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Δ-Convergence and Strong Convergence for Asymptotically Nonexpansive Mappings on a CAT(0) Space

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Abstract In this paper, we give the Δ and strong convergence theorems of the new three-step iteration for asymptotically nonexpansive mappings on a $CAT(0)$ space. Our results extend and improve the corresponding recent results announced by many authors in the literature.

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1. Introduction

A metric space X is a $CAT(0)$ space if it is geodesically connected and every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces, $\mathbb{R}$-trees (see [2]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [14]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [2]. Burago, et al. [5] contains a somewhat more elementary treatment, and Gromov [15] a deeper study.

Fixed point theory in a $CAT(0)$ space has been first studied by Kirk (see [20, 21]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. Since then the fixed point theory for various mappings in a $CAT(0)$ space has been rapidly developed and a lot of papers have appeared (see e. g., [6–13, 16, 18, 22, 23, 25–27, 31, 39, 40]).

It is worth mentioning that the results in CAT(0) spaces can be applied to any $CAT(\kappa)$ space with $\kappa \leq 0$ since any $CAT(\kappa)$ space is a $CAT(\kappa')$ space for every $\kappa' \geq \kappa$ (see [2], p. 165).

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The Mann [29] iteration process is defined by the sequence $\{x_n\}$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

Further, the Ishikawa [17] iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in (0,1). This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

Agarwal, ORegan and Sahu [1] introduced the following S-iteration process which is independent of those of the Mann iteration (1.1) and the Ishikawa iteration (1.2),

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in (0,1).

Schu [37], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

Tan and Xu [41], in 1994, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in (0,1). This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

Sahin and Basarir [35] modified S-iteration process (1.3) in a $CAT(0)$ space as follows.

Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T : C \to C$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by $x_1 \in C$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where and throughout the paper $\{\alpha_n\}, \{\beta_n\}$ are the sequence such that $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 1$. They studied modified S-iteration process (1.6) for asymptotically quasi-nonexpansive mapping in the $CAT(0)$ space and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [19]. Also, (1.6) reduces (1.3) when $T^n = T$ for all $n \geq 1$.

In 1976, Lim [28] introduced the concept of Δ-convergence in a general metric space. In 2008 Kirk and Panyanak [23] specialized Lims concept to the $CAT(0)$ space and proved that it is very similar to the weak convergence in a Banach space. Also, Dhompongsa and Panyanak [11] obtained the Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a $CAT(0)$ space for nonexpansive mappings.

In 2010, Niwongsa and Panyanak [32] proved Δ and strong convergence theorems of the following Noor iteration for an asymptotically nonexpansive mapping in $CAT(0)$ spaces.

For a given $x_1 \in C$,

$$\begin{cases} z_n = \gamma_n T^n x_n \oplus (1-\gamma_n)x_n, \\ y_n = \beta_n T^n z_n \oplus (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n)x_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in [0,1] and T is an asymptotically nonexpansive mapping on a nonempty closed bounded and convex subset of a complete $CAT(0)$ space X.

Nanjaras and Panyanak [30] proved the demiclosedness principle for asymptotically nonexpansive mappings and gave the Δ-convergence theorem of the modified Mann iteration process for mappings of this type in a $CAT(0)$ space.

Our purpose in this paper is to get some results on the strong and Δ-convergence of the following new three-step iteration process for an asymptotically nonexpansive mapping in a $CAT(0)$ space. For a given $x_1 \in C$,

$$\begin{cases} z_n = \gamma_n T^n x_n \oplus (1-\gamma_n)x_n, \\ y_n = \beta_n T^n z_n \oplus (1-\beta_n)z_n, \\ x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n)T^n z_n, \quad n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in [0,1] and T is an asymptotically nonexpansive mapping on a nonempty closed bounded and convex subset of a complete $CAT(0)$ space X.

2. Preliminaries and Lemmas

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$,$c(l) = y$, and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$. Such a triangle always exists (see [2]).

A geodesic space is said to be a $CAT(0)$ space [2] if all geodesic triangles satisfy the following comparison axiom; see, e.g., [24, 33, 34, 36, 42–44] and the references therein.

$CAT(0)$: Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d^2(x, y_0) \leq \tfrac{1}{2}d^2(x, y_1) + \tfrac{1}{2}d^2(x, y_2) - \tfrac{1}{4}d^2(y_1, y_2). \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [4]. In fact (cf. [2], p. 163), a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality.

Recall that a mapping T on a metric space X is said to be nonexpansive if

$$d(T(x), T(y)) \leq d(x, y)$$

for all $x, y \in X$. T is called asymptotically nonexpansive if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ and such that

$$d(T^n(x), T^n(y)) \leq k_n d(x, y)$$

for all $n \geq 1$ and $x, y \in X$.

A point $x \in X$ is called a fixed point of T if $x = Tx$. We shall denote with $F(T)$ the set of fixed points of T. The existence of fixed points for asymptotically nonexpansive mappings in $CAT(0)$ spaces was proved by Kirk [21] as the following statement.

Theorem 2.1. *Let C be a nonempty bounded closed and convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be asymptotically nonexpansive. Then T has a fixed point.*

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$ The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point (see [10], Proposition 7).

We now give the definition of Δ-convergence.

Definition 2.2. ([23, 28]) A sequence $\{x_n\}$ in a metric space X is said to $\Delta-$*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the $\Delta-$*limit* of $\{x_n\}$.

Recall that a subset K in a metric space X is said to be $\Delta-$*compact* ([28]) if every sequence in K has a Δ-convergent subsequence. A mapping T from a metric space X to a metric space Y is said to be *completely continuous* if $T(K)$ is a compact subset of Y whenever K is a Δ-compact subset of X. We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.3. ([23]) *Every bounded sequence in a complete $CAT(0)$ space always has a $\Delta-$convergent subsequence.*

Lemma 2.4. ([9]) *If C is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.5. ([30]) *Let C be a closed and convex subset of a complete $CAT(0)$ space X and $T : C \to X$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $lim_n d(x_n, Tx_n) = 0$ and $\Delta - lim_n x_n = x$. Then $x = Tx$.*

Lemma 2.6. ([11]) *Let (X,d) be a $CAT(0)$ space.*
(i) For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y).$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.1).
(ii) For $x,y,z \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z).$$

(iii) For $x,y,z \in X$ and $t \in [0,1]$, we have

$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y).$$

Lemma 2.7. ([46]) *Let a_n and b_n be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+b_n)a_n, \ n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

It this section, we establish some Δ-convergence and strong convergence results of the new three-step iteration process (1.8) for an asymptotically nonexpansive mapping in a $CAT(0)$ space. We will denote the set of fixed points of T by $F(T)$, that is, $F(T) = \{x \in X : Tx = x\}$.

Lemma 3.1. *Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0,1]$. For a given $x_1 \in C$, the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by (1.8). Then $lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F(T)$.*

Proof. We first note that $F(T) \neq \emptyset$ by Theorem 2.1. Setting $k_n = 1 + u_n$ for all $n \ge 1$. Using $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we have $\sum_{n=1}^{\infty} u_n < \infty$. For each $x^* \in F(T)$, we have

$$\begin{aligned} d(z_n, x^*) &= d(\gamma_n T^n x_n \oplus (1-\gamma_n)x_n, x^*) \\ &\le \gamma_n d(T^n x_n, x^*) + (1-\gamma_n)d(x_n, x^*) \\ &\le \gamma_n (1+u_n) d(x_n, x^*) + (1-\gamma_n)d(x_n, x^*) \\ &= (1+\gamma_n u_n)d(x_n, x^*) \\ &\le (1+u_n)d(x_n, x^*). \end{aligned} \tag{3.1}$$

Also

$$\begin{aligned} d(y_n, x^*) &= d(\beta_n T^n z_n \oplus (1-\beta_n)z_n, x^*) \\ &\le \beta_n d(T^n z_n, x^*) + (1-\beta_n)d(z_n, x^*) \\ &\le \beta_n (1+u_n) d(z_n, x^*) + (1-\beta_n)d(z_n, x^*) \\ &= (1+\beta_n u_n)d(z_n, x^*) \\ &\le (1+u_n)d(z_n, x^*). \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned}
d(x_{n+1}, x^*) &= d(\alpha_n T^n y_n \oplus (1-\alpha_n)T^n z_n, x^*) \\
&\le \alpha_n d(T^n y_n, x^*) + (1-\alpha_n) d(T^n z_n, x^*) \\
&\le \alpha_n (1+u_n) d(y_n, x^*) + (1-\alpha_n)(1+u_n) d(z_n, x^*) \\
&\le \alpha_n (1+u_n)(1+u_n) d(z_n, x^*) + (1-\alpha_n)(1+u_n) d(z_n, x^*) \\
&= \alpha_n (1+u_n) d(z_n, x^*) + \alpha_n u_n (1+u_n) d(z_n, x^*) + (1+u_n) d(z_n, x^*) \\
&\quad - \alpha_n (1+u_n) d(z_n, x^*) \\
&= \alpha_n u_n (1+u_n) d(z_n, x^*) + (1+u_n) d(z_n, x^*) \\
&\le u_n (1+u_n) d(z_n, x^*) + (1+u_n) d(z_n, x^*) \\
&= (u_n + u_n^2 + 1 + u_n) d(z_n, x^*) \\
&\le (1 + 2u_n + u_n^2)(1+u_n) d(x_n, x^*) \\
&= (1 + u_n + 2u_n + 2u_n^2 + u_n^2 + u_n^3) d(x_n, x^*) \\
&= (1 + (3u_n + 3u_n^2 + u_n^3)) d(x_n, x^*).
\end{aligned}$$

Since $\sum_{n=1}^{\infty} u_n < \infty,$ *using Lemma* 2.7*, we have* $lim_{n\to\infty} d(x_n, x^*)$ *exists.* ∎

Lemma 3.2. *Let C be a nonempty closed, bounded and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $\{k_n\} \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying*
(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$ *and*
(ii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.
For a given $x_1 \in C$. Consider the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.8)*. Then* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Let $x^* \in F(T)$. Setting $k_n = 1 + u_n$ for all $n \ge 1$. From $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we have $\sum_{n=1}^{\infty} u_n < \infty$. Using Lemma 2.6 and (1.8), we have

$$\begin{aligned}
d^2(z_n, x^*) &= d^2(\gamma_n T^n x_n \oplus (1-\gamma_n)x_n, x^*) \\
&\le \gamma_n d^2(T^n x_n, x^*) + (1-\gamma_n) d^2(x_n, x^*) - \gamma_n(1-\gamma_n) d^2(T^n x_n, x_n) \\
&\le \gamma_n (1+u_n)^2 d^2(x_n, x^*) + (1-\gamma_n) d^2(x_n, x^*) - \gamma_n(1-\gamma_n) d^2(T^n x_n, x_n) \\
&= \gamma_n (1 + 2u_n + u_n^2) d^2(x_n, x^*) + (1-\gamma_n) d^2(x_n, x^*) \\
&\quad - \gamma_n(1-\gamma_n) d^2(T^n x_n, x_n) \\
&\le (1 + 2u_n + u_n^2) d^2(x_n, x^*) - \gamma_n(1-\gamma_n) d^2(T^n x_n, x_n) \\
&= (1+u_n)^2 d^2(x_n, x^*) - \gamma_n(1-\gamma_n) d^2(T^n x_n, x_n). \qquad (3.3)
\end{aligned}$$

Also

$$\begin{aligned}
d^2(y_n, x^*) &= d^2(\beta_n T^n z_n \oplus (1-\beta_n) z_n, x^*) \\
&\le \beta_n d^2(T^n z_n, x^*) + (1-\beta_n) d^2(z_n, x^*) - \beta_n(1-\beta_n) d^2(T^n z_n, z_n) \\
&\le \beta_n (1+u_n)^2 d^2(z_n, x^*) + (1-\beta_n) d^2(z_n, x^*) - \beta_n(1-\beta_n) d^2(T^n z_n, z_n)
\end{aligned}$$

$$
\begin{aligned}
&= \beta_n(1+2u_n+u_n^2)d^2(z_n,x^*) + (1-\beta_n)d^2(z_n,x^*) \\
&\quad - \beta_n(1-\beta_n)d^2(T^n z_n, z_n) \\
&\leq (1+2u_n+u_n^2)d^2(z_n,x^*) \\
&= (1+u_n)^2 d^2(z_n,x^*).
\end{aligned} \tag{3.4}
$$

Using (3.3) and (3.4), we have

$$
\begin{aligned}
d^2(x_{n+1},x^*) &= d^2(\alpha_n T^n y_n \oplus (1-\alpha_n)T^n z_n, x^*) \\
&\leq \alpha_n d^2(T^n y_n, x^*) + (1-\alpha_n)d^2(T^n z_n, x^*) \\
&\quad - \alpha_n(1-\alpha_n)d^2(T^n y_n, T^n z_n) \\
&\leq \alpha_n(1+u_n)^2 d^2(y_n,x^*) + (1-\alpha_n)(1+u_n)^2 d^2(z_n,x^*) \\
&\quad - \alpha_n(1-\alpha_n)d^2(T^n y_n, T^n z_n) \\
&\leq \alpha_n(1+u_n)^2\left((1+u_n)^2 d^2(z_n,x^*)\right) + (1-\alpha_n)(1+u_n)^2 d^2(z_n,x^*) \\
&\quad - \alpha_n(1-\alpha_n)d^2(T^n y_n, T^n z_n) \\
&\leq \alpha_n(1+u_n)^4 d^2(z_n,x^*) + (1-\alpha_n)(1+u_n)^4 d^2(z_n,x^*) \\
&\quad - \alpha_n(1-\alpha_n)d^2(T^n y_n, T^n z_n) \\
&= (1+u_n)^4 d^2(z_n,x^*) - \alpha_n(1-\alpha_n)d^2(T^n y_n, T^n z_n) \\
&\leq (1+u_n)^4\left((1+u_n)^2 d^2(x_n,x^*) - \gamma_n(1-\gamma_n)d^2(T^n x_n, x_n)\right) \\
&\quad - \alpha_n(1-\alpha_n)d^2(T^n y_n, T^n z_n) \\
&\leq (1+u_n)^6 d^2(x_n,x^*) - \gamma_n(1-\gamma_n)d^2(T^n x_n, x_n) \\
&\quad - \alpha_n(1-\alpha_n)d^2(T^n y_n, T^n z_n).
\end{aligned} \tag{3.5}
$$

Since C is bounded, there exists $B_M[x_0] = \{x \in X : d(x,x_0) \leq M\}$ such that $C \subset B_M[x_0]$ for some $M > 0$. By using (3.5), we obtain the following two important inequalities:

$$
\begin{aligned}
\alpha_n(1-\alpha_n)d^2(T^n y_n, T^n z_n) &\leq d^2(x_n,x^*) - d^2(x_{n+1},x^*) \\
&\quad + M^2(6u_n + 6u_n^2 + 6u_n^3 + 6u_n^4 + 6u_n^5 + u_n^6),
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\gamma_n(1-\gamma_n)d^2(T^n x_n, x_n) &\leq d^2(x_n,x^*) - d^2(x_{n+1},x^*) \\
&\quad + M^2(6u_n + 6u_n^2 + 6u_n^3 + 6u_n^4 + 6u_n^5 + u_n^6).
\end{aligned} \tag{3.7}
$$

By (i), (3.6) and $\sum_{n=1}^{\infty} u_n < \infty$ along with the proof of Lemma 2.2 in [45] with $p = 2$ and $\omega(\lambda) = \lambda(1-\lambda)$ for $\lambda \in [0,1]$, we can obtain

$$
\lim_{n\to\infty} d(T^n y_n, T^n z_n) = 0. \tag{3.8}
$$

Similarly, using (ii), (3.7) and $\sum_{n=1}^{\infty} u_n < \infty$, we may show that

$$
\lim_{n\to\infty} d(T^n x_n, x_n) = 0. \tag{3.9}
$$

In addition using (1.8) and (3.9), we have

$$\begin{aligned} d(z_n, x_n) &= d(\gamma_n T^n x_n \oplus (1-\gamma_n)x_n, x_n) \\ &\le \gamma_n d(T^n x_n, x_n) + (1-\gamma_n)d(x_n, x_n) \\ &\le d(T^n x_n, x_n) \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.10}$$

Using (3.9), (3.10) and $\sum_{n=1}^{\infty} u_n < \infty$, we have

$$\begin{aligned} d(T^n z_n, x_n) &\le d(T^n z_n, T^n x_n) + d(T^n x_n, x_n) \\ &\le (1+u_n)d(z_n, x_n) + d(T^n x_n, x_n) \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.11}$$

From (3.8) and (3.11), we have

$$\begin{aligned} d(T^n y_n, x_n) &\le d(T^n y_n, T^n z_n) + d(T^n z_n, x_n) \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.12}$$

Using (1.8), (3.9), (3.11) and (3.12), we have

$$\begin{aligned} d(x_{n+1}, T^n x_{n+1}) &\le d(x_{n+1}, x_n) + d(T^n x_{n+1}, T^n x_n) + d(T^n x_n, x_n) \\ &\le d(x_{n+1}, x_n) + k_n d(x_{n+1}, x_n) + d(T^n x_n, x_n) \\ &= (1+k_n)d(x_{n+1}, x_n) + d(T^n x_n, x_n) \\ &= (1+k_n)d(\alpha_n T^n y_n \oplus (1-\alpha_n)T^n z_n, x_n) + d(T^n x_n, x_n) \\ &\le (1+k_n)(\alpha_n d(T^n y_n, x_n) + (1-\alpha_n)d(T^n z_n, x_n)) + d(T^n x_n, x_n) \\ &\le (1+k_n)d(T^n y_n, x_n) + (1+k_n)d(T^n z_n, x_n) + d(T^n x_n, x_n) \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.13}$$

By (3.9) and (3.13), we have

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &\le d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, Tx_{n+1}) \\ &\le d(x_{n+1}, T^{n+1}x_{n+1}) + k_1 d(T^n x_{n+1}, x_{n+1}) \\ &\to 0 \ (as\ n \to \infty), \end{aligned}$$

which implies $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ as desired. ∎

Now, we are ready to prove the Δ-convergence theorem.

Theorem 3.3. *Let C be a nonempty closed, bounded and convex subset of a complete $CAT(0)$ space X and let $T: C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0,1]$ satisfying*
(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$ and
(ii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.
For a given $x_1 \in C$, consider the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.8). Then $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. It follows from Lemma 3.2 that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Now we let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset F(T)$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.3 and 2.4 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, then $v \in F(T)$ by Lemma 2.5. We claim that $u = v$. Suppose not, since $v \in F(T)$, by Lemma 3.1 $\lim_{n\to\infty} d(x_n, v)$ exists. By the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_n d(v_n, v) &< \limsup_n d(v_n, u)\\ &\leq \limsup_n d(u_n, u)\\ &< \limsup_n d(u_n, v)\\ &= \limsup_n d(x_n, v)\\ &= \limsup_n d(v_n, v)\end{aligned}$$

a contradiction, and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 2.3 and 2.4 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in F(T)$. We can complete the proof by showing that $x = v$. Suppose not, since $\limsup_n d(x_n, v)$ exists, then by the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_n d(v_n, v) &< \limsup_n d(v_n, x)\\ &\leq \limsup_n d(x_n, x)\\ &< \limsup_n d(x_n, v)\\ &= \limsup_n d(v_n, v)\end{aligned}$$

a contradiction, and hence the conclusion follows. ∎

By using to the above same ideas and techniques, we can also obtain a strong convergence theorem for completely continuous asymptotically nonexpansive mappings. Therefore we can state the following result without proofs.

Theorem 3.4. *Let C be a nonempty closed, bounded and convex subset of a complete $CAT(0)$ space X and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying*
(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.
For a given $x_1 \in C$, consider the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.8). *Then $\{x_n\}$ converges strongly to a fixed point of T.*

Now, we give an example of an asymptotically nonexpansive mapping as in Theorem 3.3.

Let $\mathbb{R}$ be the real line with the usual norm $|.|$ and let $C = [-1, 1]$. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0, 1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0). \end{cases}$$

Clearly, $F(T) = \{0\}$. Now, we show that T is an asymptotically nonexpansive mapping. In fact, if $x, y \in [0, 1]$ or $x, y \in [-1, 0)$, then

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| \\ &= 2|\sin\frac{x}{2} - \sin\frac{y}{2}| \\ &\le |x - y| \\ &= d(x, y). \end{aligned}$$

If $x \in [0, 1]$ and $y \in [-1, 0)$ or $x \in [-1, 0)$ and $y \in [0, 1]$, then

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| \\ &= 2|\sin\frac{x}{2} + \sin\frac{y}{2}| \\ &= 4|\sin\frac{x+y}{4}\cos\frac{x-y}{4}| \\ &\le |x + y| \\ &\le |x - y| \\ &= d(x, y). \end{aligned}$$

That is, T is nonexpansive. It follows that T is an asymptotically nonexpansive mapping with $k_n = 1$ for each $n \ge 1$. Additionally, let

$$\alpha_n = \frac{n}{2n+1}, \ \beta_n = \frac{n}{3n+1}, \ \gamma_n = \frac{n}{4n+1}, \ \forall n \ge 1.$$

Therefore, the conditions of Theorem 3.3 are fulfilled. So, the convergence of the sequence $\{x_n\}$ generated by (1.8) to a point $0 \in F(T)$ can be received. For convenience, we call the iteration (1.8) as the proposed iteration process. For supporting our main theorem, the proposed iteration process is demonstrated by chosen $x_1 = 1$ and run our process within 250 iterations through an example of an asymptotically nonexpansive mapping T.

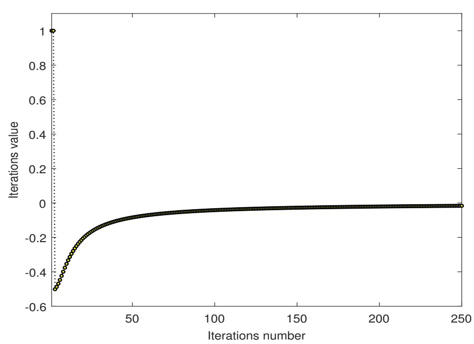

FIGURE 1. Numerical solution of an asymptotically nonexpansive mapping T by using the proposed iteration process.

Figure 1 shows the numerical solution of the proposed method. It can be seen that the sequence generated by these proposed method converge to the solution of an example.

In the remainder of this section, we give the characterization of strong convergence for the new three-step iteration process on a $CAT(0)$ space as follows.

Theorem 3.5. *Let $X, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ satisfy the hypotheses of Theorem 3.3. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf\{d(x, x^*) : x^* \in F(T)\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 3.1, for all $x^* \in F(T)$,

$$d(x_{n+1}, x^*) \leq (1 + (3u_n + 3u_n^2 + u_n^3))d(x_n, x^*).$$

This implies that

$$d(x_{n+1}, F(T)) \leq (1 + (3u_n + 3u_n^2 + u_n^3))d(x_n, F(T)).$$

By Lemma 2.7, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a positive integer n_0 such that

$$d(x_n, F(T)) < \frac{\epsilon}{4}$$

for all $n \geq n_0$. In particular, $\inf\{d(x_{n_0}, x^*) : x^* \in F(T)\} < \frac{\epsilon}{4}$. Thus, there exists $p \in F(T)$ such that

$$d(x_{n_0}, p) < \frac{\epsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p) + d(x_n, p) \leq 2d(x_{n_0}, p) \leq 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in the closed subset C of a complete $CAT(0)$ space and so it must be convergent to a point q in C. Now, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$ and closedness of $F(T)$ forces q to be in $F(T)$. This completes the proof. ■

Senter and Dotson [38] introduced the concept of *Condition* (I) as follows.

Definition 3.6. ([38], p.375) A mapping $T : C \to C$ is said to satisfy *Condition* (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$d(x, Tx) \geq f(d(x, F(T))), \forall x \in C.$$

With respect to the above definition, we have the following strong convergence theorem.

Theorem 3.7. *Let $X, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ satisfy the hypotheses of Theorem 3.3 and let T be a mapping satisfying Condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. As proved in Theorem 3.5, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Lemma 3.2, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from *Condition* (I) that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is, $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, we obtain

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

The conclusion now follows from Theorem 3.5. ∎

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