $x_0, x_1 \in C$ be arbitrary. Given $\alpha_n \in (0, +\infty)$. Let $\nu^i \in (0, \frac{1}{\sqrt{2}})$ and $\varrho, \sigma \in (0, 1)$. Set $C_0 = Q_0 = H$.

Step 0. Set n = 1.

Step 1. Given the current iterates $x_{n-1}, x_n \in C$, calculate $y_n = x_n + \alpha_n (x_n - x_{n-1})$.

Step 2. Compute

$$z_n^i = Proj_C^H (x_n - \mu_n F_i y_n), \tag{3.1}$$

where μ_n^i is chosen to be the largest $\mu^i \in \{\varrho, \varrho\sigma, \varrho\sigma^2, ...\}$ such that

$$\mu^i \left\|F_i z_n^i - F_i y_n\right\| \leq \nu^i \left\|z_n^i - y_n\right\|. \tag{3.2}$$

Step 3. If $y_n - z_n^i = 0$, for all i = 1,2,...,N , then stop and x_n is a solution of $VI(C, F)$. Otherwise, go to **Step 4.**

Step 4.

$$w_n^i = z_n^i - \mu_n^i \left(F_i z_n^i - F_i y_n\right) \tag{3.3}$$

$$\overline{w}_n = argmax \left\{\left\|w_n^i - x_n\right\| : i = 1, 2, ..., N\right\} \tag{3.4}$$

$$\nu = min\left\{\nu^i : i = 1, 2, ..., N\right\} \tag{3.5}$$

Step 5. Construct sets C_n and Q_n as

$$\begin{aligned} C_n &= \left\{p \in H : \|x_n - p\|^2 + 2\nu^2\alpha_n^2 \|x_{n-1} - x_n\|^2 \geq \|\overline{w}_n - p\|^2\right\}, \\ Q_n &= \{p \in H : \langle x_n - p, x_0 - x_n\rangle \geq 0\}, n \geq 1. \end{aligned} \tag{3.6}$$

(3.7)

Step 6. Calculate

$$x_{n+1} = Proj_{C_n \cap Q_n}^H x_0. \tag{3.8}$$

Set $n := n + 1$ and return to **Step 1.**

Lemma 3.2. *The Armijo-like search rule (3.2) always terminates and*

$$\min\left\{\varrho, \frac{\nu^i\sigma}{L_i}\right\} \leq \mu_n^i \leq \varrho. \tag{3.9}$$

Proof. Since F_i is L_i-Lipschitz continuous on H, we have

$$\frac{\nu^i}{L_i}\left\|F_i(y_n) - F_i\left(Proj_C^H\left(x_n - \mu_n^i F_i y_n\right)\right)\right\| \leq \nu^i\left\|y_n - Proj_C^H\left(x_n - \mu_n^i F_i y_n\right)\right\|.$$

This implies that (3.2) holds for all $\mu^i \leq \frac{\nu^i}{L}$, thus μ_n^i is well defined. Obviously, $\varrho \geq \mu_n^i$. If $\mu_n^i = \varrho$, this lemma is proved. If $\mu_n^i < \varrho$, one sees that $\frac{\mu_n^i}{\sigma}$ must violate inequality (3.2). Combining this with the fact that F_i is L-Lipschitz continuous on H, one has

$$\begin{aligned} & \nu^i\left\|y_n - Proj_C^H\left(x_n - \mu_n^i F_i y_n\right)\right\| \\ < & \frac{\mu_n^i}{\sigma}\left\|F_i(y_n) - F_i\left(Proj_C^H\left(x_n - \mu_n^i F_i y_n\right)\right)\right\| \\ \leq & \frac{L\mu_n^i}{\sigma}\left\|y_n - Proj_C^H\left(x_n - \mu_n^i F_i y_n\right)\right\|. \end{aligned} \tag{3.10}$$

It follows that

$$\frac{\nu^i\sigma}{L} < \mu_n^i \tag{3.11}$$

This completes the proof. ■

Lemma 3.3. *Let $\{x_n\}$, $\{y_n\}$, $\{z_n^i\}$, and $\{w_n^i\}$ be sequences generated by Algorithm 3.1 and let u be a solution of $VI(C,F)$. Then,*

$$\left\|w_n^i - u\right\|^2 \leq \|x_n - u\|^2 + 2\nu^2\alpha_n^2\|x_n - x_{n-1}\|^2. \tag{3.12}$$

Proof. Let u be a solution of the $VI(C,F)$. It follows from (3.1) that $z_n^i \in C$. Therefore, one obtains that $\left\langle F(u), z_n^i - u\right\rangle \geq 0$. By using the pseudo-monotonicity of F, one obtains that $\langle F(z_n^i), z_n^i - u\rangle \geq 0$. Using (3.1), one arrives at

$$\begin{aligned}\left\|z_n^i - u\right\|^2 &\leq \left\|u - (x_n - \mu_n^i F_i y_n)\right\|^2 - \left\|z_n^i - (x_n - \mu_n^i F_i y_n)\right\|^2 \\ &= \|x_n - u\|^2 + 2\mu_n^i \langle u - x_n, F_i y_n\rangle - \left\|z_n^i - x_n\right\|^2 - 2\mu_n^i \left\langle z_n^i - x_n, F_i y_n\right\rangle \\ &= \|x_n - u\|^2 - \left\|z_n^i - x_n\right\|^2 - 2\mu_n^i \left\langle z_n^i - u, F_i y_n\right\rangle \\ &= \|x_n - u\|^2 - \left\|z_n^i - x_n\right\|^2 - 2\mu_n^i \left\langle z_n^i - u, F_i z_n^i\right\rangle + 2\mu_n^i \left\langle z_n^i - u, F_i z_n^i - F_i y_n\right\rangle \\ &\leq \|x_n - u\|^2 - \left\|z_n^i - x_n\right\|^2 + 2\mu_n^i \left\langle z_n^i - u, F_i z_n^i - F_i y_n\right\rangle. \end{aligned} \tag{3.13}$$

In view of (3.2), (3.5), and (3.13), one has

$$\begin{aligned}\left\|w_n^i - u\right\|^2 &= \left\|z_n^i - u\right\|^2 - 2\mu_n^i \left\langle z_n^i - u, F_i z_n^i - F_i y_n\right\rangle + {\mu_n^i}^2 \left\|F_i y_n - F_i z_n^i\right\|^2 \\ &\leq \|x_n - u\|^2 - \left\|x_n - z_n^i\right\|^2 + 2\mu_n^i \left\langle u - z_n^i, F_i y_n - F_i z_n^i\right\rangle + \nu_i^2 \left\|z_n^i - z_n^i\right\|^2 \\ &\quad -2\mu_n^i \left\langle z_n^i - u, F_i z_n^i - F_i y_n\right\rangle \\ &\leq \|x_n - u\|^2 - \left\|z_n^i - x_n\right\|^2 + \nu^2 \left\|z_n^i - (x_n + \alpha_n (x_n - x_{n-1}))\right\|^2 \\ &= \|x_n - u\|^2 - \left\|z_n^i - x_n\right\|^2 + \nu^2 \left\|z_n^i - x_n\right\|^2 + \nu^2 \alpha_n^2 \|x_n - x_{n-1}\|^2 \\ &\quad -2\nu^2 \alpha_n \left\langle z_n^i - x_n, x_n - x_{n-1}\right\rangle \\ &\leq \|x_n - u\|^2 - \left\|z_n^i - x_n\right\|^2 + \nu^2 \left\|z_n^i - x_n\right\|^2 + \nu^2 \alpha_n^2 \|x_n - x_{n-1}\|^2 \\ &\quad +\nu^2 \left\|z_n^i - x_n\right\|^2 + \nu^2 \alpha_n^2 \|x_n - x_{n-1}\|^2 \\ &= \|x_n - u\|^2 - \left(1 - 2\nu^2\right) \left\|z_n^i - x_n\right\|^2 + 2\nu^2 \alpha_n^2 \|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.14}$$

This implies that

$$\left\|\overline{w}_n^i - u\right\|^2 \leq \|x_n - u\|^2 + 2\nu^2 \alpha_n^2 \|x_n - x_{n-1}\|^2. \tag{3.15}$$

for all $i = 1, 2, ..., N$. This completes the proof. ∎

The boundedness of the generated sequences will be needed, for example, to ensure the existence of weak cluster points. In light of this, we next prove the following lemma.

Lemma 3.4. *The sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded and* $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Proof. It is evident that sets C_n and Q_n are convex and closed. In view of Lemma 3.3, $VI(C, F_i) \subseteq C_n$. Let us show by the mathematical induction that $VI(C, F_i) \subseteq Q_n$ for all $n \in N$. Since $Q_0 = H$, we have $VI(C, F_i) \subseteq Q_0$. Suppose that $VI(C, F_i) \subseteq Q_n$. It is sufficient to prove that $VI(C, F_i) \subseteq Q_{n+1}$. From the facts that $VI(C, F_i) \subseteq C_n \cap Q_n$ and $x_{n+1} = Proj_{C_n\cap Q_n}^H x_0$, we conclude that $\langle x_{n+1} - \omega, x_0 - x_{n+1}\rangle \geq 0$, $\forall \omega \in VI(C, F_i)$, which, together with the definition of Q_n, yields that $\omega \in Q_{n+1}$. Since ω is chosen arbitrarily in $VI(C, F_i)$, we have $VI(C, F_i) \subseteq Q_{n+1}$. Hence, $VI(C, F_i) \subseteq C_n \cap Q_n$ and sequence $\{x_n\}$ is well defined. Let $z = Proj_{VI(C,F_i)}^H x_0$. Since $x_{n+1} = Proj_{C_n\cap Q_n}^H x_0$ and $z \in VI(C, F_i) \subseteq C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|, \tag{3.16}$$

which immediately implies that $\{x_n\}$ is bounded. It follows from (3.6), we have $x_n = Proj_{Q_n}^H x_0$, which together with (3.16), we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. Using the boundedness of $\{x_n\}$, one finds that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. In addition, it follows from $x_{n+1} \in Q_n$ and $x_n = Proj_{Q_n}^H x_0$ that

$$\|x_n - x_{n+1}\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 . \tag{3.17}$$

Due to the existence of $\lim_{n\to\infty}\|x_n - x_0\|$, it follows from the above inequality that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.18}$$

■

Theorem 3.5. *Let Assumptions (a) and (b) hold. The sequences* $\{x_n\}$,$\{y_n\}$, $\{z_n^i\}$, *and* $\{w_n^i\}$ *generated by Algorithm 3.1 converge strongly to* $z = Proj_{VI(C,F_i)}^H x_0$.

Proof. Substituting $p = x_{n+1}$ into set C_n, one obtains from (S2.44) that

$$\|\overline{w}_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + 2\nu^2\alpha_n^2\|x_n - x_{n-1}\|^2 . \tag{3.19}$$

Taking the limit as $n \to \infty$ in (3.19), if follows from (3.18) that

$$\begin{aligned}\lim_{n\to\infty}\|\overline{w}_n - x_{n+1}\|^2 &\le \lim_{n\to\infty}\|x_n - x_{n+1}\|^2 + 2\nu^2\alpha_n^2\lim_{n\to\infty}\|x_n - x_{n-1}\|^2 \\ &= 0. \end{aligned}\tag{3.20}$$

It follows from (3.18) that

$$\begin{aligned}\lim_{n\to\infty}\|\overline{w}_n - x_n\| &\le \lim_{n\to\infty}(\|\overline{w}_n - x_{n+1}\| + \|x_n - x_{n+1}\|) \\ &= \lim_{n\to\infty}\|\overline{w}_n - x_{n+1}\| = 0, \end{aligned}\tag{3.21}$$

and

$$\lim_{n\to\infty}\|\overline{z}_n - x_n\| = 0.$$

By virtue of **Step 1** in Algorithm 3.1 and (3.18), we get

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\alpha_n\|x_n - x_{n-1}\| = 0. \tag{3.22}$$

It follows from (3.22) and (3.22) that

$$\lim_{n\to\infty}\|y_n - \overline{z}_n\| \le \lim_{n\to\infty}(\|y_n - x_n\| + \|\overline{z}_n - x_n\|) = 0. \tag{3.23}$$

Since $\{x_n\}$ is bounded, one sees that there exists a sequence $\{x_{n_k}\}$, which is a subsequence of $\{x_n\}$, such that $\{x_{n_k}\}$ converges weakly to some $\widehat{x} \in H$. From (12), one finds that $\{y_{n_k}\}$ also weakly converges to $\widehat{x}$. From $z_n^i = Proj_C^H(x_n - \mu_n F_i y_n)$, one reaches

$$\left\langle z_{n_k}^i - (x_{n_k} - \mu_{n_k} F_i y_{n_k}), \omega - z_{n_k}^i \right\rangle \ge 0, \forall \omega \in C.$$

This implies that

$$\left\langle F_i y_{n_k}, \omega - z_{n_k}^i \right\rangle \ge \frac{1}{\mu_{n_k}}\left\langle z_{n_k}^i - x_{n_k}, z_{n_k}^i - \omega \right\rangle, \forall \omega \in C,$$

which is equivalent to

$$\mu_{n_k}\langle F_i y_{n_k}, \omega - y_{n_k}\rangle \ge \left\langle z_{n_k}^i - x_{n_k}, z_{n_k}^i - \omega \right\rangle - \mu_{n_k}\left\langle F_i y_{n_k}, y_{n_k} - z_{n_k}^i \right\rangle, \forall \omega \in C.$$

It follows from Lemma 3.2 that $\liminf_{n_k\to\infty}\mu_{n_k} > 0$. Fix $\omega \in C$. Taking the limit as $n_k \to \infty$ in the above inequality, one obtains from (3.22) and (3.23) that

$$\liminf_{n_k\to\infty}\langle F_i y_{n_k}, \omega - y_{n_k}\rangle \ge 0.$$

Now one chooses a positive real sequence $\{\varepsilon^i\}$, which is decreasing and tends to 0 as k tends to the infinity. For each ε_k^i, one denotes by m_k^i the smallest positive integer, which is such that

$$\langle F_i(y_{n_k}), \omega - y_{n_k}\rangle + \varepsilon_k^i \geq 0, \forall j \geq m_k^i, \text{ for all } i = 1, 2, ..., N$$

where the existence of m_k^i follows from (3.24). Since$\{\varepsilon_k^i\}$ is decreasing, it is easy to see that $\{m_k^i\}$ is increasing. In addition, for each k, $F_i\left(y_{n_{m_k^i}}\right) \neq 0$, one sets

$$t_{n_{m_k}}^i = \frac{F_i\left(y_{n_{m_k^i}}\right)}{\left\|F_i\left(y_{n_{m_k^i}}\right)\right\|^2}.$$

This implies $\left\langle F_i\left(y_{n_{m_k^i}}\right), t_{n_{m_k^i}}^i\right\rangle = 1$ for each k_0. It follows from (3.24) that

$$\left\langle F_i\left(y_{n_{m_k^i}}\right), \omega + \varepsilon_k t_{n_{m_k}}^i - y_{n_{m_k^i}}\right\rangle \geq 0.$$

By using the pseudo-monotonicity of F_i, we can induce from the above inequality that

$$\left\langle F_i\left(\omega + \varepsilon_k t_{n_{m_k}}^i\right), \omega + \varepsilon_k t_{n_{m_k}}^i - y_{n_{m_k^i}}\right\rangle \geq 0. \tag{3.24}$$

On the other hand, one has that sequence $\{y_{n_k}\}$ converges weakly to $\widehat{x}$ as $k \to \infty$. Since F_i is sequentially weakly continuous on C, one sees that $\{F_i(y_{n_k})\}$ converges weakly to $F_i(\widehat{x})$. Assume $F_i(\widehat{x}) \neq 0$ (otherwise, $\widehat{x}$ is a solution). Due to the fact that norms are sequentially weakly lower semicontinuous, one obtains that

$$\|F_i(\widehat{x})\| \leq \liminf_{k\to\infty} \|F_i(y_{n_k})\|.$$

From $\left\{y_{n_{m_k^i}}\right\} \subset \{y_{n_k}\}$ and $\varepsilon_k \to 0$ as $k \to \infty$, one obtains

$$0 \leq \lim_{k\to\infty} \left\|\varepsilon_k^i t_{n_{m_k}}^i\right\| = \lim_{k\to\infty} \frac{\varepsilon_k^i}{\left\|F_i\left(y_{n_{m_k^i}}\right)\right\|} \leq \frac{0}{F_i(\widehat{x})} = 0.$$

Therefore, one by taking the limit as $k \to \infty$ in (3.24) concludes that $\langle F_i(\omega), \omega - \widehat{x}\rangle \geq 0$. Combining this with Lemma 3.3 we get that $\widehat{x} \in VI(C, F_i)$. In view of $x_{n_k} = Proj_{Q_{n_k}}^H(x_0)$, $VI(C, F_i) \subseteq Q_{n_k}, z = Proj_{VI(C,F_i)}^H(x_0)$ and the lower semicontinuity of norms, one deduces that

$$\|x_0 - z\| \leq \|x_0 - \widehat{x}\| \leq \liminf_{k\to\infty} \|x_0 - x_{n_k}\| \leq \limsup_{k\to\infty} \|x_0 - x_{n_k}\| \leq \|x_0 - z\|.$$

Since the space is a Hilbert space, one directly obtains that $\lim_{k\to\infty} \|x_0 - x_{n_k}\| = \|x_0 - \widehat{x}\|$. Observe that $x_0 - x_{n_k} \to x_0 - \widehat{x}$ as $k \to \infty$. Lemma 3.2 sends us to $x_{n_k} \to \widehat{x}$ as $n_k \to \infty$. According to $z \in VI(C, F_i) \subseteq Q_n$, we have

$$\begin{aligned} \|z - x_{n_k}\|^2 &= \langle z - x_0, z - x_{n_k}\rangle + \langle x_0 - x_{n_k}, z - x_{n_k}\rangle. \\ &\leq \langle z - x_0, z - x_{n_k}\rangle. \end{aligned}$$

Letting $k \to \infty$, one reaches

$$\|z - \widehat{x}\|^2 \leq \langle z - x_0, z - \widehat{x}\rangle \leq 0.$$

This shows that $z = \widehat{x}$. Since $\{x_{n_k}\}$ is an arbitrary subsequence of sequence $\{x_n\}$,one finds that $x_n \to z$. From (3.21), (3.22), and (3.22), one concludes that $y_n \to z, \overline{z}_n \to z$ and $\overline{w}_n \to z$ as $n \to \infty$. This completes the proof. ∎

Remark 3.6. If N=1, Theorem 3.5 becomes to Theorem 1 of Liu and Qin [33].

4. Conclusion

In this work, we introduce a new parallel extragradient-like algorithm for solving common variational inequalities of nonmonotone and Lipschitz continuous mappings in real Hilbert spaces. The inertial technique is used to speed up the convergence and the hybrid extragradient method is used for obtaining strong convergence theorem. The results obtain in this paper extend many recent ones in the literature.

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