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Geometry of Julia Set of Complex Polynomial $z^n + c$

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Abstract: This paper is intended to estimate the upper bounded of $|c|$ such that Julia sets of complex polynomials of the form $z^n + c$ are simple closed curves when $n = 2, 3, 4, \ldots$. Moreover, we study the geometric properties of these Julia sets. We know that Julia sets of complex polynomials of the form $z^2 + c$ are simple closed curves provided $|c| < 1/4$. We expect the same phenomenon, i.e. Julia sets of complex polynomials of the form $z^n + c$ are simple closed curves if $|c|$ is small enough. However, they are far from being smooth; indeed, they contain no smooth arcs at all.

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1 Notations and Preliminaries

Let f be a function from $\mathbb{C}$ to $\mathbb{C}$ and $w \in \mathbb{C}$. We denote the iterations of a function f by $f^1 = f$ and $f^k = f^{k-1} \circ f$. We call w a *fixed point* of f provided $f(w) = w$. If $f^p(w) = w$ for some integer $p \geqslant 1$, then w is a *periodic point* of f. The least p such that $f^p(w) = w$ is called the *period* of w. Suppose f is holomorphic in a neighborhood of w and w is a periodic point of period p, with $(f^p)'(w) = \lambda$, where the prime denotes complex differentiation. The point w is called *attractive* if $|\lambda| < 1$, *repelling* if $|\lambda| > 1$, and *indifferent* if $|\lambda| = 1$. The *Julia set* $J(f)$ *of a complex polynomial* f is the closure of the set of repelling periodic points of f. The complement of the Julia set of a complex polynomial is called the *Fatou set* or *stable set* $F(f)$.

Let U be an open set in $\mathbb{C}$, and let $\{g_k : U \to \mathbb{C}\}$ be a family of complex holomorphic functions. The family $\{g_k\}$ is said to be *normal* on U if every sequence of functions selected from $\{g_k\}$ has a subsequence which converges uniformly on every compact subset of U, either to a bounded holomorphic function or to ∞. The family $\{g_k\}$ is *normal at the point* w of U if there is some open subset V of U containing w such that $\{g_k\}$ is a normal family on V. Define

$$J_0(f) = \{z \in \mathbb{C} : \text{the family } \{f^k\}_{k \geqslant 1} \text{ is not normal at } z\}$$

and

$$\begin{aligned} F_0(f) &\equiv \mathbb{C} \setminus J_0(f) \\ &= \{z \in \mathbb{C} \text{ such that there is an open set } V \text{ with} \\ &\qquad z \in V \text{ and } \{f^k\} \text{ normal on } V\}. \end{aligned}$$

The followings are the basic properties of Julia sets. For further references, see [4].

Proposition 1 *If f is a polynomial, then $J_0(f)$ is compact.*

Proposition 2 *$J_0(f)$ is non-empty.*

Proposition 3 *$J_0(f)$ is forward and backward invariant, i.e. $J_0 = f(J_0) = f^{-1}(J_0)$.*

Proposition 4 *$J_0(f^p) = J_0(f)$ for every positive integer p.*

Proposition 5 *Let f be a polynomial, let $w \in J_0(f)$ and let U be any neighborhood of w. Then $W \equiv \bigcup_{k=1}^{\infty} f^k(U)$ is the whole of $\mathbb{C}$, except possibly for a single point. Any such exceptional point is not in $J_0(f)$, and is independent of w and U.*

Proposition 6 *The following holds for all $z \in \mathbb{C}$ with, at most, one exception:*

(a) If U is an open set intersecting $J_0(f)$ then $f^{-k}(z)$ intersects U for infinitely many values of k.

(b) If $z \in J_0(f)$ then $J_0(f)$ is the closure of $\bigcup_{k=1}^{\infty} f^{-k}(z)$.

Proposition 7 *If f is a polynomial, $J_0(f)$ has empty interior.*

Proposition 8 *$J_0(f)$ is a perfect set (i.e. closed and with no isolated points and is therefore uncountable).*

Proposition 9 *If f is a polynomial, $J(f) = J_0(f)$.*

Proposition 10 *Let w be an attractive fixed point of f. Then $\partial A(w) = J(f)$. The same is true if $w = \infty$.*

It is known (see [3]) that when $n = 2$, the Julia sets of $z^2 + c$ are simple closed curves provided $|c| < 1/4$. To illustrate the ideas, let us consider briefly when $n = 3$. Recall the cubic formula which can be found in [5]. We then use the formula to solve $z^3 + c - z = 0$ to find the fixed points. The explicit formula also enable us to estimate $|c|$ in order to proceed with the similar argument when $n = 2$.

2 Main Results

For general n, we can not use the same method because there is no general formula for solving an algebraic equation of degree $n \geqslant 5$. Note that for $c = 0$, the complex polynomials of the form z^n have one attractive fixed at $z = 0$ and $n - 1$ repelling fixed points on the unit circle. When $|c|$ is small enough, we expect the result would resemble the case $c = 0$, namely, these polynomials also have one attractive fixed point near the point $z = 0$ and $n - 1$ repelling fixed points. To prove our main theorem, we will apply Rouché's theorem. We use it to compare the zeros of complex polynomials of the form $z^n - z + c$ with the zeros of complex polynomials of the form $z^n - z$. Consequently, we can estimate the upper bound of $|c|$ such that the complex polynomials of the form $z^n + c$ have exactly one attractive fixed points. Finally, we will show that if the complex polynomials of the form $z^n + c$ have only one attractive fixed point, then their Julia sets are simple closed curves. Moreover, if c is a complex number which is not real, then their Julia sets are nowhere differentiable.

Let $f_{n,c}(z) = z^n + c$, $\widetilde{f}_{n,c}(z) = z^n - z + c$ when $n = 2, 3, 4, \ldots$ and $D(a, r)$ denote the set $\{|z - a| < r\}$.

Lemma 1 *If* $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$, $n = 2, 3, 4, \ldots$, *then* $f_{n,c}(z)$ *has exactly one attractive fixed point in* $D\left(0; \dfrac{1}{\sqrt[n-1]{n}}\right)$.

Proof. Let $g(z) = z^n - z$. Consider $\xi \in \partial D\left(0; \dfrac{1}{\sqrt[n-1]{n}}\right)$. Then

$$|\widetilde{f}_{n,c}(\xi) - g(\xi)| = |\xi^n - \xi - c) - (\xi^n - \xi)| = |c| < \frac{n-1}{n\sqrt[n-1]{n}},$$

and

$$\begin{aligned}
|g(\xi)| &= |\xi^n - \xi| \\
&\geqslant ||\xi|^n - |\xi|| \\
&= \left|\left(\frac{1}{\sqrt[n-1]{n}}\right)^n - \left(\frac{1}{\sqrt[n-1]{n}}\right)\right| \\
&= \left|\frac{1}{n\sqrt[n-1]{n}} - \frac{1}{\sqrt[n-1]{n}}\right| \\
&= \frac{n-1}{n\sqrt[n-1]{n}}.
\end{aligned}$$

Thus $|\widetilde{f}_{n,c}(\xi) - g(\xi)| < |g(\xi)| \quad \forall \xi \in \partial D\left(0; \dfrac{1}{\sqrt[n-1]{n}}\right)$. By Rouché's Theorem, $\widetilde{f}_{n,c}$ and g have the same number of zeros in $D\left(0; \dfrac{1}{\sqrt[n-1]{n}}\right)$. Since g has only one zero in $D\left(0;\dfrac{1}{\sqrt[n-1]{n}}\right)$, so does $\widetilde{f}_{n,c}$. Let z_0 be a zero of $\widetilde{f}_{n,c}$ in $D\left(0;\dfrac{1}{\sqrt[n-1]{n}}\right)$. Since $|f'_{n,c}(z_0)| = |n(z_0)^{n-1}| < n\left(\dfrac{1}{\sqrt[n-1]{n}}\right)^{n-1} = 1$, z_0 is attractive. This implies $f_{n,c}$ has exactly one attractive fixed point in $D\left(0;\dfrac{1}{\sqrt[n-1]{n}}\right)$. □

Lemma 2 *Assume that* $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$ *when* $n = 2, 3, 4, \ldots$. *If* C_0 *is the circle* $|z| = \dfrac{1}{\sqrt[n-1]{n}}$, *then* $C_k = f_{n,c}^{-k}(C_0)$ *is a loop surrounding* $C_{k-1} = f_{n,c}^{-k+1}(C_0)$ *where* $k = 1, 2, 3, \ldots$

Proof. Write $C_k = f_{n,c}^{-k}(C_0)$ where $k = 1, 2, 3, \ldots$. Let C_0 be the curve $|z| = \dfrac{1}{\sqrt[n-1]{n}}$. Since

$$
\begin{aligned}
|f_{n,c}^{-1}(z)| &\geqslant ||z| - |c||^{\frac{1}{n}} \\
&> \left|\frac{1}{\sqrt[n-1]{n}} - \frac{n-1}{n\sqrt[n-1]{n}}\right|^{\frac{1}{n}} \\
&= \frac{1}{\sqrt[n-1]{n}} = |z| \qquad \text{for all } z \in C_0,
\end{aligned}
$$

C_1 is a loop surrounding C_0. The above inequality also implies that $C_0 \cap C_1 = \emptyset$. As a result, $C_{k+1} \cap C_k = \emptyset$ for all $k = 0, 1, 2, \ldots$. This is obvious since $f_{n,c}^k$ would map C_{k+1}, C_k to C_1 and C_0, respectively. As C_1 and C_0 are disjoint, so are C_{k+1} and C_k. This fact implies that C_{k+1} must be totally inside C_k or totally surrounding C_k. We claim that C_{k+1} is surrounding C_k. Since $f_{n,c}^k$ is a polynomial, $f_{n,c}^k$ maps a bounded open connected set to a bounded open connected set. Therefore, $f_{n,c}^k$ must map the interior of the loop C_{k+1} to the interior of the loop C_1 and map the interior of the loop C_k to the interior of the loop C_0. If C_{k+1} lies inside C_k, then the interior of C_{k+1} is a proper subset of the interior of C_k. By applying $f_{n,c}^k$ to those interiors, we would have that the interior of C_1 is a proper subset of the interior of C_0. This is not possible since C_1 is surrounding C_0 by the above inequality. Hence, C_{k+1} is a loop surrounding C_k for all $k = 0, 1, 2, \ldots$. □

Let w be an attractive fixed point of a complex polynomial f. Write $A(w) = \{z \in \mathbb{C} : f^k(z) \to w \text{ as } k \to \infty\}$ for the *basin of attraction* of w. E is called a *connected component* of a topological space X if it is a maximal connected subset of X. The connected component of $A(w)$ containing w is called the *immediate basin of attraction* of w and denoted by $A^*(w)$. We define the basin of attraction of infinity, $A(\infty)$, in the same way. Lemma 3 below is a well-known result and the proof will be omitted. Interested readers can consult [1].

Lemma 3 *Let* z_0 *be an attractive fixed point of* $f_{n,c}$. *If* $A^*(z_0)$ *contains all of preimages of* z_0, *then* $A^*(z_0)$ *is the only component of* $A(z_0)$.

Lemma 4 *Let* c *be a complex number such that* $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$ *and* C_0 *be the circle* $|z| = \dfrac{1}{\sqrt[n-1]{n}}$. *Then, for each* $k > 1$, $\left|f_{n,c}^{-k}(z) - f_{n,c}^{-k+1}(z)\right| < \alpha\gamma^n$ *for some constants* α *and* $\gamma > 1$.

Proof. Since $f'_{n,c}(z_0) = n(z_0)^{n-1}$ and C_0 is the circle $|z| = \dfrac{1}{\sqrt[n-1]{n}}$, there is a positive number $r > 1$ such that $|f'_{n,c}(z)| > r$ for all z outside C_0. Thus, $|(f_{n,c}^{-1}(z))'| \leqslant \dfrac{1}{|(f_{n,c})'(z)|} < \dfrac{1}{r}$ for all z outside C. For each two points z_1, z_2 outside C, let $\beta : [0,1] \to \mathbb{C} \setminus C$ be the straight line joining z_1 to z_2. Therefore,

$$\begin{aligned}
|f_{n,c}^{-1}(z_2) - f_{n,c}^{-1}(z_1)| &= \left|\int_\beta (f_{n,c}^{-1}(z))' dz\right| \\
&\leqslant \int_{z_1}^{z_2} |(f_{n,c}^{-1}(\beta(t)))'||d\beta(t)| \\
&< \frac{1}{r}\int_{z_1}^{z_2} |d\beta(t)| \\
&= \frac{1}{r}|z_1 - z_2|.
\end{aligned}$$

By simple calculations, for each $z \in C_0$ and $k \in \mathbb{N}$, $f_{n,c}^{-k}(z)$ is outside C_0. Applying the above inequality, we get that

$$\begin{aligned}
\left|f_{n,c}^{-k}(z) - f_{n,c}^{-k+1}(z)\right| &< \left(\frac{1}{r}\right)\left|f_{n,c}^{-k+1}(z) - f_{n,c}^{-k+2}(z)\right| \\
&< \left(\frac{1}{r}\right)^2 |f_{n,c}^{-k+2}(z) - f_{n,c}^{-k+3}(z)| \\
&\;\;\vdots \\
&< \left(\frac{1}{r}\right)^{k-2} |f_{n,c}^{-2}(z) - f_{n,c}^{-1}(z)|.
\end{aligned}$$

Hence, for each $k > 1$ and $z \in C_0$, $|f_{n,c}^{-k}(z) - f_c^{-k+1}(z)| < \alpha\gamma^k$ where $\gamma = \dfrac{1}{r}$ and $\alpha = r^2|f_{n,c}^{-2}(z) - f_{n,c}^{-1}(z)|$ as required. □

Lemma 5 *Let* $\{\psi_k(\theta)\}_{k=0}^{\infty}$ *be a sequence of continuous functions on an open domain* U *such that there is a positive number* $\gamma < 1$ *such that for each* $n \in \mathbb{N}$,

$$|\psi_k(\theta) - \psi_{k-1}(\theta)| < (\gamma)^k.$$

Then $\psi_k(\theta)$ *converges uniformly to a continuous function* $\psi(\theta)$ *as* $k \to \infty$.

Proof. Let $g_k(\theta) = \psi_k(\theta) - \psi_{k-1}(\theta)$. Then for each $k \in \mathbb{N}$, $|g_k(\theta)| < (\gamma)^k$. By Weierstrass M-test, $\psi_k(\theta) - \psi_0(\theta)$ converges uniformly to a continuous function $\phi(\theta)$ as k$\to \infty$. Let $\psi(\theta) = \phi(\theta) + \psi_0(\theta)$. Then $\psi(\theta)$ is also a continuous function. Hence $\psi_k(\theta)$ converges uniformly to a continuous function $\psi(\theta)$ as $k \to \infty$. □

Theorem 1 *Let c be a complex number such that $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$ where $n = 2, 3, 4, \ldots$ Then $J(f_{n,c})$ is a simple closed curve.*

Proof. Let C_0 be the circle $|z| = \dfrac{1}{\sqrt[n-1]{n}}$ and $C_k = f_{n,c}^{-k}(C_0)$ $k = 1, 2, 3, \ldots$. By Lemma 1, c and the attractive fixed point of $f_{n,c}$ are inside C_0. By Lemma 2, the inverse image C_1 is a loop surrounding C_0. Let A_1 be the annular region between C_0 and C_1. Pick $z = \dfrac{e^{i\theta}}{\sqrt[n-1]{n}} = \psi_0(\theta)$ on C_1. $\left\{f_{n,c}^{-1}(z)\right\}$ is a set of n different points. Choose $\psi_1(\theta) \in \left\{f_{n,c}^{-1}(z)\right\}$ such that the curve $\gamma_0(t)$ joining $\psi_0(\theta)$ and $\psi_1(\theta)$ is a subset of A_1 and has the shortest length (such a curve exists since A_1 is diffeomorphic to $\{r_1 \leq |z| \leq r_2\}$ for some $r_1, r_2 > 0$). Note that $f_{n,c}^{-1}(\gamma_0(t))$ are n non-intersecting curves. We choose the one, denoted by $\gamma_1(t)$, that has $\psi_1(\theta)$ as the starting point. $\gamma_0(t)$ and $\gamma_1(t)$ are then sewn together to produce a single curve. Continuing this process, we get a sequence of loops C_k, each surrounding its predecessor, and families of curves joining the points $\psi_k(\theta)$ in C_k to $\psi_{k+1}(\theta)$ in C_{k+1} for each k.

As $k \to \infty$, the curves C_k approach the boundary of the basin of attraction fixed point of $f_{n,c}$. By Lemma 3, the basin of attraction fixed point of $f_{n,c}$ has exactly one component. By Lemma 4, we get the length of the curve joining a point $\psi_k(\theta)$ to a point $\psi_{k+1}(\theta)$ converges to 0 at a geometric rate as k $\to \infty$. Therefore, $\psi_k(\theta)$ converges uniformly to a continuous function $\psi(\theta)$ as k$\to \infty$. Hence $J(f_{n,c})$ is the closed curve given by $\psi(\theta)$ $(0 \leqslant \theta \leqslant 2\pi)$.

Now, we will show that $\psi(\theta)$ parametrizes a simple curve. Assume that $\psi(\theta_1) = \psi(\theta_2)$. Let D be the region bounded by C_0 and the two paths joining $\psi_0(\theta_1)$ and $\psi_0(\theta_2)$ to this common point. The boundary of D remains bounded under iterates of $f_{n,c}$, so by the maximum modulus Theorem, D remains bounded under iterates of $f_{n,c}$. ¿From the fact that If $w \in J(f_{n,c})$ and U is a neighborhood of w, then $W \equiv \bigcup_{k=1}^{\infty} f^k(U)$ is the whole of $\mathbb{C}$, except possibly for a single point, we have the interior of D cannot contain any points of $J(f_{n,c})$. Hence $\psi(\theta_1) = \psi(\theta) = \psi(\theta_2)$ for all θ between θ_1 and θ_2. Therefore $J(f_{n,c})$ is a simple closed curve. $\square$

Theorem 2 *Suppose c is a complex number which is not real and $|c| < \dfrac{n-1}{n\sqrt[n-1]{n}}$. Then $J(f_{n,c})$ is a simple nowhere differentiable closed curve.*

Proof. By Theorem 1, $J(f_{n,c})$ is a simple closed curve. Let z_1 be a repelling fixed point of $f_{n,c}$. It is easy to check that $f'_{n,c}(z_1)$ is a complex number which is not real. We will show that z_1 does not lie in a smooth arc in $\psi(\theta)$. Suppose not. Since $J(f_{n,c})$ is invariant under $f_{n,c}$, the image of $\psi(\theta)$ would also be a smooth arc in $J(f_{n,c})$ passing through z_1. Since $\arg(f'_{n,c}(z_1)) \neq 0$ and $\neq \pi$, the tangents to these two curves would not be parallel. Hence, $\psi(\theta)$ would not be simple at z_1, which is a contradiction. This implies z_1 does not lie in a smooth arc in $\psi(\theta)$.

Since for each $z \in J(f_{n,c})$, $J_0(f)$ is the closure of $\bigcup_{k=1}^{\infty} f^{-k}(z)$, the preimages of z_1 are dense in $J(f_{n,c})$. It follows that $J(f_{n,c})$ contains no smooth arcs. $\square$

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References

[1] L. Carleson and T.W. Gamelin, *Complex Dynamics*, Springer-Verlag, 1993.

[2] R.B. Conway, *Functions of One Complex Variable*, Narosa Publishing House, New Delhi, 1982.

[3] R.L. Devancy, *An Introduction to Chaotic Dynamical Systems*, 2^{nd} ed., Addison-Wesley Publishing Company, Inc., 1989.

[4] K.J. Falconer, *Fractal Geometry*, John-Wiley & Sons, 1990.

[5] J. Rotman, *Galois Theory*, 2^{nd} ed., Springer-Verlag, New York, 1998.

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