Thai Journal of Mathematics
Volume 6 (2008) Number 2 : 295–306
www.math.science.cmu.ac.th/thaijournal

Three-Step Iteration Schemes with Errors for Generalized Asymptotically Quasi-nonexpansive Mappings

J. Nantadilok

Abstract : In this paper, we introduce new 3-step iteration schemes with errors for approximating the common fixed-point of 3 generalized asymptotically quasi-nonexpansive mappings and prove some strong convergence results for the iterative sequences with errors in real Banach spaces. The results obtained in this paper extend and improve some results announced by Lan [H.Y. Lan,Common fixed-point iterative processes with errors for generalized asymptotically quasi-nonexpansive mappings, Comput. Math. Appl. **52**(2006), 1403–1412.]

Keywords : Generalized asymptotically quasi-nonexpansive mapping, Common fixed-point, 3-step iteration schemes.
2000 Mathematics Subject Classification : 47H10, 47H09, 46B20.

1 Introduction

It is well known that the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. Since 1972, the weak and strong convergence problems of iterative sequences with errors for asymptotically nonexpansive types mapping in the Hibert space and Banach spaces setting have been studied by many authors (see, for example, [4- 12]). Zhou *et al.* [12] introduced a new class of generalized asymptotically nonexpansive mapping and gave a necessary and sufficient condition for the modified Ishikawa and Mann iterative sequences to converge to fixed points for the class of mappings. Atsushiba [1] studied the necessary and sufficient condition for the convergence of iterative sequences to a common fixed point of the finite family of asymtotically nonexpansive mappings in Banach space. Suzuki [9], Zeng and Yao [11] discussed a necessary and sufficient condition for common fixed points of two nonexpansive mappings and a finite family of nonexpansive mappings, and proved some convergence theorems for approximating a common fixed point, respectively. Recently, Lan [4] introduced a new class of generalized asymptotically quasi-nonexpansive mappings and gave necessary and sufficient condition for the 2-step modified Ishikawa iterative sequences to converge to fixed points for the class of mappings.

Inspired and motivated by the work of [4-12], a new class of 3-step iteration schemes are introduced and studied in this paper.

2 Preliminaries

Let X be a normed space, C be a nonempty closed convex subset of X, and $T_i : C \to C,\ (i = 1, 2, 3)$ be a given mapping. Then for a given w and $x_1 \in C$, we define the sequence $\{x_n\}$ in C by the 3-step iteration scheme

$$\begin{aligned} z_n &= (1 - c_n)w + c_n T_3^n x_n + v_n, \\ y_n &= (1 - b_n)w + b_n T_2^n z_n + u_n, \\ x_{n+1} &= (1 - a_n)w + a_n T_1^n y_n + w_n, \quad n \geq 1, \end{aligned} \tag{2.1}$$

is called the general modified 3-step Ishikawa iterative sequence with errors, where $\{u_n\}, \{v_n\}, \{w_n\}$ are sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}$, are sequences in $[0, 1]$ satisfying some conditions.

If we replace the same point w in all the iteration steps by x_n, then the sequnce $\{x_n\}$ defined by (2.1) becomes

$$\begin{aligned} z_n &= (1 - c_n)x_n + c_n T_3^n x_n + v_n, \\ y_n &= (1 - b_n)x_n + b_n T_2^n z_n + u_n, \\ x_{n+1} &= (1 - a_n)x_n + a_n T_1^n y_n + w_n, \quad n \geq 1, \end{aligned} \tag{2.2}$$

is called the generalized modified 3-step Ishikawa iterative sequence with errors, where $\{u_n\}, \{v_n\}, \{w_n\}$ are sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}$, are sequences in $[0, 1]$ satisfying some conditions.

If $T_i = T$ for $(i = 1, 2, 3)$, then the sequence defined by (2.2) becomes

$$\begin{aligned} z_n &= (1 - c_n)x_n + c_n T^n x_n + v_n, \\ y_n &= (1 - b_n)x_n + b_n T^n z_n + u_n, \\ x_{n+1} &= (1 - a_n)x_n + a_n T^n y_n + w_n, \quad n \geq 1, \end{aligned} \tag{2.3}$$

is called the usual modified 3-step Ishikawa iterative sequence with errors, where $\{u_n\}, \{v_n\}, \{w_n\}$ are sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}$, are sequences in $[0, 1]$ satisfying some conditions.

In the sequel, we need the following definitions and lemmas for our main results in this paper.

Definition 2.1 (See[4]). *Let X be a real Banach space, C be a nonempty subset of X and $F(T)$ denotes the set of fixed points of T. A mapping $T : C \to C$, is said to be*

(1) *asymptotically nonexpansive if there exists a sequence* $\{r_n\}$ *of positive real numbers with* $r_n \to 0$ *as* $n \to \infty$ *such that*

$$\|T^n x - T^n y\| \;\leq\; (1 + r_n)\|x - y\|, \tag{2.4}$$

(2) *asymptotically quasi-nonexpansive if* (2.4) *holds for all* $x \in C$ *and* $y \in F(T)$*;*

(3) *generalized quasi-nonexpansive with respect to* $\{s_n\}$*, if there exists a sequence* $\{s_n\} \subset [0, 1)$ *with* $s_n \to 0$ *as* $n \to \infty$ *such that*

$$\|T^n x - p\| \;\leq\; \|x - p\| + s_n\|x - T^n x\|, \tag{2.5}$$

for all $x \in C$ *and* $p \in F(T), \quad n \geq 1,$

(4) *generalized asymptotically quasi-nonexpansive with respect to sequence* $\{r_n\}$ *and* $\{s_n\} \subset [0, 1)$ *with* $r_n \to 0$ *and* $s_n \to 0$ *as* $n \to \infty$ *such that*

$$\|T^n x - p\| \;\leq\; (1 + r_n)\|x - p\| + s_n\|x - T^n x\|, \tag{2.6}$$

for all $x \in C$ *and* $p \in F(T), \quad n \geq 1,$

Remark 2.2. *It is easy to that,*

(1) *if* $s_n \equiv 0$ *for all* $n \geq 1$*, then the generalized asymptotically quasi-nonexpansive mapping reduces to the usual asymptotically quasi-nonexpansive mapping;*

(2) *if* $r_n = s_n \equiv 0$ *for all* $n \geq 1$*,then the generalized asymptotically quasi-nonexpansive mapping reduces to the usual quasi-nonexpansive mapping;*

(3) *if* $r_n \equiv 0$ *for all* $n \geq 1$*,then the generalized asymptotically quasi-nonexpansive mapping reduces to the generalized quasi-nonexpansive mapping.*

Lan in [4] has shown that the generalized asymptotically quasi-nonexpansive mapping is not a generalized quasi-nonexpansive mapping.

Lemma 2.3 (See [10])**.** *Let* $\{a_n\}, \{b_n\}$*, and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \;\leq\; (1 + \delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$*, then* $\lim_{n\to\infty} a_n$ *exists. In particular, if* $\{a_n\}$ *has a subsequence converging to zero, then* $\lim_{n\to\infty} a_n = 0$*.*

Lemma 2.4 (See [4])**.** *Let* C *be nonempty closed subset of a Banach space* X *and* $T : C \to C$ *be a generalized asymptotically quasi-nonexpansive mapping with the fixed-point set* $F(T) \neq \emptyset$*. Then* $F(T)$ *is closed subset in* C*.*

3 Main Results

In this section, we prove strong convergence theorems of the 3-step iterations (2.1) and (2.2) with errors for the generalized asymptotically quasi-nonexpansive mappings in a real Banach space.

Theorem 3.1. *Let X be a real arbitrary Banach space, C be a nonempty closed convex subset of X. Let $T_i : C \to C, (i = 1, 2, 3)$ be generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that the set $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ in C and $\sum_{n=1}^{\infty} \frac{r_n + 2s_n}{1 - s_n} < \infty$ where $r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}$, $s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}$. Assume that $\{v_n\}, \{u_n\}$ are sequences in C , where*

(i) $v_n = w_n^1(1 - c_n), \{w_n^1\}$ *is bounded;*
$u_n = w_n^2(1 - b_n), \{w_n^2\}$ *is bounded;*
and $w_n = w_n' + w_n'', \quad n \geq 1;\ \sum_{n=1}^{\infty} \|w_n'\| < \infty,\ \|w_n''\| = o(1 - a_n).$

(ii) $\sum_{n=1}^{\infty}(1 - a_n) < \infty,\ \sum_{n=1}^{\infty}(1 - b_n) < \infty,\ \sum_{n=1}^{\infty}(1 - c_n) < \infty$
Then, the iterative sequence $\{x_n\}$ defined in (2.1) converges strongly to a common fixed-point p of $T_1, T_2,$ and T_3 if and only if

$$\liminf_{n\to\infty}\ d\big(x_n, F(T_1) \cap F(T_2) \cap F(T_3)\big) = 0$$

where $d\big(x, F(T_1) \cap F(T_2) \cap F(T_3)\big)$ denotes the distance between x and the set $F(T_1) \cap F(T_2) \cap F(T_3)$.

Proof The necessity is obvious and it is omitted. Now we prove the sufficiency. For any $p \in F(T_1) \cap F(T_2) \cap F(T_3)$, by (2.1) and (2.6) we have

$$\begin{aligned} \|x_n - T_3^n x_n\| &\leq \|x_n - p\| + \|T_3^n x_n - p\| \\ &\leq \|x_n - p\| + (1 + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\| \\ &\leq (2 + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\| \\ &\leq (2 + r_n)\|x_n - p\| + s_n\|x_n - T_3^n x_n\| \end{aligned}$$

which implies that

$$\|x_n - T_3^n x_n\| \leq \frac{2 + r_n}{1 - s_n}\|x_n - p\| \tag{3.1}$$

Similarly, we have

$$\begin{aligned} \|y_n - T_1^n y_n\| &\leq \|y_n - p\| + \|T_1^n y_n - p\| \\ &\leq \|y_n - p\| + (1 + r_{1n})\|y_n - p\| + s_{1n}\|y_n - T_1^n y_n\| \\ &\leq (2 + r_n)\|y_n - p\| + s_n\|y_n - T_1^n y_n\|. \end{aligned}$$

which implies that

$$\|y_n - T_1^n y_n\| \leq \frac{2 + r_n}{1 - s_n}\|y_n - p\| \tag{3.2}$$

and also

$$\|z_n - T_2^n z_n\| \leq \frac{2+r_n}{1-s_n}\|z_n - p\| \tag{3.3}$$

From $z_n - p = (1-c_n)w + c_n T_3^n x_n + v_n - p$, it follows from (2.6) and (3.1) that

$$\begin{aligned}
\|z_n - p\| &\leq (1-c_n)\|w-p\| + c_n\|T_3^n x_n - p\| + \|v_n\| \\
&\leq (1-c_n)\|w-p\| + c_n\big((1+r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\|\big) + \|v_n\| \\
&\leq (1-c_n)\|w-p\| + c_n\big((1+r_n)\|x_n - p\| + s_n\|x_n - T_3^n x_n\|\big) + (1-c_n)\|w_n^1\| \\
&\leq c_n\big((1+r_n)\|x_n - p\| + s_n\frac{2+r_n}{1-s_n}\|x_n - p\|\big) + (1-c_n)\big(\|w-p\| + \|w_n^1\|\big) \\
&\leq \frac{1+r_n+s_n}{1-s_n}\|x_n - p\| + (1-c_n)\big(\|w-p\| + \|w_n^1\|\big). \qquad (3.4)
\end{aligned}$$

From $y_n - p = (1-b_n)w + b_n T_2^n z_n + u_n - p$, by (2.6) and (3.3) we have

$$\begin{aligned}
\|y_n - p\| &\leq (1-b_n)\|w-p\| + b_n\|T_2^n z_n - p\| + \|u_n\| \\
&\leq (1-b_n)\|w-p\| + b_n\big((1+r_{2n})\|z_n - p\| + s_{2n}\|z_n - T_2^n z_n\|\big) + \|u_n\| \\
&\leq (1-b_n)\|w-p\| + b_n\big((1+r_n)\|z_n - p\| + s_n\|z_n - T_2^n z_n\|\big) + (1-b_n)\|w_n^2\| \\
&\leq b_n\big((1+r_n)\|z_n - p\| + s_n\frac{2+r_n}{1-s_n}\|z_n - p\|\big) + (1-b_n)\big(\|w-p\| + \|w_n^2\|\big) \\
&\leq \frac{1+r_n+s_n}{1-s_n}\|z_n - p\| + (1-b_n)\big(\|w-p\| + \|w_n^2\|\big). \qquad (3.5)
\end{aligned}$$

On the other hand, by the Condition (i), we have

$\|w_n\| \leq \|w_n'\| + \|w_n''\|$, for all $n \geq 1$ with $\|w_n''\| = o(1-a_n)$ for all $n \geq 1$
and $\sum_{n=1}^{\infty}\|w_n'\| < \infty$ and so there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$
such that $\|w_n''\| = \varepsilon_n(1-a_n)$, i.e.,

$$\|w_n\| \leq \|w_n'\| + \varepsilon_n(1-a_n). \tag{3.6}$$

Therefore, it follows from (2.1) and (3.1)−(3.6) that

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|(1-a_n)w + a_nT_1^n y_n + w_n - p\| \\
&\leq (1-a_n)\|w-p\| + a_n\|T_1^n y_n - p\| + \|w_n\| \\
&\leq (1-a_n)\|w-p\| + a_n\big((1+r_{1n})\|y_n-p\| + s_{1n}\|y_n - T_1^n y_n\|\big) + \|w_n\| \\
&\leq (1-a_n)\|w-p\| + a_n\big((1+r_n)\|y_n-p\| + s_n\|y_n - T_1^n y_n\|\big) + \|w_n\| \\
&\leq (1-a_n)\|w-p\| + a_n\big((1+r_n)\|y_n-p\| + s_n\frac{2+r_n}{1-s_n}\|y_n-p\|\big) \\
&\quad + \|w'_n\| + \varepsilon_n(1-a_n) \\
&\leq (1-a_n)\|w-p\| + a_n\frac{1+r_n+s_n}{1-s_n}\|y_n-p\| + \|w'_n\| \\
&\quad + \varepsilon_n(1-a_n) \\
&\leq (1-a_n)\|w-p\| + a_n\big(\frac{1+r_n+s_n}{1-s_n}\big)^2\|z_n-p\| \\
&\quad + a_n\frac{1+r_n+s_n}{1-s_n}(1-b_n)(\|w-p\| + \|w_n^2\|) \\
&\quad + \|w'_n\| + \varepsilon_n(1-a_n) \\
&\leq (1-a_n)\|w-p\| + a_n(\frac{1+r_n+s_n}{1-s_n})^2\|z_n-p\| \\
&\quad + a_n(\frac{1+r_n+s_n}{1-s_n})(1-b_n)\|w-p\| + a_n(\frac{1+r_n+s_n}{1-s_n})(1-b_n)\|w_n^2\| \\
&\quad + \|w'_n\| + \varepsilon_n(1-a_n) \\
&\leq a_n(\frac{1+r_n+s_n}{1-s_n})^2\|z_n-p\| \\
&\quad + \{(1-a_n) + a_n(\frac{1+r_n+s_n}{1-s_n})(1-b_n)\}\|w-p\| \\
&\quad + a_n(\frac{1+r_n+s_n}{1-s_n})(1-b_n)\|w_n^2\| \\
&\quad + \|w'_n\| + \varepsilon_n(1-a_n). \\
&\leq a_n(\frac{1+r_n+s_n}{1-s_n})^3\|x_n-p\| + a_n(\frac{1+r_n+s_n}{1-s_n})^2(1-c_n)\big(\|w-p\| + \|w'_n\|\big) \\
&\quad + \{(1-a_n) + a_n\frac{1+r_n+s_n}{1-s_n})(1-b_n)\}\|w-p\| \\
&\quad + a_n\frac{1+r_n+s_n}{1-s_n}(1-b_n)\|w_n^2\| + \|w'_n\| + \varepsilon_n(1-a_n). \\
&\leq (k_n)^3\|x_n-p\| + (k_n)^2(1-c_n)\big(\|w-p\| + \|w'_n\|\big) \\
&\quad + \{(1-a_n) + k_n(1-b_n)\}\|w-p\| \\
&\quad + k_n(1-b_n)\|w_n^2\| + \|w'_n\| + \varepsilon_n(1-a_n). \\
&= (1+\delta_n)\|x_n-p\| + d_n. \qquad (3.7)
\end{aligned}
$$

where $\delta_n = k_n^3 - 1$, and

$$d_n = (k_n)^2(1-c_n)\big(\|w-p\| + \|w'_n\|\big) + \{(1-a_n) + k_n(1-b_n)\}\|w-p\| + k_n(1-b_n)\|w_n^2\| + \|w'_n\| + \varepsilon_n(1-a_n)$$

and $k_n = \dfrac{1+r_n+s_n}{1-s_n}$. Since $k_n - 1 = \dfrac{r_n+2s_n}{1-s_n}$, the assumption $\sum_{n=1}^{\infty} \dfrac{r_n+2s_n}{1-s_n} < \infty$ implies that $\lim_{n\to\infty} k_n = 1$. Therefore, it follows from the Condition (i) and (ii) and by Lemma 2.3 that

$$\sum_{n=1}^{\infty} \delta_n < \infty, \qquad \sum_{n=1}^{\infty} d_n < \infty \tag{3.8}$$

and the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that sequence $\{\|x_n - p\|\}$ is bounded.
Denote by

$$M = \sup_{n\geq 1} \|x_n - p\|.$$

Thus we can rewrite (3.7) as follow

$$\|x_{n+1} - p\| \leq \|x_n - p\| + M\delta_n + d_n, \quad n \geq 1\ . \tag{3.9}$$

Now for any positive integer $m, n \geq 1$, we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + M\delta_{n+m-1} + d_{n+m-1} \\ &\leq \|x_{n+m-2} - p\| + M\Big(\delta_{n+m-1} + \delta_{n+m-2}\Big) + \Big(d_{n+m-1} + d_{n+m-2}\Big) \\ &\leq \ldots \\ &\leq \|x_n - p\| + M\sum_{i=n}^{n+m-1} \delta_i + \sum_{i=n}^{n+m-1} d_i\ . \end{aligned} \tag{3.10}$$

By (3.7), we get

$$d\big(x_{n+1}, F(T_1)\cap F(T_2)\cap F(T_3)\big) \leq (1+\delta_n)d\big(x_n, F(T_1)\cap F(T_2)\cap F(T_3)\big) + d_n$$

and it follows from the condition $\liminf_{n\to\infty} d\big(x_n, F(T_1)\cap F(T_2)\cap F(T_3)\big) = 0$ and by Lemma 2.3 that

$$\lim_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)\cap F(T_3)) = 0. \tag{3.11}$$

Next we show that $\{x_n\}$ is a Cauchy sequence in X. In fact, from (3.8) and (3.11), it follows that for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$,

$$d(x_n, F(T_1)\cap F(T_2)\cap F(T_3)) < \frac{\varepsilon}{12}, \ \sum_{n=n_0}^{\infty} \delta_n < \frac{\varepsilon}{3M}, \ \sum_{n=n_0}^{\infty} d_n < \frac{\varepsilon}{3} \tag{3.12}$$

By the first inequality of (3.12), we know that there exists a $p_0 \in F(T_1) \cap F(T_2) \cap F(T_3)$ such that

$$\|x_{n_0} - p_0\| \ < \ \frac{\varepsilon}{6} \tag{3.13}$$

Combining (3.10), (3.12) and (3.13), for any positive integer $m \geq 1$

$$\begin{aligned}
\|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\| \\
&\leq \Big(\|x_{n_0} - p_0\| + M \sum_{i=n_0}^{n_0+m-1} \delta_i + \sum_{i=n_0}^{n_0+m-1} d_i\Big) + \|x_{n_0} - p_0\| \\
&= 2\|x_{n_0} - p_0\| + M \sum_{i=n_0}^{n_0+m-1} \delta_i + \sum_{i=n_0}^{n_0+m-1} d_i \\
&< \varepsilon,
\end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence in X. Thus the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\{x_n\}$ converges to p. Then $p \in C$, because C is a closed subset of X. By Lemma 2.4 we know that the set $F(T_1)\cap F(T_2)\cap F(T_3)$ is closed. From the continuity of $d(x, F(T_1)\cap F(T_2)\cap F(T_3))$ with

$$d(x_n, F(T_1) \cap F(T_2) \cap F(T_3)) \to 0 \ \text{ and } \ x_n \to p \text{ as } n \to \infty,$$

we get

$$d(p, F(T_1) \cap F(T_2) \cap F(T_3)) = 0$$

and so $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. This completes our proof.

Corollary 3.2. *When $c_n = 1, v_n = 0$ and $T_3 = I$ (identity) in Theorem 3.1, then we obtain the result of Lan* [4, *Theorem* 3.1].

Theorem 3.3. *Let X be real arbitrary Banach space, C be a nonempty closed convex subset of X and for $i = 1, 2, 3, T_i : C \to C$ be generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ in C. Assume that*

(i) $\{v_n\}, \{u_n\}$ in C are bounded, $w_n = w'_n + w''_n, n \geq 1$ and $\sum_{n=1}^{\infty} \|w'_n\| < \infty$, $\|w''_n\| = o(a_n)$.

(ii) $\sum_{n=1}^{\infty} a_n < \infty$.
Then, the iterative sequence $\{x_n\}$ defined in (2.2) converges strongly to a common fixed-point p of T_1, T_2, and T_3 if and only if

$$\liminf_{n\to\infty} d\big(x_n, F(T_1) \cap F(T_2) \cap F(T_3)\big) = 0.$$

Proof The necessity is obvious and is omitted. Now, we prove the sufficiency. Let

$$r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}, \ s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}.$$

Then, from $r_{in} \to 0$ and $s_{in} \to 0$ for $i = 1, 2, 3$, we know that $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$.
For any $p \in F(T_1) \cap F(T_2) \cap F(T_3)$ by (2.2) and (2.6)

$$\begin{aligned} \|x_n - T_3^n x_n\| &\le \|x_n - p\| + \|T_3^n x_n - p\| \\ &\le \|x_n - p\| + (1 + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\| \\ &= (2 + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\| \\ &\le (2 + r_n)\|x_n - p\| + s_n\|x_n - T_3^n x_n\| \end{aligned}$$

which implies that

$$\|x_n - T_3^n x_n\| \le \frac{2 + r_n}{1 - s_n}\|x_n - p\| \tag{3.14}$$

Similarly, we have

$$\|y_n - T_1^n y_n\| \le \frac{2 + r_n}{1 - s_n}\|y_n - p\| \tag{3.15}$$

and

$$\|z_n - T_2^n z_n\| \le \frac{2 + r_n}{1 - s_n}\|z_n - p\| \tag{3.16}$$

From $z_n - p = (1 - c_n)x_n + c_n T_3^n x_n - p + v_n$, by (2.6) and (3.14), we have

$$\begin{aligned} \|z_n - p\| &\le (1 - c_n)\|x_n - p\| + c_n\|T_3^n x_n - p\| + \|v_n\| \\ &\le (1 - c_n)\|x_n - p\| + c_n\big[(1 + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\|\big] + \|v_n\| \\ &\le (1 - c_n)\|x_n - p\| + c_n\big[(1 + r_n)\|x_n - p\| + s_n\|x_n - T_3^n x_n\|\big] + \|v_n\| \\ &\le (1 - c_n)\|x_n - p\| + c_n\big[(1 + r_n)\|x_n - p\| + s_n(\frac{2 + r_n}{1 - s_n})\|x_n - p\|\big] + \|v_n\| \\ &= \Big[1 - c_n + c_n(\frac{1 + r_n + s_n}{1 - s_n})\Big]\|x_n - p\| + \|v_n\| \\ &= \Big[1 + c_n(\frac{1 + r_n + s_n}{1 - s_n} - 1)\Big]\|x_n - p\| + \|v_n\| \\ &\le (\frac{1 + r_n + s_n}{1 - s_n})\|x_n - p\| + \|v_n\| \\ &= k_n\|x_n - p\| + \|v_n\| \end{aligned} \tag{3.17}$$

where $k_n = \frac{1+r_n+s_n}{1-s_n} \to 1$, since $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$

Similarly, by (2.6), (3.16) and (3.17), we have

$$\begin{aligned}
\|y_n - p\| &\leq (1-b_n)\|x_n - p\| + b_n\|T_2^n z_n - p\| + \|u_n\| \\
&\leq (1-b_n)\|x_n - p\| + b_n\Big[(1+r_{2n})\|z_n - p\| + s_{2n}\|z_n - T_2^n z_n\|\Big] + \|u_n\| \\
&\leq (1-b_n)\|x_n - p\| + b_n\Big[(1+r_n)\|z_n - p\| + s_n\|z_n - T_2^n z_n\|\Big] + \|u_n\| \\
&\leq (1-b_n)\|x_n - p\| + b_n\Big[(1+r_n)\|z_n - p\| + s_n(\frac{2+r_n}{1-s_n})\|z_n - p\|\Big] + \|u_n\| \\
&= (1-b_n)\|x_n - p\| + b_n\Big[(k_n)\Big(k_n\|x_n - p\| + \|v_n\|\Big)\Big] + \|u_n\| \\
&= \Big[1 + b_n(k_n^2 - 1)\Big]\|x_n - p\| + b_n k_n\|v_n\| + \|u_n\| \\
&\leq (k_n^2)\|x_n - p\| + k_n\|v_n\| + \|u_n\|. \qquad (3.18)
\end{aligned}$$

where $k_n = \dfrac{1+r_n+s_n}{1-s_n}$
On the other hand, by Condition (i) we have

$$\|w_n\| \leq \|w_n'\| + \|w_n''\| \text{ with } \|w_n''\| = o(a_n) \text{ for all } n \geq 1$$

and $\sum_{n=1}^{\infty}\|w_n'\| \leq \infty$, and so there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ such that

$$\|w_n\| \leq \|w_n'\| + \varepsilon_n a_n. \qquad (3.19)$$

Therefore, it follows from (2.6) and (3.14)−(3.19) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-a_n)x_n + a_n T_1^n y_n + w_n - p\| \\
&\leq (1-a_n)\|x_n - p\| + a_n\|T_1^n y_n - p\| + \|w_n\| \\
&\leq (1-a_n)\|x_n - p\| + a_n\Big[(1+r_{1n})\|y_n - p\| + s_{1n}\|y_n - T_1^n y_n\|\Big] \\
&\quad + \|w_n'\| + \|w_n''\| \\
&\leq (1-a_n)\|x_n - p\| + a_n\Big[(1+r_n)\|y_n - p\| + s_n\|y_n - T_1^n y_n\|\Big] \\
&\quad + \|w_n'\| + \varepsilon_n a_n \\
&\leq (1-a_n)\|x_n - p\| + a_n\Big[(1+r_n)\|y_n - p\| + s_n\frac{2+r_n}{1-s_n}\|y_n - p\|\Big] \\
&\quad + \|w_n'\| + \varepsilon_n a_n \\
&= (1-a_n)\|x_n - p\| + a_n\Big[\frac{1+r_n+s_n}{1-s_n}\Big]\|y_n - p\| \\
&\quad + \|w_n'\| + \varepsilon_n a_n
\end{aligned}$$

$$\begin{aligned}
&\leq (1-a_n)\|x_n-p\| + a_n k_n\Big[(k_n^2)\|x_n-p\| + k_n\|v_n\| + \|u_n\|\Big] \\
&\quad + \|w_n'\| + \varepsilon_n a_n \\
&= \Big[1 + a_n(k_n^3-1)\Big]\|x_n-p\| + a_n k_n^2\|v_n\| + a_n k_n\|u_n\| \\
&\quad + \|w_n'\| + \varepsilon_n a_n \\
&\leq (k_n^3)\|x_n-p\| + a_n k_n^2\|v_n\| + a_n k_n\|u_n\| + \|w_n'\| + \varepsilon_n a_n \\
&\leq (1+\delta_n)\|x_n-p\| + d_n.
\end{aligned}$$

where $\delta_n = k_n^3 - 1$ and

$$d_n = a_n k_n^2\|v_n\| + a_n k_n\|u_n\| + \|w_n'\| + \varepsilon_n a_n.$$

We can see that $\sum_{n=1}^{\infty}\delta_n < \infty$, and $\sum_{n=1}^{\infty} d_n < \infty$. By Lemma 2.3, we can conclude that $\lim_{n\to\infty}\|x_n - p\|$ exists. This implies that sequence $\{\|x_n - p\|\}$ is bounded. The rest of proof follows as those of theorem 3.1 and therefore is omitted. This completes the proof.

Corollary 3.4. *When $c_n = 1, v_n = 0$ and $T_3 = I$ (identity) in Theorem 3.3, then we obtain the result of Lan* [4, *Theorem* 3.2].

In Theorem 3.3, if $T_1 = T_2 = T_3 = T$, we obtain the following result:

Theorem 3.5. *Let X be a real arbitrary Banach space, C be a nonempty closed convex subset of X and $T : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping with respect to $\{r_n\}$ and $\{s_n\}$ such that $F(T) \neq \emptyset$ in C. Assume that*

(i) $\{v_n\}, \{u_n\} \subset C$ are bounded, $w_n = w_n' + w_n''$, $n \geq 1$, and $\sum_{n=1}^{\infty}\|w_n'\| < \infty$, $\|w_n''\| = o(a_n)$.

(ii) $\sum_{n=1}^{\infty} a_n < \infty$.
Then, the iterative sequence $\{x_n\}$ defined in (2.3) converges strongly to a common fixed-point p of T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1)$.

Acknowledgement(s) : The author is grateful to the referee for valuable suggestions that helped towards the improvement of this paper.

References

[1] S. Atsushiba, *Strong convergence of of iterative sequences for asymptotically nonexpansive mappings in Banach spaces,* Sci. Math. Japan. **57(2)** (2003), 377–388.

[2] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings,* Proc. Amer. Math. Soc. **35** (1972), 171–174.

[3] M.K. Gosh and L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings* , J. Math. Anal.Appl. **207** (1997) 96–103.

[4] H.Y. Lan, *Common fixed-point iterative processes with errors for generalized asymptotically quasi-nonexpansive mappings,* Comput. Math. Appl. **52** (2006), 1403–1412.

[5] Y.C. Lin, *Three-step iterative convergence theorems with errors in Banach spaces,* Taiwanese J. Math. **10(1)** (2006), 75–86.

[6] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings,* J. Math. Anal. Appl. **259** (2001), 1–8.

[7] S. Plubtieng, R. Wangkeeree and R. Punpaeng, *On the convergence of modified Noor iterations with errors for asymtotically nonexpansive mappings,* J. Math. Anal. Appl. **322** (2006), 1018–1029.

[8] S. Suantai, *Weak and strong convergence criteria of Noor iterations for asymtotically nonexpansive mappings,* J. Math. Anal. Appl. **311** (2005), 506–517.

[9] T. Suzuki, *Common fixed points of two nonexpansive mappings in Banach spaces,* Bull. Austral. Math. Soc. **69**, (2004), 1–18.

[10] K.K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process,* J. Math. Anal. Appl. **178** (1993), 301–308.

[11] L.C. Zeng and J.C. Yao, *Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings,* Nonlinear Anal. Series A: Theory and Applications **64(1)**,(2006), 2507–2515.

[12] H.Y. Zhou, Y.J. Cho and M. Grabiec, *Iterative processes for generalized asymptotically nonexpansive mappings in Banach spaces,* Pan Amer. Math. J.**13** (2003), 99–107.

(Received 19 Otober 2008)

Jamnian Nantadilok
Department of Mathematics, Faculty of Science, Lampang Rajabhat University, Lampang 52100,Thailand
e-mail : `jamnian52@lpru.ac.th`