Thai Journal of **Math**ematics
Volume 19 Number 3 (2021)
Pages 842–853

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Fractional-Order Delay Differential Equation with Separated Conditions

Piyachat Borisut[1] and Chaiwat Auipa-arch[2,*]

[1] *Faculty of Liberal Arts, Rajamangala University of Technology Rattanakosin, Samphanthawong, Bangkok 10100, Thailand*
e-mail : piyachat.b@rmutr.ac.th (P. Borisut)

[2] *Department of Mathematics, Faculty of Education, Valaya Alongkorn Rajabhat University under the Royal Potronage, Pathumthani 13180, Thailand*
e-mail : chaiwat.aui@vru.ac.th (C. Auipa-arch)

Abstract In this paper, we prove the existence and uniqueness solution of fractional delay differential equation and separated condition of the from:

$$ {}_{C}D^{q}_{0+}u(t) = f(t, u(t), u(g(t))),\ t \in [0,T],\ 1 < q < 2, $$

$$ \begin{aligned} a_1 u(0) + b_1 {}_{C}D^{p}_{0+}u(T) &= \eta_1,\ a_2 u(0) + b_2 {}_{C}D^{p}_{0+}u(T) = \eta_2,\ 0 < p \leq 1, \\ u(t) &= \psi(t),\ t \in [-h, 0) \end{aligned} $$

where ${}_{C}D^{q}_{0+}, {}_{C}D^{p}_{0+}$ are the Caputo fraction derivative of order q, p and we consider $a_1, a_2, b_1, b_2, \eta_1, \eta_2 \in \mathbb{R}$, the functions $f \in C([0,T]\times\mathbb{R}\times\mathbb{R}, \mathbb{R}),\ g \in C((0,T], [-h,T])$ with $g(t) \leq t$ and $h > 0,\ \psi(t)$ is continuous and bounded via fixed point theorem of Schaefers and Boyd-Wong nonlinear contraction. Also we give example as an application to illustrate the results obtained.

MSC: 26A33; 34A34; 34B15; 47H01; 54H25
Keywords: Fractional delay differential equation; separated conditions; fixed point theorems.

Submission date: 27.04.2021 / Acceptance date: 11.07.2021

1. Introduction

Fractional calculus is an old mathematical problem and has been always thought of as a pure mathematical problem for nearly three centuries. Though having a long history, it was not applied to physics and engineering for a long period of time. However, in the last few decades, fractional calculus began to attract increasing attention of scientists from an application point of view. Fractional differential equations have an important role in numerous fields of study carried out by mathematicians, physicists and engineers. They have used it basically to developed the mathematical modeling, many physical applications and engineering disciplines such as viscoelasticity, control, porous media, phenomena

*Corresponding author.

Published by The Mathematical Association of Thailand.

in eletromagnetics etc., (see [1–3]). The major differences between fractional order differential operator and classical calculus is its nonlocal behavior, that is the feature future state based on the fractional differential operator depends on its current and past states. More details on the fundamental concepts of fractional calculus, fractional differential equations and fractional integral equations can be found in books like A. A. Kilbas, H. M Srivastava and J. J. Trujillo [1], K. S Miller and B. Ross [2], and J. Banas and K. Goebel [4]. Fractional integro-differential equations involving the Caputo-Fabrizio derivative have been studied by many researchers from differential points of view (see [5–8] and the references therein). However, in real world systems, delay is very often encountered in many practical systems such as automatic control, biology and hydraulic networks, economics, and long transmission lines.

Delayed differential equations are correspondingly used to describe such dynamical systems. In recent years, delayed FDEs begin to arouse the attention of many researchers [9–11]. To simulate these equations is an important technique in the research, accordingly, finding effective numerical methods for the delayed FDEs is a necessary process. Qualitative theory of differential equations have significant application, and the existence of solutions and of positive solutions of fractional differential equations, which respect the initial and boundary value, have also received considerable attention. In order to study such type of problems different kind of techniques such as fixed point theorems [12–14], fixed point index [14, 15], upper and lower solutions method [16], coincidence theory [17] etc., are in vogue. For instance, in [18–20], the authors investigate the existence of solutions of boundary value problems.

$$
{}_C D_{0+}^{q} u(t) = -f(t, u(t), {}_C D_{0+}^{q} u(t)),\ 0 < t < 1,
$$

$$
au(0) - bu'(0) = 0,\ y(1) = \int_0^1 k(s) g(t, u(s)) ds + \mu,
$$

where $1 < q < 2$, ${}_C D_{0+}^{q}$ is the Caputo fraction derivative order q, $(E, \|\cdot\|)$ be Banach space, $f : [0,1] \times C([0,1], E) \times E \to E$, $k \in C([0,1], E)$,$k \neq 0$, $a, b \in \mathbb{R}^+$, $a + b > 0$ and $\frac{a}{a+b} < q - 1$.

In [21], the authors investigated the existence and uniqueness of solutions of the nonlocal fractional integral condition.

$$
\begin{aligned}
{}_{RL} D_{0+}^{q} x(t) &= f(t, x(t)),\ t \in [0, T], \\
x(0) = 0, \quad x(T) &= \sum_{i=1}^{n} \alpha_i \, {}_H I_{0+}^{p_i} x(\eta_i),
\end{aligned}
$$

where $1 < q \leq 2$, ${}_{RL} D_{0+}^{q}$ is the Riemann-Liouville fractional derivative of order q, ${}_H I_{0+}^{p_i}$ is Hadamard fractional integral of order $p_i > 0$, $\eta_i \in (0, T)$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, and $\alpha_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$ are real constants such that $\displaystyle\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{p_i}} \neq T^{q-1}$.

In [22], the authors study and investigate the $\psi-$Hilfer fractional differential equation with nonlocal multi point condition of the form:

$$
\begin{cases}
D_{a^+}^{q,p;\psi} u(t) = f(t, u(t), D_{a^+}^{q,p;\psi} u(t)),\ t \in [a, b], \\
I_{a^+}^{1-r;\psi} u(a) = \displaystyle\sum_{i=1}^{m} \beta_i u(\eta_i),\ q \leq r = q + p - qp < 1,\ \eta_i \in [a, b],
\end{cases}
$$

where $0 < q < 1,\ 0 \leq p \leq 1,\ m \in \mathbb{N},\ \beta_i \in \mathbb{R},\ i = 1, 2, \ldots, m,\ -\infty < a < b < \infty,\ D_{a^+}^{q,p;\psi}$ is the $\psi-$ Hilfer fractional derivative, $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $I_{a^+}^{1-r;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $1 - r$.

Inspired by the above papers in [18–22], the objective of this paper is to derive the existence and uniqueness solution of fractional delay differential equation and separated condition of the from:

$$ {}_C D_{0^+}^{q} u(t) = f(t, u(t), u(g(t))),\ t \in [0, T],\ 1 < q < 2, \tag{1.1} $$

$$ \begin{aligned} a_1 u(0) + b_1 {}_C D_{0^+}^{p} u(T) &= \eta_1,\ a_2 u(0) + b_2 {}_C D_{0^+}^{p} u(T) = \eta_2,\ 0 < p \leq 1, \\ u(t) &= \psi(t),\ t \in [-h, 0) \end{aligned} $$

where ${}_C D_{0^+}^{q}$ and ${}_C D_{0^+}^{p}$ are the Caputo fraction derivative of order q, p and we consider $a_1, a_2, b_1, b_2, \eta_1, \eta_2 \in \mathbb{R}$, the functions $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ g \in C((0, T], [-h, T])$ with $g(t) \leq t$ and $h > 0,\ \psi(t)$ is continuous and bounded via fixed point theorem of Schaefers and Boyd-Wong nonlinear contraction. This study gives us several ideas for solving boundary valued problem for fractional differential equation by using some suitable fixed point theorems.

The current paper is organized as follows: Section 1 contains the introduction; in Section 2, some basic definitions of fractional differential equations are introduced. In Section 3, the main results are devided into two parts; existance result via Schaefer's fixed point theorem is considered in Section 3.1; the study of existence and uniqueness result via Boyd and Wong fixed point theorem is presented in Section 3.2. As an application, we present example to illustrate the results obtained. Finally, a conclusion is presented in Section 4.

2. Preliminaries

We need the following lemmas that will be used to prove our main results.

Definition 2.1. [23] The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$ {}_{RL} I_{0^+}^{q} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) ds, $$

where $\Gamma(\cdot)$ denotes the Gamma function defined by

$$ \Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds. $$

Definition 2.2. [24] The Caputo fractional derivative of order $q > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$ \left({}_C D_{0^+}^{q} f \right)(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n-q-1} f^{(n)}(s) ds, $$

where n is the smallest integer greater than or equal to q.

Lemma 2.3. [23] Let $n - 1 < q < n$. If $f \in C^n([a, b])$, then

$$ {}_{RL} I_{0^+}^{q} ({}_C D_{0^+}^{q} x)(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, $$

where $c_i \in \mathbb{R}$,$i = 1, 2, \ldots, n$, n is the smallest integer greater than or equal to q.

Proposition 2.4. [23] If $q, \rho > 0$ then

(1) If $f(t) = k \neq 0$, k is a constant, then ${}_CD^q_{0^+}k = 0$ and ${}_{RL}D^q_{0^+}k = \frac{t^{-q}k}{\Gamma(1-q)}$.
(2) ${}_{RL}D^q_{0^+}t^{q-1} = 0$.
(3) For $\rho > 1$, we have ${}_{RL}I^q_{0^+}t^\rho = \frac{\Gamma(\rho+1)}{\Gamma(q+\rho+1)}t^{q+\rho}$.
(4) For $\rho > -1+q$, we have ${}_{RL}D^q_{0^+}t^\rho = \frac{\Gamma(\rho+1)}{\Gamma(1+\rho-q)}t^{\rho-q}$.
(5) For $\rho > 0$, we have ${}_CD^q_{0^+}t^\rho = \frac{\Gamma(\rho+1)}{\Gamma(1+\rho-q)}t^{\rho-q}$.

Lemma 2.5. [24]**(Arzela-Ascoli theorem)** Let $M \subseteq C[a,b]$. M is relatively compact in $C[a,b]$ if and only if M is
(1) uniformly bounded (meaning that it is a bounded set in $C[a,b]$),
(2) equicontinuous on $[a,b]$, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|t_2 - t_1| < \delta$ implies $|f(t_2) - f(t_1)| < \epsilon$ for any $f \in M$.

Definition 2.6. [24] Let X and Y be normed spaces. The mapping $A : X \to Y$ is said to be completely continuous, if $A(C)$ is a relatively compact subset of Y for every bounded subset C of X.

Theorem 2.7. [25] **(Schaefer's fixed point theorem)**
Let $A : X \to X$ be a completely continuous operator. If the set $E(A) = \{x \in X : x = \lambda^* Ax$ for some $\lambda^* \in [0,1]\}$ is bounded, then A has fixed points.

Definition 2.8. [25] Let E be a Banach space and let $A : E \to E$ be a mapping. A is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\epsilon) < \epsilon$ for all $\epsilon > 0$ with the property:

$$\|Ax - Ay\| \leq \Psi(\|x - y\|), \text{ for all } x, y \in E.$$

Theorem 2.9. [25] **(Boyd and Wong fixed point theorem)**
Let E be a Banach space and let $A : E \to E$ be a nonlinear contraction. Then A has a unique fixed point in E.

3. Main Results

In this section, we are going to study the existence of solution for problem (4.1) by using Schaefer's fixed point theorem. Firstly, we establish the following lemma to ensure a solution for $u(t)$.

Lemma 3.1. *Let $1 < q < 2$, assume $y(t) \in C[0,T]$, then the following equation*

$$ {}_CD^q_{0^+}u(t) = y(t), \ t \in [0,T], \ 1 < q < 2, \tag{3.1}$$

$$\begin{aligned} a_1u(0) + b_1{}_CD^p_{0^+}u(T) &= \eta_1, \ a_2u(0) + b_2{}_CD^p_{0^+}u(T) = \eta_2, \ 0 < p \leq 1, \\ u(t) &= \psi(t), \ t \in [-h, 0) \end{aligned}$$

where ${}_CD^q_{0^+}, {}_CD^p_{0^+}$ are the Caputo fraction derivative of order q, p and we consider $a_1, a_2, b_1, b_2, \eta_1, \eta_2 \in \mathbb{R}$, $h > 0, \psi(t)$ is continuous and bounded, has a solution

$$u(t) = \begin{cases} \psi(t), \ t \in [-h, 0), \\ {}_{RL}I^q_{0^+}y(t) + \dfrac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1} + \dfrac{\Gamma(2-q)t}{b_1T^{1-q}}\left(\eta_1 - a_1\left(\dfrac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\right)\right. \\ \left. -b_1{}_CD^p_{0^+}({}_{RL}I^q_{0^+}y(T))\right), \ t \in [0,T]. \end{cases}$$

Proof. We may apply Lemma 2.3 to reduce equation (3.1) to an equivalent integral equation.

$$u(t) = {}_{RL}I^q_{0+}y(t) + c_0 + c_1 t.$$

By $u(0) = c_0$ and ${}_C D^p_{0+}u(T) = {}_C D^q_{0+}({}_{RL}I^q_{0+}y(T)) + c_1\frac{\Gamma(2)T^{1-q}}{\Gamma(2+q)}$, we can get

$$a_1c_0 + b_1{}_C D^q_{0+}({}_{RL}I^q_{0+}y(T)) + \frac{c_1b_1T^{1-q}}{\Gamma(2-q)} = \eta_1$$
$$a_2c_0 + b_2{}_C D^q_{0+}({}_{RL}I^q_{0+}y(T)) + \frac{c_1b_2T^{1-q}}{\Gamma(2-q)} = \eta_2.$$

So,

$$\begin{aligned} c_0 &= \frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1} \\ c_1 &= \frac{\Gamma(2-q)}{b_1T^{1-q}}\Big[\eta_1 - a_1\left(\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\right) - b_1{}_C D^q_{0+}({}_{RL}I^q_{0+}y(T))\Big]. \end{aligned}$$

Hence,

$$u(t) = \begin{cases} \psi(t),\ t \in [-h, 0], \\ {}_{RL}I^q_{0+}y(t) + \dfrac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1} + \dfrac{\Gamma(2-q)t}{b_1T^{1-q}}\left(\eta_1 - a_1\left(\dfrac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\right)\right. \\ \left.-b_1{}_C D^p_{0+}({}_{RL}I^q_{0+}y(T))\right),\ t \in (0, T]. \end{cases}$$

Let $X := C[-h, T]$ with the sup norm and define the operator $A : C[-h, T] \to C[-h, T]$ as follows,

$$Au(t) = \begin{cases} \psi(t),\ t \in [-h, 0], \\ {}_{RL}I^q_{0+}f(t, u(t), u(g(t))) + \dfrac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1} + \dfrac{\Gamma(2-q)t}{b_1T^{1-q}}\left(\eta_1 - a_1\left(\dfrac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\right)\right. \\ \left.-b_1{}_C D^p_{0+}({}_{RL}I^q_{0+}f(T, u(T), u(g(T))))\right),\ t \in (0, T]. \end{cases}$$

then the equation (4.1) has a solution if and only if the operator A has a fixed point. ∎

3.1. Existance result via Schaefer's fixed point theorem

We begin with an existance result via Schaefer's fixed point theorem.

Theorem 3.2. Suppose that
$(H1)$ $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ g \in (I, [-h, T]),\ g(t) \le t,\ h \ge 0.$
$(H2)$ There exists $k > 0$ such that

$$\Big|f(t, u_1, u_2) - f(t, v_1, v_2)\Big| \le k\Big(|u_1 - v_1| + |u_2 - v_2|\Big) \text{ for all } t \in [0, T].$$

$(H3)$ There exist continuous functions $\theta_1, \theta_2, \theta_3 : [0,T] \to \mathbb{R}^+$ with

$$\begin{aligned}\bar{\theta}_1 &= \sup_{[0,T]} \theta_1(t) < 1,\\ \bar{\theta}_2 &= \sup_{[0,T]} \theta_2(t) < 1,\\ \bar{\theta}_3 &= \sup_{[0,T]} \theta_3(t) < 1.\end{aligned}$$

If $\Big|f(t,u,v)\Big| \leq \theta_1(t) + \theta_2(t)|u|+\theta_3(t)|v|$ for all $t \in [0,T]$, u, $v \in \mathbb{R}$, and $\Big(\frac{T^q}{\Gamma(q+1)} + \frac{\Gamma(2-q)T^{2q-p+1}}{\Gamma(q-p+1)}\Big)(\bar{\psi}_2+\bar{\psi}_3) < 1$, then the problem (4.1) has at least one solution in $C[-h,T]$.

Proof. The proof will be give in several steps.
Consider the operator $A : C[-h,T] \to C[-h,T]$, defined by:

$$Au(t) = \begin{cases} \psi(t),\ t \in [-h,0), \\ {}_{RL}I^q_{0^+} f(t,u(t),u(g(t))) + \dfrac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1} + \dfrac{\Gamma(2-q)t}{b_1T^{1-q}}\Bigg(\eta_1 - a_1\Bigg(\dfrac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Bigg) \\ -b_1{}_C D^p_{0^+}({}_{RL}I^q_{0^+} f(T,u(T),u(g(T)))\Bigg),\ t \in [0,T]. \end{cases}$$

Step 1: A is continuous.
Let $\{u_n\}$ be sequence such that $u_n \to u$ in $C[-h,T]$ where $n \to \infty$, then for each $t \in [-h,0)$, we have

$$\Big|Au_n(t) - Au(t)\Big| = \Big|\psi(t) - \psi(t)\Big| = 0,\ \text{for all } u_n, u \in C([-h,0),\mathbb{R}).$$

For all $t \in [0,T]$, we have

$$\begin{aligned}\Big|Au_n(t) - Au(t)\Big| &\leq {}_{RL}I^q_{0^+}\Big|f(t,u(t),u(g(t))) - f(t,u_n(t),u_n(g(t)))\Big| \\ &\quad + \frac{\Gamma(2-q)t}{T^{1-q}}{}_C D^p_{0^+}({}_{RL}I^q_{0^+}\Big|f(T,u(T),u(g(T))) - f(T,u_n(T),u_n(g(T)))\Big|) \\ &\leq k{}_{RL}I^q_{0^+}\Big(\Big|u(t) - u_n(t)\Big| + \Big|u(g(t)) - u_n(g(t))\Big|\Big) \\ &\quad + \frac{k\Gamma(2-q)T}{T^{1-q}}{}_C D^p_{0^+}({}_{RL}I^q_{0^+}\Big(\Big|u(T) - u_n(T)\Big| + \Big|u(g(T)) - u_n(g(T))\Big|\Big) \\ &\leq 2k\Big(1 + T^q\Gamma(2-q)\Big)\Big(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-p+1}}{\Gamma(q-p+1)}\Big)\|u_n - u\|.\end{aligned}$$

So, $\|Au_n - Au\|_{C[0,T]} \to 0$ as $n \to \infty$. Thus show that the operator A is continuous in $C[-h,T]$.
Step 2: $A(B_r) \subset B_r$.
Let u belong to B_r. In order to prove that $Au \in B_r$, it suffices to show that $|Au(t)| \leq r$

for $t \in [-h, T]$. However, for $t \in [-h, 0)$, we get $|Au(t)| = |\psi(t)| < r_1$. For $t \in [0, T]$, we get

$$\begin{aligned} |Au(t)| &\leq {}_{RL}I^q_{0^+}|f(t, u(t), u(g(t)))| + \Big|\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Big| + \frac{\Gamma(2-q)T^q}{b_1}\Big(|\eta_1| \\ &\quad + \Big|a_1\Big(\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Big)\Big| + b_1{}_CD^p_{0^+}\Big({}_{RL}I^q_{0^+}|f(T, u(T), u(g(T)))|\Big)\Big) \\ &\leq {}_{RL}I^q_{0^+}\Big(\bar{\theta}_1 + (\bar{\theta}_2 + \bar{\theta}_3)\|u\|\Big) + \Big|\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Big| + \frac{\Gamma(2-q)T^q}{b_1}\Big(|\eta_1| \\ &\quad + \Big|a_1\Big(\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Big)\Big| + b_1{}_CD^p_{0^+}\Big({}_{RL}I^q_{0^+}\Big(\bar{\theta}_1 + (\bar{\theta}_2 + \bar{\theta}_3)\|u\|\Big)\Big) \\ &\leq \Big|\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Big| + \frac{\Gamma(2-q)T^q}{b_1}\Big(|\eta_1| + \Big|a_1\Big(\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Big)\Big|\Big) + \Big(\frac{T^q}{\Gamma(q+1)} \\ &\quad + \frac{\Gamma(2-q)T^{2q-p+1}}{\Gamma(q-p+1)}\Big)\bar{\theta}_1 + \Big(\frac{T^q}{\Gamma(q+1)} + \frac{\Gamma(2-q)T^{2q-p+1}}{\Gamma(q-p+1)}\Big)(\bar{\theta}_2 + \bar{\theta}_3)\|u\| \\ &\leq r_2, \end{aligned}$$

where, $r_2 \geq \frac{\alpha+\beta}{\gamma}$ with

$$\begin{aligned} \alpha &:= \Big|\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Big|, \\ \beta &:= \frac{\Gamma(2-q)T^q}{b_1}\Big(|\eta_1| + \Big|a_1\Big(\frac{b_2\eta_1 - b_1\eta_2}{a_1b_2 - a_2b_1}\Big)\Big| + \Big(\frac{T^q}{\Gamma(q+1)} + \frac{\Gamma(2-q)T^{2q-p+1}}{\Gamma(q-p+1)}\Big)\bar{\theta}_1, \\ \gamma &:= 1 - \Big(\frac{T^q}{\Gamma(q+1)} + \frac{\Gamma(2-q)T^{2q-p+1}}{\Gamma(q-p+1)}\Big)(\bar{\theta}_2 + \bar{\theta}_3). \end{aligned}$$

Choose $r = \max\{r_1, r_2\}$. Thus, we have $A(B_r) \subset B_r$.

Step 3: $A(B_r)$ is uniformly bounded and equicontinuous.

From step 2, we get $A(B_r) = \Big\{Au : u \in B_r\Big\} \subset B_r$. Hence, for each $u \in B_r$, we get $\|Au\| \leq r$, which means that $A(B_r)$ is uniformly bounded. Let $\tau_1,\ \tau_2 \in [0, T],\ \tau_1 < \tau_2$ and choose $u \in B_r$ we have;

$$\begin{aligned} \Big|Au(\tau_2) - Au(\tau_1)\Big| &\leq \Big|\frac{1}{\Gamma(q)}\int_0^{\tau_2}(\tau_2 - s)^{q-1}f(s, u(s), u(g(s)))ds \\ &\quad - \frac{1}{\Gamma(q)}\int_0^{\tau_1}(\tau_1 - s)^{q-1}f(s, u(s), u(g(s)))ds\Big| \\ &\quad + (\tau_2 - \tau_1)\frac{\Gamma(2-q)T^q}{b_1}\Big(\eta_1 - a_1[b_2\eta_1 - b_1\eta_2] \\ &\quad + b_1{}_CD^p_{0^+}({}_{RL}I^q_{0^+}f(T, u(T), u(g(T))))\Big). \\ \Big|Au(\tau_2) - Au(\tau_1)\Big| &\leq \frac{1}{\Gamma(q)}\int_0^{\tau_1}\Big[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\Big]\Big(\bar{\theta}_1 + (\bar{\theta}_2 + \bar{\theta}_3)|u|\Big)ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}\Big(\bar{\theta}_1 + (\bar{\theta}_2 + \bar{\theta}_3)|u|\Big)ds \\ &\quad + (\tau_2 - \tau_1)\frac{\Gamma(2-q)T^q}{b_1}\Big(\eta_1 - a_1[b_2\eta_1 - b_1\eta_2] \\ &\quad + b_1{}_CD^p_{0^+}({}_{RL}I^q_{0^+}f(T, u(T), u(g(T))))\Big). \end{aligned}$$

$$\begin{aligned}\Big|Au(\tau_2)-Au(\tau_1)\Big| &\le \frac{\Big(\bar\theta_1+(\bar\theta_2+\bar\theta_3)\|u\|\Big)}{\Gamma(q+1)}\Big(\tau_2^q-\tau_1^q-(\tau_2-\tau_1)^q\Big)+\frac{(\tau_2-\tau_1)^q}{\Gamma(q+1)}\|u\|\\ &+(\tau_2-\tau_1)\frac{\Gamma(2-q)T^q}{b_1}\Big(\eta_1-a_1[b_2\eta_1-b_1\eta_2]\\ &+b_1{}_CD^p_{0+}({}_{RL}I^q_{0+}f(T,u(T),u(g(T))))\Big).\end{aligned}$$

As $\tau_1\to\tau_2$, the right hand side of the above inequality tends to zero. It is clear that if $\tau_1,\ \tau_2\in[-h,0)$ then $\|Au(\tau_2)-Au(\tau_1)\|=0$. So, $A(B_r)$ is equicontinuous and uniformly bounded. Hence as a consequence of Arzalà -Ascoli theorem, we can conclude that $A: C[-h,T]\to C[-h,T]$ is completely continuous.

Step 4: $E(A)$ is bounded set,

where $E(A)=\Big\{u\in C[-h,T]:u=\lambda^*(Au),\ 0<\lambda^*<1\Big\}$ and choose $u\in E(A),\ u=\lambda^*(Au)$ for some $0<\lambda^*<1$. So for each $t\in[0,T]$, we get

$$\begin{aligned}u(t) &= \lambda^*{}_{RL}I^q_{0+}f(t,u(t),u(g(t)))+\frac{b_2\eta_1-b_1\eta_2}{a_1b_2-a_2b_1}+\frac{\Gamma(2-q)t}{b_1T^{1-q}}\Big(\eta_1-a_1\Big(\frac{b_2\eta_1-b_1\eta_2}{a_1b_2-a_2b_1}\Big)\\ &-b_1{}_CD^p_{0+}({}_{RL}I^q_{0+}f(T,u(T),u(g(T))))\Big)\end{aligned}$$

it follows from step 2 and for each $t\in[0,T]$, we get

$$|u(t)|=|\lambda^*(Au)(t)|\le|(Au)(t)|\le r.$$

It is clear that if $t\in[-h,0)$ then $|u(t)|=|\lambda^*(Au)(t)|\le|(Au)(t)|=|\psi(t)|\le r$. Hence the set $E(A)$ is bounded set. As a consequence of Schaefer's fixed point theorem, A has a fixed point which is a solution of problem (4.1). ■

3.2. Existence and Uniqueness Result Via Boyd and Wong Fixed Point Theorem.

Theorem 3.3. *Let $f:[-h,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption:*

$$|f(t,u(t),u(g(t))-f(t,v(t),v(g(t))|\le\frac{\beta(t)|u-v|}{B+|u-v|},\ \text{for } t\in[-h,T],\ u,v\ge0,$$

where $\beta(t):[a,b]\to\mathbb{R}$ is continuous and B is the constant defined by $B:={}_{RL}I^q_{a^+}\beta(t)+\Gamma(2-q)T^q{}_CD^p_{0+}({}_{RL}I^q_{0+}f(T,u(T),u(g(T))))<1$. Then the problem (4.1) *has a unique solution on $C[-h,T]$.*

Proof. Consider a continuous non-decreasing function $\Psi:\mathbb{R}^+\bigcup\{0\}\to\mathbb{R}^+\bigcup\{0\}$ by $\Psi(\epsilon)=\frac{\beta(s)\epsilon}{B+\epsilon}$, $\forall\epsilon>0$, such that $\Psi(0)=0$ and $\Psi(\epsilon)<\epsilon\ \forall\epsilon>0$. For any $u,v\in C[-h,T]$

and for each $t \in [-h, T]$, yields.

$$
\begin{aligned}
|Au(t) - Av(t)| &\leq {}_{RL}I^q_{a^+}\Big|f(s, u(s), u(g(s)) - f(s, v(s), v(g(s))\Big| \\
&\quad + \Gamma(2-q)T^{q-p}{}_C D^p_{0^+}\big({}_{RL}I^q_{0^+}\Big|f(s, u(s), u(g(s)) - f(s, v(s), v(g(s))\Big)\\
&\leq {}_{RL}I^q_{a^+}\frac{\beta(s)|u-v|}{B+|u-v|}(t) + \Gamma(2-q)T^q{}_C D^p_{0^+}\big({}_{RL}I^q_{0^+}\frac{\beta(s)|u-v|}{B+|u-v|}\big)(T) \\
&\leq \frac{\Psi(\|u-v\|)}{B}\Big[{}_{RL}I^q_{a^+}\beta(s)(t) \\
&\quad + \Gamma(2-q)T^q{}_C D^p_{0^+}\big({}_{RL}I^q_{0^+}f(T, u(T), u(g(T)))\big)\Big] \\
&\leq \Psi(\|u-v\|).
\end{aligned}
$$

This implies that $\|Tu - Tv\| \leq \Psi(\|u-v\|)$. There for A is a non-linear contraction. Hence, by theorem (Boyd and Wong). The operator A has a unique fixed point, which is the unique solution of the problem (4.1). ■

Example 3.4. Consider the following fractional boundary value problems

$$
\begin{cases}
{}^C D^{\frac{7}{5}}_{0^+}u(t) = \frac{\sin^2 t}{999+9^t}\left(\frac{|u(t)|}{1+|u(t)|}\right) + \frac{\cos^2 t}{999+e^t}\left(\frac{|{}^C D^{\frac{7}{5}}_{0^+}u(t)|}{1+|{}^C D^{\frac{7}{5}}_{0^+}u(t)|}\right) + \frac{1}{2},\ t \in [0, \pi], \\
u(0) = \pi,\ u(\pi) = {}_{RL}\, I^{\frac{1}{3}}_{0^+}u(\frac{\pi}{2}).
\end{cases}
\tag{3.2}
$$

By comparing problem (4.1) and (3.2), we obtain the following parameters: $q = 7/5,\ p = 1/3,\ \eta = \pi,\ \kappa = \pi/2,\ f(t, u(t), {}^C D^{\frac{7}{5}}_{0^+}u(t)) = \frac{\sin^2 t}{999+9^t}\left(\frac{|u(t)|}{1+|u(t)|}\right) + \frac{\cos^2 t}{999+e^t}\left(\frac{|{}^C D^{\frac{7}{5}}_{0^+}u(t)|}{1+|{}^C D^{\frac{7}{5}}_{0^+}u(t)|}\right) + \frac{1}{2}$. As, $\Big|f(t, u, v) - f(t, u^*, v^*)\Big| \leq \frac{1}{1{,}000}\Big|u - u^*\Big| + \frac{1}{1{,}000}\Big|v - v^*\Big|$ with $M = N = 1/1{,}000$ and $f(t, u(t), {}^C D^{\frac{7}{5}}_{0^+}u(t)) \leq \frac{1}{2} + \frac{1}{1{,}000}\Big|u(t)\Big| + \frac{1}{1{,}000}\Big|v(t)\Big|$. Therefore the condition of Theorem 3.2 is satisfied with $\frac{TM\Gamma(p+2))}{(T\Gamma(p+2)-\kappa^{p+1})(1-N)}\left\{\frac{\kappa^{q+p}}{\Gamma(q+p+1)} + \frac{T^q}{\Gamma(q+1)}\right\} \approx 0.0049 < 1$. Hence, the problem (3.2) has at least one solution on $[0, \pi]$, if we choose $\beta(t) = 0.002$. Then, we find

$B \approx 0.0202$, clealy clealy,

$$
\begin{aligned}
\Big|f(t,u(t),^C D^{\frac{7}{5}}_{0+}u(t)) - f(t,v(t),^C D^{\frac{7}{5}}_{0+}v(t))\Big| &\leq \frac{1}{1,000}\Big(\frac{\Big|u-v\Big|}{1+|u-v\Big|} \\
&\quad + \frac{\Big|^C D^{\frac{7}{5}}_{0+}u(t) - {}^C D^{\frac{7}{5}}_{0+}v(t)\Big|}{1+\Big|^C D^{\frac{7}{5}}_{0+}u(t) - {}^C D^{\frac{7}{5}}_{0+}v(t)\Big|}\Big) \\
&\leq \frac{1}{1,000}\Big(\frac{\Big|u-v\Big|}{1+|u-v\Big|} + \frac{\frac{1}{999}\Big|u-v\Big|}{1+\frac{1}{999}|u-v\Big|}\Big) \\
&\leq \frac{1}{1,000}\Big(\frac{\Big|u-v\Big|}{1+|u-v\Big|} + \frac{\Big|u-v\Big|}{999+|u-v\Big|}\Big) \\
&\leq \frac{1}{500}\Big(\frac{|u-v|}{0.0202+|u-v|}\Big).
\end{aligned}
$$

Hence, by Theorem 3.3, problem (3.2) has a unique solution on $(0, \pi)$.

4. Conclusions

We have derived the existence and uniqueness solution of fractional delay differential equation and separated condition of the from:

$$
{}_C D^q_{0+}u(t) = f(t,u(t),u(g(t))),\ t \in [0,T],\ 1 < q < 2,
$$

$$
\begin{aligned}
a_1 u(0) + b_1 {}_C D^p_{0+}u(T) &= \eta_1,\ a_2 u(0) + b_2 {}_C D^p_{0+}u(T) = \eta_2,\ 0 < p \leq 1, \\
u(t) &= \psi(t),\ t \in [-h,0)
\end{aligned}
$$

where ${}_C D^q_{0+}$ and ${}_C D^p_{0+}$ are the Caputo fraction derivative of order q, p and we consider $a_1, a_2, b_1, b_2, \eta_1, \eta_2 \in \mathbb{R}$, the functions $f \in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g \in C((0,T],[-h,T])$ with $g(t) \leq t$ and $h > 0$, $\psi(t)$ is continuous and bounded via fixed point theorem of Schaefers and Boyd-Wong nonlinear contraction. Moreover, one example is given to illustrate the results obtained. Our results are new and develop the previous results in [18–22].

Acknowledgements

The first author would like to thank Rajamangala University of Technology Rattanakosin and the second author would like to thank Valaya Alongkorn Rajabhat University under the Royal Potronage, Pathumthani for giving us the opportunity to do research. Also, the authors are grateful to the referees for many useful comments and suggestions which have improved the presentation of this paper.

References

[1] I. Podlubny, Fractional differential equations. Academic Press, New york, 1999.
[2] K.S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley, New York, 1993.

[3] A.A. Kilbas, H. M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies 204 (2006) 7–10.

[4] J. Banas, K.Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics 60, Macel Dekker, New York, 1980.

[5] Sh. Rezapour, V. Hedayati, On a Caputo fractional differential inclusion with integral boundary condition for convex-compact and nonconvex-compact valued multifunctions, Kragujevac Journal of Mathematics 41 (1) (2017) 143–158.

[6] Sh. Rezapour, M. Shabibi, A singular fractional differential equation with Riemann-Liouville integral boundary condition, Journal of Advanced Mathematical Studies 8 (1) (2015) 80–88.

[7] D. Baleanu, S. Rezapour, Z. Saberpour, On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivation, Journal of Advanced Mathematical Studies (2019).

[8] D. Baleanu, A. Mousalou, S. Rezapour, A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative, Advances in Difference Equations (2017).

[9] Y. Chen, K. L. Moore, Analytical stability bound for a class of delayed fractional-order dynamic systems, Nonlinear Dynamics 29 (1) (2002) 191–200.

[10] P. Lanusse, H. Benlaoukli, D. Nelson-Gruel, A. Oustaloup, Fractional-order control and interval analysis of SISO systems with time-delayed state, IET Control Theory & Applications 2 (1) (2008) 16–23.

[11] X. Zhang, Some results of linear fractional order time-delay system, Applied Mathematics and Computation 197 (1) (2008) 407–411.

[12] Z. Han, H. Lu, C. Zhang, Positive solutions for eigenvalue problems of fractional differential equations with generalized P-Laplacian, Appl. Math. Comp. 257 (2014) 526–536.

[13] H.R. Marasi, H. Afshari, C.B. Zhai, Some existence and uniqueness results for nonlinear fractional partial differential equations, Rocky Mountain J. Math. To appear.

[14] X. Zhang, L. Wang, Q. Sun, Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, Appl. Math. Comp. 226 (2014) 708–718.

[15] Y. Sun, Positive solutions of Sturm-Liouville boundary value problems for singular nonlinear second-order impulsive integro-differential equation in Banach spaces, Boundary Value Problems 201 (2012) 86.

[16] S. Liang, J. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, Nonlinear Anal. 71 (2009) 5545–5550.

[17] T. Chen, W. Liu, Z. Hu, A boundary value problem for fractional differential equation with P-Laplacian operator at resonance, Nonlinear Anal. 75 (2012) 3210–3217.

[18] N. Kosmatov, Integral equations and initial value problems for nonlinear differential equations of fractional order, Nonlinear Anal. 70 (2009) 2521–2529.

[19] J. Deng, L. Ma, Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, Appl. Math. Lett. 23 (2010) 676–680.

[20] K. Sathiyanathan, V. Krishnaveni, Nonlinear Implicit Caputo Fractional Differential Equations with Integral Boundary Conditions in Banach Space,Global Journal of Pure and Applied Mathematics 13 (2017) 3895–3907.

[21] J. Tariboon, S. K. Ntouyas, W. Sudsutad, Nonlocal Hadamard fractional integral conditions for nonlinear Riemann-Liouville fractional differential equations, Boundary Value Problems a Springer Open Journal 253 (2014).
[22] P. Borisut, P. Kumam, I. Ahmed, W. Jirakitpuwapat, Existence and uniqueness for ψHilfer fractional differential equation with nonlocal multipoint condition, Mathematical Methods in the Applied Sciences (2019) 1–15.
[23] A.A. Kibas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North Holland athematics Studies 204 (2006) 123–145.
[24] Y. Zhou, J. Wang, L. Zhang, Basic Theory of Fractional Differential Equations, World Scientific (2013) 18–20.
[25] B. Ahmad, A. Alsacdi, S.K. Ntouyas, J. Tariboon, Hadamard-Type Fractional Differential Equations Inclusions and Inequalities, Springer International Publishing 3–11.