Thai Journal of Mathematics
Volume 19 Number 3 (2021)
Pages 793–803

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

A modification of Extragradient method for solving fixed point, variational inequality, and equilibrium problems without the monotonicity

Thidaporn Seangwattana[1,*]

[1] *Faculty of Science Energy and Environment, King Mongkut's University of Technology North Bangkok, Rayong Campus (KMUTNB), Rayong 21120, Thailand.*
e-mail : thidaporn.s@sciee.kmutnb.ac.th (T. Seangwattana)

Abstract The propose of this work is to modify an Extragradient method (in [2]) for finding a common solution of fixed point, variational inequality, and equilibrium problems without the monotone assumption of a bifunction in a Hilbert space. A weak convergence theorem is presented by the proposed method. When reducing some mappings in the method, it can find solutions of various problems without the monotonicity of a bifunction.

MSC: 46M01

Keywords: Extragradient method; Nonmonotone; Armijo linesearch; Variational Inequality; Fixed point problems; Equilibrium problems

Submission date: 16.04.2021 / Acceptance date: 07.07.2021

1. Introduction

Suppose that $\mathbb{H}$ is a real Hilbert space, $C \subset \mathbb{H}$ is a nonempty closed convex set, and Ω is an open convex subset in $\mathbb{H}$ containing C. In 1992, Blum, Muu and Oettli [14, 15] presented the equilibrium problem (shortly, $EP(C,F)$) that is to find $x \in C$ such that

$$F(x,y) \geq 0, \forall\, y \in C, \tag{1.1}$$

where $F : \Omega \times \Omega \to \mathbb{R}$ is a bifunction satisfying $F(x,x) = 0$ for every $x \in C$. The Minty equilibrium problem (shortly, $MEP(C,F)$) is to find $u \in C$ such that

$$F(y,u) \geq 0, \forall\, y \in C. \tag{1.2}$$

We denote the solution set of $EP(C,F)$ and $MEP(C,F)$ by S_{EP} and S_{MEP}, respectively. Furthermore, there is a simple formulation of $EP(C,F)$ which can apply in applied mathematics: variational inequality problem, fixed point problem, saddle point problem, and

*Corresponding author.

others (for example [14, 15, 18, 19]). The classical variational inequality problem (shortly, $VI(C,A)$) is to find $\hat{x} \in C$ such that

$$\langle A\hat{x}, v - \hat{x}\rangle \geq 0, \forall v \in C, \tag{1.3}$$

where a mapping $A : C \to \mathbb{H}$. Later, the solution method of equilibrium problems have been usually extended from those for variational inequality problem (for more detail see in [3, 29–32]) because if $F(x,y) = \langle Ax, y - x\rangle$ where a mapping $A : C \to \mathbb{H}$ for every $x, y \in C$ then $z \in EP(C,F)$ if and only if $\langle Az, y - z\rangle \geq 0$ for every $y \in C$, i.e., z is a solution of the variational inequality problem. There are a lot of iterative processes for finding a solution of the variational inequality problem [4, 17, 20–29]. Among them, the Extragradient method which was introduced by Korpelevich [17], that is,

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda A x_n) \\ x_{n+1} = P_C(x_n - \lambda A\bar{x}_n), \end{cases}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{k})$, and A is a monotone and $k-$Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C,A) \neq \emptyset$, then the sequences which generated by this method converge to the same point $z \in VI(C,A)$. After, authors extended this method to equilibrium problem [33, 34] that is an important method. However, it always requires the condition $S_{EP} \subset S_{MEP}$. This condition is guaranteed under the pseudomonotonicity assumption of bifunction F on C, that is, if $x, y \in C$, $F(x,y) \geq 0$, then $F(y,x) \leq 0$. Thus, if S_{EP} is not contained in S_{MEP}, then the existing extragradient method cannot be applied for $EP(C,F)$ directly. In 2015, Ye and He [8] suggested a new method which is called a double projection method under the only assumption that the solution set of dual variational inequality is nonempty. They also proved that their method can solve the solution set of variational inequality problem without the monotonicity of A, and gave some numerical experiments. Recently, motivated by the idea of a double projection method [8], Dinh and Kim [1] proposed projection algorithms for solving equilibrium problems where the bifunction is not required to be satified any monotone property. Under assumptions on the continuity, convexity of the bifunction and the nonemptyness of the solution set of Minty equilibrium problem. They also proved a weak convergence theorem which is generated by their proposed algorithms.

Another interseting problem is the problem for finding a common element of the set of fixed point of a nonexpansive mapping and the solution set of the variational inequality problem for an inverse strongly-monotone mapping which was presented by Takahashi and Toyoda [16]. A mapping S of C into itself is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.4}$$

The fixed point problem is to find a point $x \in C$ such that $x = Tx$ where a mapping $T : C \to C$. We denote by $F(S)$ the set of fixed point of S. By the way, their process obtained a weak convergence theorem for two sequences. In 2006, Nadezhkina and Takahashi [7] presented the iterative algorithm for finding the common solution of $F(S) \cap VI(C,A)$, that is,

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n) \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda A y_n), \ \forall n \geq 0. \end{cases}$$

Furthermore, they proved that the sequence $\{x_n\}$ and $\{y_n\}$, generated by their method, converge weakly to $z \in F(S) \cap VI(C,A)$. Thereby, Jaiboon, Kumam, and Humphries [2] introduced a new iterative algorithm for finding the common solution of fixed point, equilibrium, and variational inequality problems and proved the following weak convergence theorem.

Assumptions A bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(C1) $F(x, x) = 0$ for every $x \in C$;

(C2) F is monotone, i.e, $F(x, y) + F(y, x) \leq 0$ for every $x, y \in C$;

(C3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(C4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Theorem 1.1. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (C1)-(C4) and let A be a monotone $k-$Lipschit continous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \cap EP(C, F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$, and $\{u_n\}$ are given by*

$$\begin{cases} u_n \in C; F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \\ y_n = P_C(x_n - \lambda A u_n) \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda A y_n), \forall n \geq 0, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (a, b) \subset (0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, \frac{1}{k})$ and $\{r_n\} \subset (0, \infty)$ satisfying the following conditions:

(i) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} \|r_{n+1} - r_n\| < \infty$, *and*

(ii) $\lim_{n\to\infty}\lambda_n = 0$, *then $\{x_n\}, \{y_n\}$, and $\{u_n\}$ converge weakly to the same point $p \in F(S) \cap VI(C, A) \cap EP(C, F)$, where $p = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)\cap EP(C,F)}x_n$.*

Motivated by [1], [2] and [3], we introduce a modification of Extragradient method for solving fixed point, variational inequality, and equilibrium problems without the monotonicity of the bifunction F. Our method is a combination between the projection algorithm [1] for solving nonmonotone equilibrium problems in Hilbert space and the Extragradient method [2] for solving the common solution of fixed point, variational inequality, and equilibrium problems. Moreover, we prove a weak convergence theorem which is generated by this method.

The paper is organized as follows. Section 2 contains some preliminaries on the metric projection and equilibrium problems. Section 3 introduces a modification of extragradient method for solving the common solution of $F(S) \cap VI(C, A) \cap EP(C, F)$. When reducing some mappings in the method, it can find solutions of various problems without the monotonicity.

2. Preliminaries

In this section, we contain definitions and useful lemmas for using in the next section. A unique nearest point in C, denoted by $P_C(x)$, is for every $x \in \mathbb{H}$ such that

$$\|x - P_C(x)\| \leq \|x - y\|, \ \forall\, y \in C.$$

P_C is called the metric projection of $\mathbb{H}$ onto C.

Lemma 2.1. [12] *For any $x \in \mathbb{H}$ and $z \in C$,*

(A) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|P_C(x) - x\|^2$;

(B) $\langle P_C(x) - x, z - P_C(x)\rangle \geq 0$.

Lemma 2.2. [7] *Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for every $n = 0, 1, 2, \ldots,$ and $\{v_n\}, \{w_n\}$ sequences in $\mathbb{H}$ such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c, \quad \limsup_{n\to\infty} \|w_n\| \leq c, \text{ and } \lim_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n) w_n\| = c,$$

for some $c \geq 0$. Then $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 2.3. [7] *Let $\{x_n\}$ be a sequence in H. Suppose that for each $u \in C$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|$$

for every $n = 0, 1, 2, \ldots$. Then, the sequence $\{P_C x_n\}$ converges strongly to some $z \in C$.

Lemma 2.4. [13] *Let $\mathbb{H}$ be a Hilbert space, C a closed convex subset of $\mathbb{H}$, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - T)x_n\}$ converges strong to y, then $(I - T)x = y$.*

Opial Condition [2] For every $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for any $y \in \mathbb{H}$ with $y \neq x$.

Definition 2.5. A bifunction $\phi : C \times C \to \mathbb{R}$ is said to be jointly weakly continuous on $C \times C$ if for all $x, y \in C$ and $\{x_n\}, \{y_n\}$ are two sequences in C converging weakly to x and y respectively, then $\phi(x_n, y_n)$ converges to $\phi(x, y)$. In the sequel, we need the following assumptions:

$(A1)$ $F(x, \cdot)$ is convex on Ω for every $x \in C$;
$(A2)$ F is joint weakly continuous on $\Omega \times \Omega$.

For each $z, x \in C$, by $\partial_2 F(z, x)$ we denote the subdifferential of the convex function $F(z, \cdot)$ at x, i.e.,

$$\partial_2 F(z, x) := \{w \in \mathbb{H} : F(z, y) \geq F(z, x) + \langle w, y - x\rangle, \forall y \in C\}.$$

In paticular,

$$\partial_2 F(z, z) = \{w \in H : F(z, y) \geq \langle w, y - z\rangle, \forall y \in C\}.$$

Lemma 2.6. [1] *Suppose the bifunction F satisfies the assumptions $(A1), (A2)$. If $\{x_n\} \subset C$ is bounded, $\rho > 0$, and $\{u_n\}$ is a sequence such that*

$$u_n = \arg\min\{F(x_n, y) + \frac{\rho}{2}\|y - x_n\|^2 : y \in C\},$$

then $\{u_n\}$ is bounded.

3. Main results

This section, we propose two algorithms for finding the common solution. Firstly, Algorithm 1 for solving fixed point, variational inequality, and equilibrium problems without the monotonicity of the bifunction F is presented.

Algorithm 1 Pick $x_0 \in C$ and choose $\eta \in (0, 1), \rho > 0, \{\alpha_n\} \subset (0, 1)$,
$\{\lambda_n\} \subset (0, 1/k)$ and $C_0 = C$.

Step 1. Compute

$$y_n = P_C(u_n - \lambda_n A u_n), \tag{3.1}$$

where $u_n = \arg\min\{F(x_n, y) + \frac{\rho}{2}\|y - x_n\|^2, y \in C\}$.
If $y_n = x_n$, then stop. Otherwise, do Step 2.
Step 2. (Armijo linesearch rule) Find m_n as the smallest positive integer number m satisfying

$$\begin{cases} z^{n,m} = (1-\eta^m)x_n + \eta^m u_n \\ w^{n,m} \in \partial_2 F(z^{n,m}, z^{n,m}) \\ \langle w^{n,m}, x_n - u_n\rangle \geq \frac{\rho}{2}\|u_n - x_n\|^2. \end{cases} \tag{3.2}$$

Step 3. Set $\eta_n = \eta^{m_n}, z_n = z^{n,m_n}, w_n = w^{n,m_n}$. Take

$$H_n = \{x \in \mathbb{H} : \langle w_n, x - z_n\rangle \leq 0\}, C_{n+1} = C_n \cap H_n. \tag{3.3}$$

Step 4. Compute

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_{C_{n+1}}(x_n - \lambda_n Ay_n), \tag{3.4}$$

and go to Step 1 with n is replaced by $n+1$.

Lemma 3.1. *If $S_{MEP} \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 1 is well defined in the sense that there exists $m > 0$ satisfying the inequality in (3.2) for every $w^{n,m} \in \partial_2 F(z^{n,m}, z^{n,m}), C_n$ is nonempty closed convex, and*

$$\langle w^n, x_n - z_n\rangle \geq \frac{\eta_n \rho}{2}\|x_n - u_n\|^2 \tag{3.5}$$

for each iteration n.

Proof. The proof is similar to Lemma 3.1 in [1] ■

Theorem 3.2. *Let $F : C \times C \to \mathbb{R}$ be a function satisfying (A1),(A2) and $A : C \to \mathbb{H}$ be a monotone k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping. If $F(S) \cap VI(A, C) \cap EP(C, F) \neq \emptyset, S_{MEP} \neq \emptyset$ and $\lim_{n\to\infty} \lambda_n = 0$ then $\{x_n\}, \{y_n\}$ and $\{u_n\}$ which generated by Algorithm 1 converges weakly to the same point $u = \lim_{n\to\infty} P_{F(S)\cap VI(A,C)\cap EP(F)} x_n$.*

Proof. Let $b_n = P_{C_{n+1}}(x_n - \lambda_n Ay_n)$ for every $n = 0, 1, 2, \ldots$. Let $u \in F(S) \cap VI(C, A) \cap EP(F)$. From Lemma 2.1 (A), we have

$$\begin{aligned} \|b_n - u\|^2 &\leq \|x_n - \lambda_n Ay_n - u\|^2 - \|x_n - \lambda_n Ay_n - b_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - b_n\|^2 + 2\lambda_n\langle Ay_n, u - b_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - b_n\|^2 + 2\lambda_n(\langle Ay_n - Au, u - y_n\rangle \\ &\quad + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - b_n\rangle) \\ &\leq \|x_n - u\|^2 - \|x_n - b_n\|^2 + 2\lambda_n\langle Ay_n, y_n - b_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - b_n\rangle - \|y_n - b_n\|^2 \\ &\quad + 2\lambda_n\langle Ay_n, y_n - b_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - b_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, b_n - y_n\rangle. \end{aligned}$$

Thank to Lemma 2.1 (B), we receive

$$\begin{aligned} \langle x_n - \lambda_n Ay_n - y_n, b_n - y_n\rangle &= \langle x_n - \lambda_n Ax_n - y_n, b_n - y_n\rangle \\ &\quad + \langle \lambda_n Ax_n - \lambda_n Ay_n, b_n - y_n\rangle \\ &\leq \langle \lambda_n Ax_n - \lambda_n Ay_n, b_n - y_n\rangle \\ &\leq \lambda_n k\|x_n - y_n\|\,\|b_n - y_n\|. \end{aligned}$$

This implies that

$$\begin{aligned}
\|b_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - b_n\|^2 \\
&\quad +2\lambda_n k\|x_n - y_n\| \, \|b_n - y_n\| \\
&\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - b_n\|^2 \\
&\quad +\lambda_n^2 k^2\|x_n - y_n\|^2 + \|y_n - b_n\|^2 \\
&\leq \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\| \\
&\leq \|x_n - u\|^2.
\end{aligned}$$

We also have

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sb_n - u\|^2 \\
&= \|\alpha_n(x_n - u) + (1-\alpha_n)(Sb_n - u)\|^2 \\
&\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\|Sb_n - u\|^2 \\
&\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|b_n - u\|^2 \\
&\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)(\|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2) \\
&= \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\
&\leq \|x_n - u\|^2.
\end{aligned}$$

Since the sequence $\{\|x_n - u\|\}$ is a bounded and nonincreasing sequence, $\lim_{n\to\infty}\|x_n - u\|$ exists, that is there exists

$$c = \lim_{n\to\infty}\|x_n - u\|. \tag{3.6}$$

Moreover, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Thus the sequence $\{x_n\}$, $\{b_n\}$ are bounded. According to Lemma 2.6, it obtians that $\{u_n\}, \{z_n\}, \{w_n\}$ are also bounded. Furthermore, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.7}$$

Note that

$$\begin{aligned}
\|y_n - b_n\| &= \|P_C(u_n - \lambda_n Au_n) - P_C(x_n - \lambda_n Ay_n)\| \\
&\leq \|(u_n - \lambda_n Au_n) - (x_n - \lambda_n Ay_n)\| \\
&\leq \|u_n - x_n\| + \lambda_n\|Au_n - Ay_n\| \\
&\leq \|u_n - x_n\| + k\lambda_n\|u_n - y_n\|.
\end{aligned}$$

Since $\lambda_n \to 0$ as $n \to \infty$, we get

$$\|y_n - b_n\| \leq \|u_n - x_n\|. \tag{3.8}$$

Therefore $\lim_{n\to\infty}\|y_n - b_n\| = 0$ and $\lim_{n\to\infty}\|x_n - b_n\| = 0$.

Now, we are going to prove that $\{x_n\}$ converges weakly to some point u. Assume that $\bar{u}$ and u are two weak accumulation points of $\{x_n\}$. There exist $\{x_{n_i}\} \subset \{x_n\}$ and $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{u}$ and $x_{n_j} \rightharpoonup u$. By (3.6), it yields $\lim_{n\to\infty}\|x_n - \bar{u}\|^2 = \gamma$

and $\lim_{n\to\infty}\|x_n-u\|^2=\kappa$. We can see that

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\|x_n-\bar{u}\|^2=\lim_{j\to\infty}\|x_{n_j}-\bar{u}\|^2\\ &= \lim_{j\to\infty}(\|x_{n_j}-u\|^2+2\langle x_{n_j}-u,u-\bar{u}\rangle+\|u-\bar{u}\|^2)\\ &= \lim_{j\to\infty}(\|x_{n_j}-u\|^2+\|u-\bar{u}\|^2)\\ &= \lim_{n\to\infty}(\|x_n-u\|^2+\|u-\bar{u}\|^2)\\ &= \lim_{n\to\infty}(\|x_n-\bar{u}\|^2+2\|u-\bar{u}\|^2)\\ &= \gamma+2\|u-\bar{u}\|^2. \qquad (3.9)\end{aligned}$$

Obviously, $\|u-\bar{u}\|=0$. This means that $\{x_n\}$ converges weakly to u.

After that we will show that $u\in EP(C,F)$. We know that $\{w_n\}$ is bounded. There exists $L>0$ such that $\|w_n\|\leq L$ for all k. Combining with Lemma 3.1 (3.5), we have

$$\|x_{n+1}-x_n\|=d(x_n,C_n)\geq d(x_n,H_k)=\frac{|\langle w_n,x_n-z_n\rangle|}{\|w_n\|}\geq\frac{\eta_n\rho}{2L}\|x_n-u_n\|^2. \quad (3.10)$$

Thus

$$\lim_{n\to\infty}\eta_n\|x_n-u_n\|^2=0. \quad (3.11)$$

We will consider two cases.

In the case of $\limsup_{n\to\infty}\eta_n>0$. There exists $\eta'>0$ and a subsequence $\{\eta_{n_i}\}\subset\{\eta_n\}$ such that $\eta_{n_i}>\eta'$. From (3.11), we have

$$\lim_{i\to\infty}\|x_{n_i}-u_{n_i}\|=0. \quad (3.12)$$

Since $x_n\rightharpoonup u$ and (3.12), we get that $u_{n_i}\rightharpoonup u$ as $i\to\infty$. By the definition of u_{n_i}, we obtain $0\in\partial_2F(x_{n_i},u_{n_i})+\rho(u_{n_i}-x_{n_i})+N_C(u_{n_i})$. There exists $v_{n_i}\in\partial_2F(x_{n_i},u_{n_i})$ such that

$$\langle v_{n_i},y-u_{n_i}\rangle+\rho\langle u_{n_i}-x_{n_i},y-u_{n_i}\rangle\geq 0,\forall y\in C. \quad (3.13)$$

We combine (3.13) with

$$F(x_{n_i},y)-F(x_{n_i},u_{n_i})\geq\langle v_{n_i},y-u_{n_i}\rangle,\forall y\in C.$$

This implies that

$$F(x_{n_i},y)-F(x_{n_i},u_{n_i})+\rho\langle u_{n_i}-x_{n_i},y-u_{n_i}\rangle\geq 0,\forall y\in C. \quad (3.14)$$

In addition, $\langle u_{n_i}-x_{n_i},y-u_{n_i}\rangle\leq\|u_{n_i}-x_{n_i}\|\|y-u_{n_i}\|$, we receive

$$F(x_{n_i},y)-F(x_{n_i},u_{n_i})+\rho\|u_{n_i}-x_{n_i}\|\|y-u_{n_i}\|\geq 0. \quad (3.15)$$

Taking $i\to\infty$, we have $F(u,y)-F(u,u)\geq 0$. So $F(u,y)\geq 0$ for all $y\in C$ which means that $u\in EP(C,F)$.

In the case of $\lim_{n\to\infty}\eta_n=0$. By the boundedness of $\{u_n\}$, there exists $\{u_{n_i}\}\subset\{u_n\}$ such that $u_{n_i}\rightharpoonup u^*$ as $i\to\infty$. We replace y by x_{n_i} in (3.14). So

$$F(x_{n_i},u_{n_i})+\rho\|u_{n_i}-x_{n_i}\|^2\leq 0. \quad (3.16)$$

Whereas, by the Armijo linesearch rule, for $m_{n_i}-1$,
there is $w^{n_i,m_{n_i}-1}\in\partial_2F(z^{n_i,m_{n_i}-1},z^{n_i,m_{n_i}-1})$ such that

$$\langle w^{n_i,m_{n_i}-1},x_{n_i}-u_{n_i}\rangle<\frac{\rho}{2}\|u_{n_i}-x_{n_i}\|^2. \quad (3.17)$$

By the convexity of $F(z^{n_i,m_{n_i}-1},\cdot)$, we see that

$$\begin{aligned} F(z^{n_i,m_{n_i}-1},u_{n_i}) &\geq F(z^{n_i,m_{n_i}-1},z^{n_i,m_{n_i}-1})+\langle w^{n_i,m_{n_i}-1},u_{n_i}-z^{n_i,m_{n_i}-1}\rangle \\ &= (1-\eta^{n_i,m_{n_i}-1})\langle w^{n_i,m_{n_i}-1},u_{n_i}-x_{n_i}\rangle \\ &\geq -(1-\eta^{n_i,m_{n_i}-1})\frac{\rho}{2}\|u_{n_i}-x_{n_i}\|. \end{aligned}$$

By (3.16) and (3.17), we obtain

$$F(z^{n_i,m_{n_i}-1},u_{n_i}) \geq -(1-\eta^{n_i,m_{n_i}-1})\frac{\rho}{2}\|u_{n_i}-x_{n_i}\|^2 \geq \frac{1}{2}(1-\eta^{n_i,m_{n_i}-1})F(x_{n_i},u_{n_i}).$$

Since $z^{n_i,m_{n_i}-1}=(1-\eta^{m_{n_i}-1})x_{n_i}+\eta^{m_{n_i}-1}u_{n_i}$, $\eta^{n_i,m_{n_i}-1}\to 0, x_{n_i}\rightharpoonup u$, and $u_{n_i}\rightharpoonup u^*$, it can imply that $z^{n_i,m_{n_i}-1}\rightharpoonup u$ as $i\to\infty$. Without loss of generallity, we may suppose that $\lim_{i\to\infty}\|u_{n_i}-x_{n_i}\|^2=0$ because $\{\|u_{n_i}-x_{n_i}\|^2\}$ is bounded. Therefore

$$F(u,x^*) \geq -\frac{\rho}{2}\lim_{i\to\infty}\|u_{n_i}-x_{n_i}\|^2 \geq \frac{1}{2}F(u,u^*).$$

That is $F(u,u^*)=0$ and $\lim_{i\to\infty}\|u_{n_i}-x_{n_i}\|^2=0$. From the Case 1, u is a solution of $EP(C,F)$. Now, we want to show that $u\in VI(C,A)$ but the same argument as in the proof of Theorem 3.1 in [7]. We can prove that $u\in VI(C,A)$.

We will show that $u\in F(S)$. Since $\|Sb_n-u\|\leq\|b_n-u\|\leq\|x_n-u\|$ and (3.6), we have $\limsup_{n\to\infty}\|Sb_n-u\|\leq c$. Consider

$$\lim_{n\to\infty}\|\alpha_n(x_n-u)+(1-\alpha_n)(Sb_n-u)\| = \lim_{n\to\infty}\|x_{n+1}-u\| = c.$$

From Lemma 2.2, it can imply that $\lim_{n\to\infty}\|Sb_n-x_n\|=0$. Moreover, we have

$$\begin{aligned} \|Sx_n-x_n\| &\leq \|Sx_n-Sb_n\|+\|Sb_n-x_n\| \\ &\leq \|x_n-b_n\|+\|Sb_n-x_n\|. \end{aligned}$$

Thus $\lim_{n\to\infty}\|Sx_n-x_n\|=0$. By using the demiclosedness of $I-S$, we obtain that $x_{n_i}\rightharpoonup u$ and $\lim_{n\to\infty}\|Sx_n-x_n\|=0$. So $u\in F(S)$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup u'\in EP(C,F)\cap VI(C,A)\cap F(S)$. We are going to show that $u=u'$. Assume that $u\neq u'$. The Opial condition yields

$$\begin{aligned} \lim_{n\to\infty}\|x_n-u\| &= \liminf_{n\to\infty}\|x_{n_i}-u\| < \liminf_{n\to\infty}\|x_{n_i}-u'\| \\ &= \lim_{n\to\infty}\|x_n-u'\| = \liminf_{n\to\infty}\|x_{n_j}-u'\| \\ &< \liminf_{n\to\infty}\|x_{n_j}-u\| = \lim_{n\to\infty}\|x_n-u\|. \end{aligned}$$

This is a contradiction. Therefore $u=u'$. That is

$$x_n\rightharpoonup u\in EP(C,F)\cap VI(C,A)\cap F(S).$$

Next, we will prove that $u=\lim_{n\to\infty}P_{EP(C,F)\cap VI(C,A)\cap F(S)}x_n$. Suppose that $z_n=P_{EP(C,F)\cap VI(C,A)\cap F(S)}x_n$. For every $u\in EP(C,F)\cap VI(C,A)\cap F(S)$, we have

$$\begin{aligned} \|x_{n+1}-u\| &= \|\alpha_n(x_n-u)+(1-\alpha_n)(SP_C(x_n-\lambda_n y_n)-u)\| \\ &\leq \|x_n-u\|. \end{aligned}$$

By using Lemma 2.3, it obtains that $\{z_n\}$ converges strongly to some $u^*\in EP(C,F)\cap VI(C,A)\cap F(S)$. Since $\langle u-z_n,z_n-x_n\rangle\geq 0$. So $\langle u-u^*,p_0-p\rangle\geq 0$. We can conclude that $u=u^*=\lim_{n\to\infty}P_{EP(C,F)\cap VI(C,A)\cap F(S)}x_n$. ∎

By setting $S = I_H$, we obtain the following algorithm for solving the equilibrium and variational inequality problems without the monotonicity of the bifunction F.
Algorithm 2 Pick $x_0 \in C$ and choose $\eta \in (0,1), \rho > 0, \{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0, 1/k)$ and $C_0 = C$.
Step 1. Compute

$$y_n = P_C(u_n - \lambda_n A u_n), \tag{3.18}$$

where $u_n = \arg\min\{F(x_n, y) + \frac{\rho}{2}\|y - x_n\|^2, y \in C\}$.
If $y_n = x_n$, then stop. Otherwise, do Step 2.
Step 2. (Armijo linesearch rule) Find m_n as the smallest positive integer number m satisfying

$$\begin{cases} z^{n,m} = (1 - \eta^m)x_n + \eta^m u_n \\ w^{n,m} \in \partial_2 F(z^{n,m}, z^{n,m}) \\ \langle w^{n,m}, x_n - u_n\rangle \geq \frac{\rho}{2}\|u_n - x_n\|^2. \end{cases} \tag{3.19}$$

Step 3. Set $\eta_n = \eta^{m_n}, z_n = z^{n,m_n}, w_n = w^{n,m_n}$. Take

$$H_n = \{x \in \mathbb{H} : \langle w_n, x - z_n\rangle \leq 0\}, C_{n+1} = C_n \cap H_n. \tag{3.20}$$

Step 4. Compute

$$x_{n+1} = P_{C_{n+1}}(x_n - \lambda_n A y_n), \tag{3.21}$$

and go to Step 1 with n is replaced by $n + 1$.

By Theorem 3.2, we obtain the following corollary.

Corollary 3.3. *Let $F : C \times C \to \mathbb{R}$ be a function satisfying (A1),(A2) and $A : C \to \mathbb{H}$ be a monotone k-Lipschitz continuous mapping. Suppose that $VI(A, C) \cap EP(C, F) \neq \emptyset, S_{MEP} \neq \emptyset$ and $\lim_{n\to\infty} \lambda_n = 0$. Then $\{x_n\}, \{y_n\}$ and $\{u_n\}$ which generated by Algorithm 2 converges weakly to $u = \lim_{n\to\infty} P_{VI(A,C)\cap EP(F)} x_n$.*

Remark 3.4. When setting $A = 0$, Algorithm 2 can imply the projection algorithms for solving nonmonotone equilibrium problem in Hilbert space (see in [1])

4. Conclusions

A modification of Extragradient method for finding a common solution of variational inequality, equilibrium and fixed point problems has been proposed, in which the bifunction F is a nonmonotone mapping on C, A is a monotone $k-$Lipschitz continuous mapping, and S is a nonexpansive mapping. When setting the solution of dual equilibrium is nonempty, we can obtain a weak convergence theorem which generated by our method. Furthermore, we have another algorithm for solving the common solution of variational inequality and equilibrium problems without the monotonicity of F.

Acknowledgements

The author would like to thank King Mongkut's University of Technology North Bangkok and Prof. Dr. Somyot Plubiteng for providing valuable suggestions. This research was funded by King Mongkuts University of Technology North Bangkok. Contract no. KMUTNB-64-DRIVE-8.

References

[1] B.V. Binh, D.S. Kim, Projection algorithms for solving nonmonotone equilibrium problems in Hilbert space, J. Comput. App. Math. 302 (2016) 106–117.

[2] C. Jaiboon, P. Kumam, U. Humphries, Weak convergence theorem by an extragradient method for variational inequality, equilibrium and fixed point problems, Bull. Malays. Math. Sci. Soc. 32 (2) (2009) 173–185.

[3] Y.R. Ye, A new double projection algorithm for variational inequalities, J. Comput. Appl. Math. 185 (2006) 166–173.

[4] T. Seangwattana, S. Plubtieng, T. Yuying, An extragradient method without monotonicity, Thai Journal of Mathematics 18 (1) (2020) 94–103.

[5] W. Takahashi, M. Toyoda, Weak convergence Theorems for nonexpansive mapping and monotone mapping, J. Optim. Theory. Appl. 118 (2003) 417–428.

[6] S. Karamardian, Complementarity problems over cones with monotone and pseudomonotone maps, J. Optim. Theory. Appl. 18 (1976) 445–454.

[7] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an Extragradient method for nonexpansive mappings and monotone mapping, J. Optim. Theory Appl. 128 (2006) 191–201.

[8] M. Ye, Y. He, A double projection method for solving variational inequalities without monotonicity, Comput Optim Appl. 60 (2015) 141–150.

[9] G.M. Korpelevich, The extragradient method for finding suddle points and other problems, Matecon 17 (1976) 747–756.

[10] P.N., Anh, A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems, Bull. Malays. Math. Sci. Soc. 36 (2013) 107–116.

[11] W.R. Mann, Mean value methods in iteration, Proceedings of the American Mathematical Society, 4 (1953) 506–510.

[12] E.H. Zarantonello, Projections on convex sets in Hilblert spaces and Spectral theory, Academic Press, New York, 1971.

[13] K. Goebel, W.A. Kirk, Topics in metric fixed point theory, Cambridge Univ. Press, Cambridge, 1990.

[14] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 127–149.

[15] L.D. Muu, W. Oettli, Convergence of an adaptive penalty scheme for finding constrained equilibria, Nonlinear Anal. TMA 18 (1992) 1159–1166.

[16] W. Takahashi, M. Toyoda, Weak convergence Theorems for nonexpansive mapping and monotone mapping, J. Optim. Theory. Appl. 118 (2003) 417–428.

[17] G.M. Korpelevich, The extragradient method for finding suddle points and other problems, Matecon 17 (1976) 747–756.

[18] G. Bigi, M. Castellani, M. Pappalardo, M. Passacantando, Existence and solution methods for equilibria, European J. Global Optim. 57 (2013) 1213–1227.

[19] A.N. Iusem, W. Sosa, Iterative algorithms for equilibrium problems, Optimization 52 (2003) 301–316.

[20] P. Kumam, S. Plubtieng, Viscosity approximation methods for monotone mappings and a countable family of nonexpansive mappings, Math. Slovaca 61 (2) (2011) 257–274.

[21] J. Abubakar, P. Kumam, A.H. Ibrahim, A. Padcharoen, Relaxed Inertial Tsengs Type Method for Solving the Inclusion Problem with Application to Image Restoration, Mathematics 8 (5) (2021)818.

[22] D. Kitkuan, P. Kumam, V. Berinde, A. Padcharoen, Adaptive algorithm for solving the SCFPP of demicontractive operators without a priori knowledge of operator norms, Ann. Univ. Ovidius Math. Ser. 27 (2019) 153–175.

[23] A. Padcharoen, D. Kitkuan, W. Kumam, P. Kumam, Tseng methods with inertial for solving inclusion problems and application to image deblurring and image recovery problems, Computational and Mathematical Methods 3 (3) (2021) e1088.

[24] P. Sadeewong, T. Saleewong, P. Kumam, Y.J. Cho, The Modified Viscosity Iteration with m-Generalized Hybrid Mappings and (a, b)-Monotone Mappings for Equilibrium Problems, Thai J. Math. 16 (1) (2018) 243–265.

[25] Y. Song, K. Muangchoo-in, P. Kumam, Y.J. Cho, Successive approximations for common fixed points of a family of α-nonexpansive mappings, J. Fixed Point Theory Appl. 20 (10) (2018).

[26] C. Jaiboon, W. Chantarangsi, P. Kumam, A Convergence theorem based on a hybrid relaxed extragradient method for generalized equilibrium problems and fixed point problems of a finite family of nonexpansive mappings, Nonlinear Analysis: Hybrid Systems 4 (1) (2010) 199–215.

[27] K. Muangchoo-in, P. Kumam, Y.J. Cho, Approximating Common Fixed Points of Two α-Nonexpansive Mappings, Thai J. Math. 16 (1) (2018) 139–145.

[28] S. Wang, M. Zhao, P. Kumam, Y.J. Cho, A viscosity extragradient method for an equilibrium problem and fixed point problem in Hilbert space, J. Fixed Point Theory Appl. 20 (19) (2018).

[29] M.V. Solodov, B.F. Svaiter, A new projection method for variational inequality problem, SIAM J. Control Optim. 37 (3) (1999) 765–776.

[30] F. Facchinei, J.S. Pang, Finite-dimensional variational inequalities and complementarity problems, Springer-Verlag, New York, 2003.

[31] M.V. Solodov, B.F. Svaiter, A new projection method for variational inequality problem, SIAM J. Control Optim. 37 (3) (1999) 765–776.

[32] N.N. Tam, J.C. Yao, N.D. Yen, Solution methods for pseudomonotone variational inequalites, J. Optim. Theory Appl. 38 (2008) 253–273.

[33] P.N. Anh, L.T.H. An, The subgradient extragradient method extended to equilibrium problem, Optimization 64 (2015) 225–248.

[34] D.Q. Tran, M.L. Dung, V.N. Nguyen, Extragradient algorithms extended to equilibrium problems, Optimization 57 (2008) 749–776.