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On the Novel Existence Results of Solutions for Fractional Langevin Equations Associating with Nonlinear Fractional Orders

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Abstract The Langevin equation is a core premise of the Brownian motion, which describes the development of essential processes in continuously changing situations. As a generalization of the classical one, the fractional Langevin equation offers a fractional Gaussian mechanism with two indices as parametrization, which is more flexible to model fractal systems. This paper deals with a nonlinear fractional Langevin equation involving two fractional orders with nonlocal integral boundary conditions. Our goal is to find the existence and uniqueness of a solution to the proposed Langevin equation by using the appropriate fixed point method. Some examples are also presented to illustrate the importance of our results in the existing literature.

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1. Introduction

Differential equations are widely used to explain the dynamical behavior of physical systems. If, for example, physical structures have memory and genetic properties, such as complex networks [22], stock market [21], anomalous diffusion [20], bacterial chemotaxis [19] and viscoelastic deformation [18], then fractional differential equations may be used to describe these models (see, for example [7, 37–39]).

Fractional order models are sometimes more accurate than integer-order models due to the additional degrees of freedom provided by fractional-order models. Additionally, fractional differential equations are one of the best ways for describing the inherited characteristics of diverse materials and techniques [49]. The existence of a memory term in such models not only accounts for the process's past but also for its current and future growth. For some recent work on fractional differential equations, see [1, 43–48].

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The theory and application of fractional differential equations have experienced extensive development (for more detail, see [30, 31]). It has been evident to researchers in recent decades that the analysis of various forms of fractional differential equations is of particular significance (see, for more detail [32, 33]). Most papers and books on fractional calculus are based on the linear fractional differential equations' solvability [27–29]. Different interpretations of fractional derivatives exist. Many of them have integral operators with various regularity properties, and some of them include singular kernels, for example Riemann-Liouville [33], Caputo [33], Caputo Fabrizio [34], etc (for more detail, see [35]). Many recent publications have used nonlinear computational techniques such as the Leray Schauder theorem, fixed-point analysis, and stabilization to solve the nonlinear fractional differential equations [23–26].

In the literature, several complex fractional modeling methods are used, including (but not limited to) the eminent Caputo and RiemannLiouville operators (see, for example [38, 39]). Various new generalizations of the Hadamard, Hilfer and CaputoHadamard operators were employed in this century, and various modeling attempts have been conducted by utilizing these new operators (see, for example [36, 37]). Five years ago, a radical formulation of a fractional frame without singularity was proposed by Fabrizio and Caputo [34]. A new operator has been created named as the fractional Caputo-Fabrizio operator. Nieto and Losada [40] focused on some significant computational aspects shortly after this work. The implementation of nonsingular operators led to numerous research articles about fractional modeling on this topic (for more detail, see [41, 42]).

Paul Langevin first introduced the Langevin equation in 1908 in order to better explain Brownian motion [2]. In this, Newton's second rule was extended to a Brownian particle to develop the stochastic mechanics' law named Langevin equation, which is applied to a molecule to describe the F = ma of thermodynamics. The Langevin equation is commonly adopted to discuss the progression of natural processes in oscillating conditions [4, 5, 12]. For example, if an arbitrary oscillation force is believed to be white noise, then Brownian motion is defined in terms of the Langevin equation. Nevertheless, the ordinary Langevin equations do not offer an exact explanation of the components for structures of a dynamic culture. Therefore, it is easier to substitute the natural derivative with fractional and analyze the Langevin equation in terms of the fractional derivative. For some latest work regarding the fractional Langevin equation, see [14–16, 41].

In the earlier 1990s, Mainardi developed the fractional Langevin equation [3]. There are several multiple kinds of fractional Langevin equations that have been investigated in [4–6]. The standard fractional Langevin equation that included only one fractional memory kernel was examined in [6]; the nonlinear Langevin equation with two fractional orders was studied in [4, 13]; the nonlinear Langevin equation that contained both a fractional memory kernel and a fractional derivative was analyzed in [9, 10].

Anti-periodic boundary value problems arise in the mathematical modeling of a wide range of scientific phenomena [50] and have garnered significant attention in recent years. For further information on anti-periodic boundary conditions, including examples and details, see [51, 52].

Here, we deal with the following fractional Langevin equation with nonlocal integral boundary conditions and two different fractional orders in different intervals.

$$D^m \left(D^\ell + \vartheta\right) \sigma(\zeta) = \varphi(\zeta, \sigma(\zeta)), \quad \zeta \in (0,1), \tag{1.1}$$

$$\begin{cases} \sigma(0)+\sigma(1)=\kappa_1 \int\limits_0^{\upsilon} \sigma(\varsigma)d\varsigma, \\ D^{\ell}\sigma(0)+D^{\ell}\sigma(1)=\kappa_2 \int\limits_0^{\upsilon} \sigma(\varsigma)d\varsigma, \\ \mathcal{D}^{2\ell}\sigma(0)+\mathcal{D}^{2\ell}\sigma(1)=\kappa_3 \int\limits_0^{\upsilon} \sigma(\varsigma)d\varsigma, \end{cases} \tag{1.2}$$

where D^{ℓ} and D^{m} are the Caputo fractional derivative of order $0<\ell<1$ and $1<m\leq 2$, $\vartheta,\kappa_1,\kappa_2,\kappa_3\in\mathbb{R}$, $0<\upsilon<1$, $\varphi:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuously differentiable function and $\mathcal{D}^{2\ell}$ is the sequential fractional derivative discussed in [31] and defined by

$$\begin{cases} \mathcal{D}^{\ell}\sigma = D^{\ell}\sigma, \\ \mathcal{D}^{k\ell}\sigma=\mathcal{D}^{\ell}\mathcal{D}^{(k-1)\ell}\sigma, \quad (k=2,3,\dots). \end{cases} \tag{1.3}$$

Our intent is to find the existence and uniqueness of a solution to the fractional Langevin equation (1.1) with (1.2) by using the fixed point theorems due to Banach [17] and Krasnoselskii [8]. We first discuss some basic results related to the fractional derivatives and integral. After that, we examine the existence results for the proposed nonlinear fractional Langevin equation. The first conclusion is focused on the concept of the Banach fixed point theorem, while the second finding is relied on a fixed point theorem by Krasnoselskii. At the end, some examples are given to demonstrate the importance of our findings in this area of research.

2. Preliminaries

In the continuation, the subsequent concepts and established results will be required.

Definition 2.1 ([11]). The Riemann-Lioville fractional integral of order $\ell>0$ for a continuous function $\sigma:[0,\infty)\to\mathbb{R}$ is defined as

$$\mathbb{I}^{\ell}\sigma(\zeta)=\frac{1}{\Gamma(\ell)}\int_0^{\zeta}(\zeta-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma$$

equipped with that the right-hand-side integral exists, where $\Gamma(\ell)$ defined by

$$\Gamma(\ell)=\int_0^{\infty}\zeta^{\ell-1}e^{-\zeta}d\zeta, \quad \ell>0.$$

Definition 2.2 ([33]). For a function $\sigma:[0,\infty)\to\mathbb{R}$, the Caputo fractional derivative of order $\ell>0$ is defined as

$$D^{\ell}\sigma(\zeta)=\frac{1}{\Gamma(\eta-\ell)}\int_0^{\zeta}(\zeta-\varsigma)^{\eta-\ell-1}\sigma^{(n)}(s)d\varsigma \quad (\eta-1<\ell<\eta,\ \eta=[\ell]+1),$$

where $[\ell]$ represents the integer part of ℓ.

Lemma 2.3 ([11, 33]). *Let $\sigma\in L^1([0,1])$ and $\ell,m>0$.*

(i) If $\ell\in\mathbb{N}$, then $\mathbb{I}^{\ell}\sigma(\zeta)=\frac{1}{(\ell-1)!}\int_0^{\zeta}(\zeta-\varsigma)^{\ell-1}\sigma(\varsigma)ds$.
(ii) If $\ell\in\mathbb{N}$, then $D^{\ell}\sigma(\zeta)=\sigma^{(\ell)}(\zeta)$.
(iii) $D^{\ell}\mathbb{I}^{\ell}\sigma(\zeta)=\sigma(\zeta)$.

(iv) $\mathbb{I}^{\ell}\mathbb{I}^{m}\sigma(\zeta)=\mathbb{I}^{\ell+m}\sigma(\zeta)$.

Lemma 2.4 ([11]). *Let $n\in\mathbb{N}$ and $n-1<\ell\leq n$. For a continuous function $\sigma:[0,\infty)\to\mathbb{R}$, we have*

$$\mathbb{I}^{\ell}D^{\ell}\sigma(\zeta)=\sigma(\zeta)+a_0+a_1\zeta+a_2\zeta^2+...+a_{n-1}\zeta^{n-1},$$

where $a_i\in\mathbb{R}$ for all $i=1,2,\ldots,n-1$.

We now consider the following linear fractional Langevin equation

$$D^{m}\left(D^{\ell}+\vartheta\right)\sigma(\zeta)=\psi(\zeta),\ \ 0<\zeta<1,\ 0<\ell<1,\ 1<m\leq 2, \tag{2.1}$$

enhanced with the nonlocal integral boundary conditions

$$\begin{cases} \sigma(0)+\sigma(1)=\kappa_1\int\limits_0^{\upsilon}\sigma(\varsigma)d\varsigma, \\ D^{\ell}\sigma(0)+D^{\ell}\sigma(1)=\kappa_2\int\limits_0^{\upsilon}\sigma(\varsigma)d\varsigma, \\ \mathcal{D}^{2\ell}\sigma(0)+\mathcal{D}^{2\ell}\sigma(1)=\kappa_3\int\limits_0^{\upsilon}\sigma(\varsigma)d\varsigma, \end{cases} \tag{2.2}$$

where $\sigma\in L^1([0,1])$ is an unknown function, $0<\upsilon<1$ and $\vartheta,\kappa_i\in\mathbb{R}$, for $i=1,2,3$.

Lemma 2.5. *σ is a solution of (2.1) with the condition (2.2) if and only if it is a solution of the following nonlinear integral equation*

$$\begin{aligned}\sigma(\zeta) \;=\; & \frac{1}{\Gamma(\ell+m)}\int_0^{\zeta}(\zeta-\varsigma)^{\ell+m-1}\psi(\varsigma)d\varsigma-\frac{\vartheta}{\Gamma(\ell)}\int_0^{\zeta}(\zeta-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \\ & -\frac{1}{2\Gamma(\ell+m)}\int_0^1(1-\varsigma)^{\ell+m-1}\psi(\varsigma)d\varsigma+\frac{\vartheta}{2\Gamma(\ell)}\int_0^1(1-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \\ & +\left(\frac{1-2\zeta^{\ell}}{4\Gamma(\ell+1)\Gamma(m)}\right)\int_0^1(1-\varsigma)^{m-1}\psi(\varsigma)d\varsigma \\ & +\frac{\Gamma(2-\ell)}{\Gamma(\ell+2)}\left(\frac{1-\ell}{4}+\frac{\zeta^{\ell}(1+\ell)}{2}-\zeta^{\ell+1}\right)\int_0^1\frac{(1-\varsigma)^{m-\ell-1}}{\Gamma(m-\ell)}\psi(\varsigma)d\varsigma \\ & +\frac{A_0}{4\Gamma(\ell+2)}\int\limits_0^{\upsilon}\sigma(\varsigma)d\varsigma,\end{aligned} \tag{2.3}$$

where

$$A_0=\left[\begin{array}{c}\left\{2\Gamma(\ell+2)+(1+\ell)\vartheta(2\zeta^{\ell}-1)\right\}\kappa_1 \\ +\left\{(1+\ell)(2\zeta^{\ell}-1)+\vartheta\Gamma(2-\ell)\left(2\zeta^{\ell}(2\zeta-\ell-1)-(1-\ell)\right)\right\}\kappa_2 \\ +\left\{\Gamma(2-\ell)(2\zeta^{\ell}(2\zeta-\ell-1)-(1-\ell))\right\}\kappa_3\end{array}\right].$$

Proof. Let σ be a solution of (2.1) with the condition (2.2). As argued in [7] with Lemmas 2.3 and 2.4, the general solution of

$$D^{m}\left(D^{\ell}+\vartheta\right)\sigma(\zeta)=\psi(\zeta)$$

can be written as

$$\sigma(\zeta)=\mathbb{I}^{\ell+m}\psi(\zeta)-\vartheta\mathbb{I}^{\ell}\sigma(\zeta)+a_1+a_2\frac{\zeta^{\ell}}{\Gamma(\ell+1)}+a_3\frac{\zeta^{\ell+1}}{\Gamma(\ell+2)}. \tag{2.4}$$

Using the boundary condition (2.2), we have

$$a_3 = -\frac{\Gamma(2-\ell)}{\Gamma(m-\ell)}\int_0^1(1-\varsigma)^{m-\ell-1}\psi(\varsigma)d\varsigma+(\vartheta\kappa_2+\kappa_3)\Gamma(2-\ell)\int\limits_0^{\upsilon}\sigma(\varsigma)d\varsigma,$$

$$\begin{aligned} a_2 &= \frac{-1}{2\Gamma(m)}\int_0^1(1-\varsigma)^{m-1}\psi(\varsigma)d\varsigma+\frac{\Gamma(2-\ell)}{2\Gamma(m-\ell)}\int_0^1(1-\varsigma)^{m-\ell-1}\psi(\varsigma)d\varsigma \\ &\quad+\left(\frac{\vartheta\kappa_1+(1-\vartheta\Gamma(2-\ell))\kappa_2-\Gamma(2-\ell)\kappa_3}{2}\right)\int\limits_0^{\upsilon}\sigma(\varsigma)d\varsigma, \\ a_1 &= \frac{-1}{2\Gamma(\ell+m)}\int_0^1(1-\varsigma)^{\ell+m-1}\psi(\varsigma)d\varsigma+\frac{\vartheta}{2\Gamma(\ell)}\int_0^1(1-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \\ &+ \frac{1}{4\Gamma(\ell+1)\Gamma(m)}\int_0^1(1-\varsigma)^{m-1}\psi(\varsigma)d\varsigma+\frac{(1-\ell)\Gamma(2-\ell)}{4\Gamma(m-\ell)\Gamma(\ell+2)}\int_0^1(1-\varsigma)^{m-\ell-1}\psi(\varsigma)d\varsigma \\ &+ \frac{1}{4\Gamma(\ell+2)}\times\left\{\begin{array}{c}(2\Gamma(\ell+2)-(1+\ell)\vartheta)\kappa_1 \\ -((1+\ell)+\vartheta(1-\ell)\Gamma(2-\ell))\,\kappa_2 \\ -((1-\ell)\Gamma(2-\ell))\kappa_3\end{array}\right\}\int\limits_0^{\upsilon}\sigma(\varsigma)d\varsigma. \end{aligned}$$

Replacing the values of a_1, a_2, a_3 in (2.4), we have the desired solution.

On the other hand, it is easy to prove that, if σ is a solution of the integral equation (2.3), then σ is also a solution of the problem (2.1) with the condition (2.2). ■

Lemma 2.6. *For all $\ell\in(0,1)$, we have*

$$V := \max_{\zeta\in[0,1]}\left|\zeta^{\ell}(2\zeta-\ell-1)\right| = \begin{cases} 1-\ell & \text{if } \ell\le\frac{1}{2}, \\ \left(\frac{\ell}{2}\right)^{\ell} & \text{if } \ell>\frac{1}{2}. \end{cases} \tag{2.5}$$

In the proof of our main results, we shall adopt the Banach contraction mapping principle and the Krasnoselskii fixed point theorem which is described below.

Theorem 2.7 ([8])**.** *Let $\mathcal{O}$ be a closed convex and nonempty subset of Banach space $\mathcal{B}$. Suppose that $\mathcal{G}_1:\mathcal{O}\to\mathcal{B}$ and $\mathcal{G}_2:\mathcal{O}\to\mathcal{B}$ are two operators satisfying the following conditions:*

(1) $\mathcal{G}_1a+\mathcal{G}_2b\in\mathcal{O}$ whenever $a,b\in\mathcal{O}$;
(2) $\mathcal{G}_1$ is compact and continuous on $\mathcal{O}$;
(3) $\mathcal{G}_2$ is a Banach contraction mapping on $\mathcal{O}$.

Then there exists $a\in\mathcal{O}$ such that $a=\mathcal{G}_1a+\mathcal{G}_2a$.

3. Main Results

Let $\mathcal{X}=[0,1]$ and $\mathcal{B}=C(\mathcal{X},\mathbb{R})$ be the normed space of all continuous functions from $\mathcal{X}$ into $\mathbb{R}$ endowed with the norm $\|\cdot\|:\mathcal{B}\to[0,\infty)$ defined by

$$\|\sigma\|=\sup\{|\sigma(\zeta)|,\zeta\in\mathcal{X}\}$$

for all $\sigma\in\mathcal{B}$.

We add the following hypotheses before stating and proving the key findings. Suppose that the following conditions hold:

$(\tilde{H}_1)$: the function $\varphi:\mathcal{X}\times\mathbb{R}\to\mathbb{R}$ is jointly continuous;

($\tilde{H}_2$): the function φ satisfies

$$|\varphi(\zeta,\sigma_1)-\varphi(\zeta,\sigma_2)|\leq Ļ\,|\sigma_1-\sigma_2|\,,\ \forall\zeta\in\mathcal{X},\ \sigma_1,\sigma_2\in\mathbb{R}$$

where $Ļ$ is the Lipschitz constant;

($\tilde{H}_3$): there exists a continuous function $\phi:\mathcal{X}\to[0,\infty)$ such that

$$|\varphi(\zeta,\mu)|\leq\phi(\zeta),\quad\forall(\zeta,\mu)\in(\mathcal{X},\mathbb{R}).$$

For the computational convenience, we set

$$A_0 = \begin{bmatrix} \left\{2\Gamma(\ell+2)+(1+\ell)\vartheta(2\zeta^\ell-1)\right\}\kappa_1 \\ +\left\{(1+\ell)(2\zeta^\ell-1)+\vartheta\Gamma(2-\ell)\left(2\zeta^\ell(2\zeta-\ell-1)-(1-\ell)\right)\right\}\kappa_2 \\ +\left\{\Gamma(2-\ell)(2\zeta^\ell(2\zeta-\ell-1)-(1-\ell))\right\}\kappa_3 \end{bmatrix}, \tag{3.1}$$

$$A_1 = \begin{bmatrix} (2\Gamma(\ell+2)+(1+\ell)|\vartheta|)\,\kappa_1 \\ +((1+\ell)+|\vartheta|\Gamma(2-\ell)\,(2V+1-\ell))\,\kappa_2 \\ +(\Gamma(2-\ell)(2V+1-\ell))\,\kappa_3 \end{bmatrix}, \tag{3.2}$$

$$\Theta = \Xi_1+Ļ\Theta_1, \tag{3.3}$$

$$\begin{aligned}\Xi &= Ļ\left(\frac{1}{2\Gamma(\ell+m+1)}+\frac{1}{4\Gamma(m+1)\Gamma(\ell+1)}+\frac{\Gamma(2-\ell)(2V+1-\ell)}{4\Gamma(\ell+2)\Gamma(m-\ell+1)}\right)\\ &\quad+\frac{|\vartheta|}{2\Gamma(\ell+1)}+\frac{A_1 v}{4\Gamma(\ell+2)},\end{aligned} \tag{3.4}$$

where

$$\Theta_1 := Ļ\left(\frac{3}{2\Gamma(\ell+m+1)}+\frac{1}{4\Gamma(m+1)\Gamma(\ell+1)}+\frac{\Gamma(2-\ell)(2V+1-\ell)}{4\Gamma(\ell+2)\Gamma(m-\ell+1)}\right) \tag{3.5}$$

$$\Xi_1 := \frac{3|\vartheta|}{2\Gamma(\ell+1)}+\frac{A_1 v}{4\Gamma(\ell+2)} \tag{3.6}$$

From Lemma 2.5, we can write

$$\sigma=\mathcal{G}(\sigma)=\mathcal{G}_1(\sigma)+\mathcal{G}_2(\sigma), \tag{3.7}$$

where the operator $\mathcal{G}_i:\mathcal{B}\to\mathcal{B}$ for all $i=1,2$ can be described as

$$(\mathcal{G}_1\sigma)(\zeta)=\frac{1}{\Gamma(\ell+m)}\int_0^\zeta(\zeta-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma-\frac{\vartheta}{\Gamma(\ell)}\int_0^\zeta(\zeta-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \tag{3.8}$$

and

$$\begin{aligned}(\mathcal{G}_2\sigma)(\zeta) &= -\frac{1}{2\Gamma(\ell+m)}\int_0^1(1-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma+\frac{\vartheta}{2\Gamma(\ell)}\int_0^1(1-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma\\ &+\left(\frac{1-2\zeta^\ell}{4\Gamma(\ell+1)\Gamma(m)}\right)\int_0^1(1-\varsigma)^{m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma+\frac{\Gamma(2-\ell)}{\Gamma(\ell+2)}\times\\ &\quad\left(\frac{1-\ell+2\zeta^\ell(1+\ell)-4\zeta^{\ell+1}}{4\Gamma(m-\ell)}\right)\int_0^1(1-\varsigma)^{m-\ell-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma\\ &+\frac{A_0}{4\Gamma(\ell+2)}\int_0^v\sigma(\varsigma)d\varsigma.\end{aligned} \tag{3.9}$$

Theorem 3.1. *Consider the problem (1.1) with (1.2). Assume that the assumptions $(\tilde{H}_1)$ and $(\tilde{H}_2)$ hold. Then the boundary value problem (1.1) with (1.2) has a unique solution if $\Theta < 1$, where Θ is given by (3.3).*

Proof. We define $\mathcal{G} : \mathcal{B} \to \mathcal{B}$ for each $\sigma \in \mathcal{B}$ by

$$\begin{aligned}(\mathcal{G}\sigma)(\zeta) &= \frac{1}{\Gamma(\ell+m)} \int_0^\zeta (\zeta-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma - \frac{\vartheta}{\Gamma(\ell)} \int_0^\zeta (\zeta-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \\ &\quad -\frac{1}{2\Gamma(\ell+m)} \int_0^1 (1-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma + \frac{\vartheta}{2\Gamma(\ell)} \int_0^1 (1-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \\ &\quad + \left(\frac{1-2\zeta^\ell}{4\Gamma(\ell+1)\Gamma(m)}\right) \int_0^1 (1-\varsigma)^{m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma + \frac{\Gamma(2-\ell)}{\Gamma(\ell+2)} \times \\ &\quad \left(\frac{1-\ell+2\zeta^\ell(1+\ell)-4\zeta^{\ell+1}}{4\Gamma(m-\ell)}\right) \int_0^1 (1-\varsigma)^{m-\ell-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma \\ &\quad +\frac{A_0}{4\Gamma(\ell+2)} \int_0^\upsilon \sigma(\varsigma)d\varsigma.\end{aligned}$$

We set $\sup_{\zeta\in\mathcal{X}} |\varphi(\zeta,0)| = \mathcal{M}$ and choose

$$\tilde{r} \geq \frac{\mathcal{M}\Theta_1}{1-\Theta}, \tag{3.10}$$

where Θ_1 is given by (3.5). We now prove that $\mathcal{G}\mathcal{B}_{\tilde{r}} \subset \mathcal{B}_{\tilde{r}}$, where $\mathcal{B}_{\tilde{r}} = \{\sigma \in \mathcal{B} : \|\sigma\| \leq \tilde{r}\}$. For $\sigma \in \mathcal{B}_{\tilde{r}}$, we get

$$\begin{aligned}\|\mathcal{G}\sigma\| &= \sup_{\zeta\in\mathcal{X}} \Bigg| \frac{1}{\Gamma(\ell+m)} \int_0^\zeta (\zeta-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma - \frac{\vartheta}{\Gamma(\ell)} \int_0^\zeta (\zeta-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \\ &\quad -\frac{1}{2\Gamma(\ell+m)} \int_0^1 (1-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma + \frac{\vartheta}{2\Gamma(\ell)} \int_0^1 (1-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \\ &\quad + \left(\frac{1-2\zeta^\ell}{4\Gamma(\ell+1)\Gamma(m)}\right) \int_0^1 (1-\varsigma)^{m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma + \frac{\Gamma(2-\ell)}{\Gamma(\ell+2)} \times \\ &\quad \left(\frac{1-\ell+2\zeta^\ell(1+\ell)-4\zeta^{\ell+1}}{4\Gamma(m-\ell)}\right) \int_0^1 (1-\varsigma)^{m-\ell-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma \\ &\quad +\frac{A_0}{4\Gamma(\ell+2)} \int_0^\upsilon \sigma(\varsigma)d\varsigma \Bigg| \\ &\leq \frac{(\mathfrak{L}\tilde{r}+\mathcal{M})}{\Gamma(\ell+m)} \sup_{\zeta\in\mathcal{X}} \int_0^\zeta (\zeta-\varsigma)^{\ell+m-1}d\varsigma + \frac{|\vartheta|\,\tilde{r}}{\Gamma(\ell)} \sup_{\zeta\in\mathcal{X}} \int_0^\zeta (\zeta-\varsigma)^{\ell-1}d\varsigma + \frac{(\mathfrak{L}\tilde{r}+\mathcal{M})}{2\Gamma(\ell+m)} \times \\ &\quad \int_0^1 (1-\varsigma)^{\ell+m-1}d\varsigma + \frac{|\vartheta|\tilde{r}}{2\Gamma(\ell)} \int_0^1 (1-\varsigma)^{\ell-1}d\varsigma + \frac{(\mathfrak{L}\tilde{r}+\mathcal{M})}{4\Gamma(\ell+1)\Gamma(m)} \int_0^1 (1-\varsigma)^{m-1}d\varsigma \\ &\quad +\frac{\Gamma(2-\ell)(\mathfrak{L}\tilde{r}+\mathcal{M})}{\Gamma(\ell+2)\Gamma(m-\ell)} \left(\frac{1-\ell}{4}+\frac{V}{2}\right) \int_0^1 (1-\varsigma)^{m-\ell-1}d\varsigma + \frac{A_1\tilde{r}}{4\Gamma(\ell+2)} \int_0^\upsilon d\varsigma \\ &\leq \tilde{r},\end{aligned}$$

which shows that $\|\mathcal{G}\sigma\| \leq \tilde{r}$. Let $\sigma_1, \sigma_2 \in \mathcal{B}$. For each $\zeta \in \mathcal{X}$, we have

$$
\begin{aligned}
\|\mathcal{G}\sigma_1-\mathcal{G}\sigma_2\| &= \sup_{\zeta\in\mathcal{X}}|(\mathcal{G}\sigma_1)(\zeta)-(\mathcal{G}\sigma_2)(\zeta)| \\
&\leq \frac{1}{\Gamma(\ell+m)}\sup_{\zeta\in\mathcal{X}}\int_0^{\zeta}(\zeta-\varsigma)^{\ell+m-1}|\varphi(\varsigma,\sigma_1(\varsigma))-\varphi(\varsigma,\sigma_2(\varsigma))|\,d\varsigma \\
&\quad +\frac{|\vartheta|}{\Gamma(\ell)}\sup_{\zeta\in\mathcal{X}}\int_0^{\zeta}(\zeta-\varsigma)^{\ell-1}|\sigma_1(\varsigma)-\sigma_2(\varsigma)|\,d\varsigma \\
&\quad +\frac{1}{2\Gamma(\ell+m)}\int_0^1(1-\varsigma)^{\ell+m-1}|\varphi(\varsigma,\sigma_1(\varsigma))-\varphi(\varsigma,\sigma_2(\varsigma))|\,d\varsigma \\
&\quad +\frac{|\vartheta|}{2\Gamma(\ell)}\int_0^1(1-\varsigma)^{\ell-1}|\sigma_1(\varsigma)-\sigma_2(\varsigma)|\,d\varsigma \\
&\quad +\frac{1}{4\Gamma(\ell+1)\Gamma(m)}\int_0^1(1-\varsigma)^{m-1}|\varphi(\varsigma,\sigma_1(\varsigma))-\varphi(\varsigma,\sigma_2(\varsigma))|\,d\varsigma \\
&\quad +\frac{\Gamma(2-\ell)}{\Gamma(\ell+2)\Gamma(m-\ell)}\left(\frac{1-\ell}{4}+\frac{V}{2}\right)\times \\
&\quad \int_0^1(1-\varsigma)^{m-\ell-1}|\varphi(\varsigma,\sigma_1(\varsigma))-\varphi(\varsigma,\sigma_2(\varsigma))|\,d\varsigma \\
&\quad +\frac{A_1}{4\Gamma(\ell+2)}\int_0^{v}|\sigma_1(\varsigma)-\sigma_2(\varsigma)|\,d\varsigma \\
&\leq \Theta\|\sigma_1-\sigma_2\|,
\end{aligned}
$$

where Θ is given by (3.3) and depends on the constants associated to the problem. Hence, $\mathcal{G}$ is a Banach contraction mapping if $\Theta<1$. Consequently, the completion of the theorem relies on the Banach contraction mapping principle. This fills out the proof. ■

Theorem 3.2. *Consider the problem (1.1) with (1.2). Assume that the assumptions ($\tilde{H}_1$), ($\tilde{H}_2$) and ($\tilde{H}_3$) hold. Then the boundary value problem (1.1) with (1.2) has at least one solution if $\Xi<1$, where Ξ is given by (3.4).*

Proof. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be defined as in (3.8) and (3.9). We set $\sup_{\zeta\in\mathcal{X}}|\phi(\zeta)|\leq\|\phi\|$. Let $\mathcal{B}_{\tilde{r}}=\{\sigma\in\mathcal{B}:\|\sigma\|\leq\tilde{r}\}$ be the closed ball described for

$$\tilde{r}\geq\Theta_1\|\phi\|(1-\Xi_1)^{-1},$$

where Θ_1 and Ξ_1 are given by (3.5) and (3.6), respectively. Then, for $\sigma_1,\sigma_2\in\mathcal{B}_{\tilde{r}}$, we have

$$
\begin{aligned}
\|\mathcal{G}_1\sigma_1+\mathcal{G}_2\sigma_2\| &\leq \Theta_1\|\phi\|+\tilde{r}\Xi_1 \\
&\leq \tilde{r}
\end{aligned}
$$

which concludes that $\mathcal{G}_1\sigma_1+\mathcal{G}_2\sigma_2\in\mathcal{B}_{\tilde{r}}$. In view of condition ($\tilde{H}_2$), $\mathcal{G}_2$ may be shown to be a Banach contraction mapping if $\Xi<1$. Continuity of φ indicates that the operator

$\mathcal{G}_1$ is continuous. Further, $\mathcal{G}_1$ is bounded uniformly on $\mathcal{B}_{\tilde{r}}$ as

$$\begin{aligned}
\|(\mathcal{G}_1\sigma)(\zeta)\| &= \sup_{\zeta\in\mathcal{X}}\left|\frac{1}{\Gamma(\ell+m)}\int_0^{\zeta}(\zeta-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma-\frac{\vartheta}{\Gamma(\ell)}\int_0^{\zeta}(\zeta-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma\right| \\
&\le \frac{\|\phi\|}{\Gamma(\ell+m)}\sup_{\zeta\in\mathcal{X}}\int_0^{\zeta}(\zeta-\varsigma)^{\ell+m-1}d\varsigma+\frac{|\vartheta|\,\|\sigma_1\|}{\Gamma(\ell)}\sup_{\zeta\in\mathcal{X}}\int_0^{\zeta}(\zeta-\varsigma)^{\ell-1}d\varsigma \\
&\le \frac{\|\phi\|}{\Gamma(\ell+m+1)}+\frac{|\vartheta|\,\tilde{r}}{\Gamma(\ell+1)}.
\end{aligned}$$

Let us suppose that $0\le\zeta_1<\zeta_2\le 1$. For $\sigma\in\mathcal{B}_{\tilde{r}}$, we obtain

$$\begin{aligned}
\|(\mathcal{G}_1\sigma)(\zeta_2)-(\mathcal{G}_1\sigma)(\zeta_1)\| &= \left\|\frac{1}{\Gamma(\ell+m)}\int_0^{\zeta_2}(\zeta_2-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma\right. \\
&\quad -\frac{\vartheta}{\Gamma(\ell)}\int_0^{\zeta_2}(\zeta_2-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma \\
&\quad -\frac{1}{\Gamma(\ell+m)}\int_0^{\zeta_1}(\zeta_1-\varsigma)^{\ell+m-1}\varphi(\varsigma,\sigma(\varsigma))d\varsigma \\
&\quad \left.+\frac{\vartheta}{\Gamma(\ell)}\int_0^{\zeta_1}(\zeta_1-\varsigma)^{\ell-1}\sigma(\varsigma)d\varsigma\right\| \\
&\le \frac{\|\phi\|}{\Gamma(\ell+m+1)}\left(\zeta_2^{\ell+m}-\zeta_1^{\ell+m}\right) \\
&\quad +\frac{|\vartheta|\,\tilde{r}}{\Gamma(\ell+1)}\left(2(\zeta_2-\zeta_1)^{\ell}+\zeta_1^{\ell}-\zeta_2^{\ell}\right),
\end{aligned}$$

which is not dependent on σ and approaches to zero as $\zeta_2\to\zeta_1$. Thus, $\mathcal{G}_1$ is relatively compact on $\mathcal{B}_{\tilde{r}}$. Consequently, $\mathcal{G}_1$ is compact by the Arzelá-Ascoli theorem. Hence all of the Theorem 2.7's premises are fulfilled. Therefore the proposed problem (1.1) with (1.2) has at least one solution on $\mathcal{B}_{\tilde{r}}$. This fills out the proof. ■

4. Some illustrative examples

Here, we present the following examples to support our key findings.

Example 4.1. Consider the following boundary value problem

$$\begin{cases}
D^{\frac{3}{2}}\left(D^{\frac{2}{5}}+\frac{1}{8}\right)\sigma(\zeta)=\varphi(\zeta,\sigma(\zeta)), \quad 0<\zeta<1, \\
\sigma(0)+\sigma(1)=2\int_0^{\frac{1}{4}}\sigma(\varsigma)d\varsigma, \\
D^{\frac{2}{5}}\sigma(0)+D^{\frac{2}{5}}\sigma(1)=3\int_0^{\frac{1}{4}}\sigma(\varsigma)d\varsigma, \\
\mathcal{D}^{\frac{4}{5}}\sigma(0)+\mathcal{D}^{\frac{4}{5}}\sigma(1)=5\int_0^{\frac{1}{4}}\sigma(\varsigma)d\varsigma,
\end{cases} \tag{4.1}$$

where $\ell = \frac{2}{5}$, $m = \frac{3}{2}$, $\vartheta = \frac{1}{8}$, $v = \frac{1}{4}$, $\kappa_1 = 2$, $\kappa_2 = 3$, $\kappa_3 = 5$ and $\varphi(\zeta, \sigma(\zeta))$ will be evaluated on the premises of the theorems. With the provided data, we have $V = \frac{3}{5}$, $\Theta_1 \approx 1.343$ and $\Xi_1 \approx 0.771$.

Case 1 (Banach fixed point theorem): We define $\varphi : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\zeta, \mu) = Ļ(1 + \zeta \sin(\zeta\mu)) \tag{4.2}$$

for all $\zeta \in \mathcal{X}$ and $\mu \in \mathbb{R}$, where $Ļ > 0$ is a constant. It is straightforward that $\varphi : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$|\varphi(\zeta, \sigma_1) - \varphi(\zeta, \sigma_2)| = Ļ\, |\zeta \sin(\zeta\sigma_1) - \zeta \sin(\zeta\sigma_2)| \leq Ļ\, |\sigma_1 - \sigma_2| \tag{4.3}$$

for all $\zeta \in \mathcal{X}$ and $\sigma_1, \sigma_1 \in \mathbb{R}$, where $Ļ$ is the Lipschitz constant. Note that the conditions $(\tilde{H}_1)$ and $(\tilde{H}_2)$ are satisfied. Also, if we pick $L < \frac{229}{1343}$, then all the conditions of Theorem 3.1 are fulfilled. Thus, if $L < \frac{229}{1343}$, then the proposed problem (4.1) has a unique solution on $\mathcal{X}$.

Case 2 (Krasnoselskii's fixed point theorem): For Theorem 3.2, we define φ as indicated in (4.2) and $\phi(\zeta) = 1 + \zeta$. Clearly, the conditions $(\tilde{H}_1)$, $(\tilde{H}_2)$ and $(\tilde{H}_3)$ hold. Also, if we choose $Ļ < \frac{185}{398}$, then all Theorem 3.2 premises are fulfilled. Thus, the proposed problem (4.1) has at least one solution on $\mathcal{X}$ if $Ļ < \frac{185}{398}$.

Example 4.2. Consider the boundary value problem given below

$$\begin{cases} D^{\frac{6}{5}}\left(D^{\frac{1}{3}} + \dfrac{1}{25}\right)\sigma(\zeta) = \varphi(\zeta, \sigma(\zeta)), \quad 0 < \zeta < 1, \\ \sigma(0) + \sigma(1) = \dfrac{7}{16}\displaystyle\int_0^{\frac{1}{6}} \sigma(\varsigma)d\varsigma, \\ D^{\frac{1}{3}}\sigma(0) + D^{\frac{1}{3}}\sigma(1) = \dfrac{13}{29}\displaystyle\int_0^{\frac{1}{6}} \sigma(\varsigma)d\varsigma, \\ \mathcal{D}^{\frac{2}{3}}\sigma(0) + \mathcal{D}^{\frac{2}{3}}\sigma(1) = \dfrac{3}{17}\displaystyle\int_0^{\frac{1}{6}} \sigma(\varsigma)d\varsigma, \end{cases} \tag{4.4}$$

where $\ell = \frac{1}{3}$, $m = \frac{6}{5}$, $\vartheta = \frac{1}{25}$, $v = \frac{1}{6}$, $\kappa_1 = \frac{7}{16}$, $\kappa_2 = \frac{13}{29}$, $\kappa_3 = \frac{3}{17}$ and $\varphi(\zeta, \sigma(\zeta))$ will be evaluated on the premises of the theorems. With the provided data, we have $V = \frac{2}{3}$, $\Theta_1 \approx 1.755$ and $\Xi_1 \approx 0.155$.

Case 1 (Banach fixed point theorem): We define $\varphi : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\zeta, \mu) = Ļ(|\mu| \cos(\zeta) + 1), \tag{4.5}$$

for all $\zeta \in \mathcal{X}$ and $\mu \in \mathbb{R}$, where $Ļ > 0$ is a constant. It is straightforward that $\varphi : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$|\varphi(\zeta, \sigma_1) - \varphi(\zeta, \sigma_2)| \leq Ļ\, |\sigma_1 - \sigma_2| \tag{4.6}$$

for all $\zeta \in \mathcal{X}$ and $\sigma_1, \sigma_1 \in \mathbb{R}$, where $Ļ$ is the Lipschitz constant. Note that the conditions $(\tilde{H}_1)$ and $(\tilde{H}_2)$ are satisfied. Also, if we pick $L < \frac{0.845}{1.755}$, then all the conditions of Theorem 3.1 are fulfilled. Thus, if $L < \frac{0.845}{1.755}$, then the proposed problem (4.4) has a unique solution on $\mathcal{X}$.

Case 2 (Krasnoselskii's fixed point theorem): For Theorem 3.2, we define φ as indicated in (4.5) and $\phi(\zeta) = 1 + \zeta$. Clearly the conditions $(\tilde{H}_1)$, $(\tilde{H}_2)$ and $(\tilde{H}_3)$ hold. Also, if we choose $Ļ < \frac{89}{102}$, then all Theorem 3.2 premises are fulfilled. Thus, the proposed problem (4.4) has at least one solution on $\mathcal{X}$ if $Ļ < \frac{89}{102}$.

Example 4.3. Consider the following boundary value problem

$$\begin{cases} D^{\frac{9}{5}}\left(D^{\frac{1}{4}} + \dfrac{1}{11}\right)\sigma(\zeta) = \varphi(\zeta, \sigma(\zeta)), \quad 0 < \zeta < 1, \\ \sigma(0) + \sigma(1) = \dfrac{1}{3}\displaystyle\int_0^{\frac{1}{2}} \sigma(\varsigma)d\varsigma, \\ D^{\frac{1}{4}}\sigma(0) + D^{\frac{1}{4}}\sigma(1) = \dfrac{4}{11}\displaystyle\int_0^{\frac{1}{2}} \sigma(\varsigma)d\varsigma, \\ \mathcal{D}^{\frac{1}{2}}\sigma(0) + \mathcal{D}^{\frac{1}{2}}\sigma(1) = \dfrac{3}{2}\displaystyle\int_0^{\frac{1}{2}} \sigma(\varsigma)d\varsigma, \end{cases} \tag{4.7}$$

where $\ell = \frac{1}{4}$, $m = \frac{9}{5}$, $\vartheta = \frac{1}{11}$, $\upsilon = \frac{1}{2}$, $\kappa_1 = \frac{1}{3}$, $\kappa_2 = \frac{4}{11}$, $\kappa_3 = \frac{3}{2}$ and $\varphi(\zeta, \sigma(\zeta))$ will be evaluated on the premises of the theorems. With the provided data, we have $V = \frac{3}{4}$, $\Theta_1 \approx 1.212$ and $\Xi_1 \approx 0.638$.

Case 1 (Banach fixed point theorem): We define $\varphi : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\zeta, \mu) = Ļ(|\mu| + \cos(\zeta)), \tag{4.8}$$

for all $\zeta \in \mathcal{X}$ and $\mu \in \mathbb{R}$, where $Ļ > 0$ is a constant. It is straightforward that $\varphi : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$|\varphi(\zeta, \sigma_1) - \varphi(\zeta, \sigma_2)| \leq Ļ\, |\sigma_1 - \sigma_2| \tag{4.9}$$

for all $\zeta \in \mathcal{X}$ and $\sigma_1, \sigma_1 \in \mathbb{R}$, where $Ļ$ is the Lipschitz constant. Note that the conditions $(\tilde{H}_1)$ and $(\tilde{H}_2)$ are satisfied. Also, if we pick $L < \frac{181}{606}$, then all the conditions of Theorem 3.1 are fulfilled. Thus, if $L < \frac{181}{606}$, then the proposed problem (4.7) has a unique solution on $\mathcal{X}$.

Case 2 (Krasnoselskii's fixed point theorem): For Theorem 3.2, we define φ as indicated in (4.8) and $\phi(\zeta) = 1 + \zeta$. Clearly the conditions $(\tilde{H}_1)$, $(\tilde{H}_2)$ and $(\tilde{H}_3)$ hold. Also, if we choose $Ļ < \frac{22}{35}$, then all Theorem 3.2 premises are fulfilled. Thus, the proposed problem (4.7) has at least one solution on $\mathcal{X}$ if $Ļ < \frac{22}{35}$.

5. Conclusion

A broad range of generalizations of the Langevin equation has been proposed in recent years, including fractional derivatives [3–6]. This paper proposed the nonlinear fractional Langevin equation having two fractional orders at different intervals and nonlocal integral boundary conditions. We used the fixed-point theorems due to Banach and Krasnoselskii to figure out the existence and uniqueness of a solution to the suggested boundary value problem. Three illustrative examples are also presented to show the significance of our results in the particular literature. Our method is simple and can be extended to several real-world problems.

In the end, we raise the stability of the following fractional differential equation as an open problem for the interested readers

$$D^m\left(D^\ell+\vartheta\right)\sigma(\zeta)=\varphi(\zeta,\sigma(\zeta)),\quad \zeta\in(0,1),$$

associated with

$$\begin{cases} \sigma(0)+\sigma(1)=\kappa_1\displaystyle\int_0^{\upsilon}\sigma(\varsigma)d\varsigma, \\ D^\ell\sigma(0)+D^\ell\sigma(1)=\kappa_2\displaystyle\int_0^{\upsilon}\sigma(\varsigma)d\varsigma, \\ \mathcal{D}^{2\ell}\sigma(0)+\mathcal{D}^{2\ell}\sigma(1)=\kappa_3\displaystyle\int_0^{\upsilon}\sigma(\varsigma)d\varsigma, \end{cases}$$

where D^ℓ and D^m are the Caputo fractional derivative of order $0<\ell<1$ and $1<m\leq 2$, $\vartheta,\kappa_1,\kappa_2,\kappa_3\in\mathbb{R}$, $0<\upsilon<1$, $\varphi:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuously differentiable function and $\mathcal{D}^{2\ell}$ is the sequential fractional derivative.

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