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Simpson's Second Type Integral Inequalities for Twice Differentiable Convex Functions

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Abstract In this paper, the authors used a new auxiliary integral identity involving twice differentiable function based on a three step quadratic kernel $\omega(\mathsf{s})$ and obtained several new integral inequalities of Simpson's $\frac{3}{8}$ type for functions whose second derivative absolute value power q are φ-convex and φ-quasiconvex function via Hlder's and power mean inequality. Then we also obtain some Simpson's second type integral inequalities as special cases of our main results and also provided some applications to special means.

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1. Introduction and Preliminaries

A set $\mathcal{C} \subset \mathbb{R}$ is said to be convex, if

$$(1-\mathsf{s})z_1 + \mathsf{s}z_2 \in \mathcal{C}, \quad \forall z_1, z_2 \in \mathcal{C}, \mathsf{s} \in [0,1].$$

Similarly a function $\mathfrak{F} : \mathcal{C} \to \mathbb{R}$ is said to be convex, if

$$\mathfrak{F}((1-\mathsf{s})z_1 + \mathsf{s}z_2) \leq (1-\mathsf{s})\mathfrak{F}(z_1) + \mathsf{s}\mathfrak{F}(z_2), \quad \forall z_1, z_2 \in \mathcal{C}, \mathsf{s} \in [0,1].$$

In recent times, the integral inequalities models have been very useful in approximation theory and is applied to describe the rate of increase of competing mathematical analysis. Applications of these models are also found in fractional calculus and ordinary differential equations. The concept of convexity may be considered as very simple, but it has several applications in our daily lives through its various branches of pure and applied sciences. The theory of integral inequalities has also been influenced greatly by the convexity concept. In the literature, several renowned integral inequalities can be easily derived by

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applying the class of convex functions. Very many problems have been studied by using the integral inequalities approach. Recently, variants of integral inequalities have been studied by applying different techniques. One of the most studied and recorded integral inequality among several others is the Simpsons integral inequality type. Simpsons integral inequality has been applied successively in several models of fractional and ordinary differential equations, see [1, 5, 6, 9, 11]. There are also several applications of it in other branches of mathematics. Interested readers can find some interesting information and applications regarding Simpson's inequality in survey article written by Dragomir et al. [5]. This inequality reads as:
Let $\mathfrak{F} : I = [z_1, z_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a four time continuously differentiable on I°, where I° is the interior of I and $\|\mathfrak{F}^{(4)}\|_\infty < \infty$. Then following inequality is known as Simpson's inequality

$$\left| \frac{1}{3} \left[\frac{\mathfrak{F}(z_1) + \mathfrak{F}(z_2)}{2} + 2\mathfrak{F}\left(\frac{z_1 + z_2}{2} \right) \right] - \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} \mathfrak{F}(x)dx \right|$$
$$\leq \frac{1}{2880} \|\mathfrak{F}^{(4)}\|_\infty (z_2 - z_1)^4.$$

Recently many researchers have obtained a variety of new refinements of Simpson's inequality using novel and innovative techniques [2–4, 7, 8, 10, 12].

Definition 1.1. [6]. A function $\mathfrak{F} : \mathcal{C} \to \mathbb{R}$ is called convex with respect to φ, if

$$\mathfrak{F}((1 - \mathsf{s})z_1 + \mathsf{s}z_2) \leq \mathfrak{F}(z_1) + \mathsf{s}\varphi(\mathfrak{F}(z_1), \mathfrak{F}(z_2)),$$

for all z_1, $z_2 \in \mathcal{C}$ and $\mathsf{s} \in [0, 1]$. Furthermore, $\mathfrak{F}$ is called φ-quasiconvex, if

$$\mathfrak{F}((1 - \mathsf{s})z_1 + \mathsf{s}z_2) \leq \max\{\mathfrak{F}(z_1), \mathfrak{F}(z_1) + \varphi(\mathfrak{F}(z_1), \mathfrak{F}(z_2))\}, \quad \forall z_1, z_2 \in \mathcal{C}, \mathsf{s} \in [0, 1].$$

If we take $\varphi(z_1, z_2) = z_2 - z_1$ in above definition, then the definition of φ-convex and φ-quasiconvex are deduced to the definition of convex and quasi convex functions, respectively. It is easy to see that every φ-convex function is φ-quasi convex.

Remark 1.2. [3].

(1) Let $\mathfrak{F}(z) = z^2$, then $\mathfrak{F}$ is convex and φ-convex with $\varphi(z_1, z_2) = 2z_1 + z_2$.
(2) Let $\mathfrak{F} : [z_1, z_1] \to \mathbb{R}$ $0 < z_1 < z_2$, with $\mathfrak{F}(z) = \frac{1}{z^2}$. We observe that $\mathfrak{F}$ is convex on $[z_1, z_2]$ and therefore is φ-quasiconvex with $\varphi(z_1, z_2) = z_1 - z_2$.
(3) Let $\mathfrak{F} : [z_1, z_1] \to \mathbb{R}$ $0 < z_1 < z_2$, with $\mathfrak{F}(z) = \frac{2}{z^3}$. The function $\mathfrak{F}$ is convex on $[z_1, z_2]$ and therefore is φ-quasiconvex with $\varphi(z_1, z_2) = z_1 - z_2$.
(4) Let $\mathfrak{F} : [z_1, z_1] \to \mathbb{R}$ $0 < z_1 < z_2$, with $\mathfrak{F}(z) = 2$. The function $\mathfrak{F}$ is φ-quasiconvex with $\varphi(z_1, z_2) = z_1 - z_2$.
(5) Let $\mathfrak{F}(z) = z^3$, then $\mathfrak{F}$ is not convex but is φ-convex with $\varphi(z_1, z_2) = 3z_2^2(z_1 - z_2) + 3z_2(z_1 - z_2)^2 + (z_1 - z_2)^3$.

The aim of this article is to obtain some new refinements of Simpson's inequality for twice differentiable convex functions via φ-convex and φ-quasiconvex functions. In order to show the importance of our obtained results, we also present some nice and interesting applications to Simpson's $\frac{3}{8}$-rule and also provided some applications to special mean and shown in figures to demonstrate the explanation of the readers. We hope that the ideas and techniques of this paper will inspire interested readers working in the field of integral inequalities and its applications.

2. Results and Discussions

In this section, we will discuss our main results.

2.1. New Auxiliary Result

In order to establish our main results, we need the following auxiliary result.

Lemma 2.1. *Suppose that $\mathfrak{F} : I^\circ \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on I° and $z_1, z_2 \in I^\circ$ with $z_1 < z_2$. If $\mathfrak{F}'' \in L[z_1, z_2]$, then*

$$\begin{aligned}
&\frac{1}{8}\left[(z_2 - z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right. \\
&\quad \left. +\mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right) + \mathfrak{F}(z_2)\right] \\
&- \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} \mathfrak{F}(u)\mathrm{d}u \\
&= (z_2 - z_1)^2 \int_0^1 \omega(s)\mathfrak{F}''((1-s)z_1 + sz_2)\mathrm{d}s,
\end{aligned} \tag{2.1}$$

where $\omega(s)$ is defined by

$$\omega(s) = \begin{cases} \frac{s}{2}\left(\frac{1}{4} - s\right), & s \in \left[0, \frac{1}{3}\right) \\ \frac{s}{2}(1-s), & s \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ (1-s)\left(\frac{s}{2} - \frac{3}{8}\right), & s \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Proof. It is easy to see that

$$\begin{aligned}
&(z_2 - z_1)^2 \int_0^1 \omega(s)\mathfrak{F}''((1-s)z_1 + sz_2)\mathrm{d}s \\
&= (z_2 - z_1)^2 \int_0^{\frac{1}{3}} \frac{s}{2}\left(\frac{1}{4} - s\right)\mathfrak{F}''((1-s)z_1 + sz_2)\mathrm{d}s \\
&\quad + (z_2 - z_1)^2 \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{s}{2}(1-s)\mathfrak{F}''((1-s)z_1 + sz_2)\mathrm{d}s \\
&\quad + (z_2 - z_1)^2 \int_{\frac{2}{3}}^{1} (1-s)\left(\frac{s}{2} - \frac{3}{8}\right)\mathfrak{F}''((1-s)z_1 + sz_2)\mathrm{d}s.
\end{aligned} \tag{2.2}$$

Applying integration by parts to the first integral in right side of the above equality, we have

$$\begin{aligned}
&\int_0^{\frac{1}{3}} \frac{s}{2}\left(\frac{1}{4} - s\right)\mathfrak{F}''((1-s)z_1 + sz_2)\mathrm{d}s \\
&= \left.\frac{\frac{s}{2}\left(\frac{1}{4} - s\right)\mathfrak{F}'((1-s)z_1 + sz_2)}{(z_2 - z_1)}\right|_0^{\frac{1}{3}} \\
&\quad - \frac{1}{(z_2 - z_1)}\int_0^{\frac{1}{3}}\left(\frac{1}{8} - s\right)\mathfrak{F}'((1-s)z_1 + sz_2)\mathrm{d}s \\
&= -\frac{1}{72(z_2 - z_1)}\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)
\end{aligned} \tag{2.3}$$

$$-\left.\frac{\left(\frac{1}{8}-\mathsf{s}\right)\mathfrak{F}((1-\mathsf{s})z_1+\mathsf{s}z_2)}{(z_2-z_1)^2}\right|_0^{\frac{1}{3}}-\frac{1}{(z_2-z_1)^2}\int_0^{\frac{1}{3}}\mathfrak{F}((1-\mathsf{s})z_1+\mathsf{s}z_2)d\mathsf{s}$$
$$=-\frac{1}{72(z_2-z_1)}\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)+\frac{5}{24(z_2-z_1)^2}\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+\frac{1}{8(z_2-z_1)^2}\mathfrak{F}(z_1)$$
$$-\frac{1}{(z_2-z_1)^2}\int_0^{\frac{1}{3}}\mathfrak{F}((1-\mathsf{s})z_1+\mathsf{s}z_2)d\mathsf{s}.$$

Changing the variable $u=(1-\mathsf{s})z_1+\mathsf{s}z_2$, we have

$$\int_0^{\frac{1}{3}}\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)d\mathsf{s} \tag{2.4}$$
$$=-\frac{1}{72(z_2-z_1)}\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)+\frac{1}{(z_2-z_1)^2}\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+\frac{1}{8(z_2-z_1)^2}\mathfrak{F}(z_1)$$
$$-\frac{1}{(z_2-z_1)^3}\int_{z_1}^{\frac{2z_1+z_2}{3}}\mathfrak{F}(u)du.$$

If we similarly calculate the other integrals in the right side of (2.2), we obtain

$$\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{\mathsf{s}}{2}(1-\mathsf{s})\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)d\mathsf{s}d\mathsf{s} \tag{2.5}$$
$$=\frac{1}{9(z_2-z_1)}\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)-\frac{1}{9(z_2-z_1)^2}\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)$$
$$+\frac{1}{6(z_2-z_1)^2}\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)+\frac{1}{6(z_2-z_1)^2}\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)$$
$$-\frac{1}{(z_2-z_1)^3}\int_{\frac{2z_1+z_2}{3}}^{\frac{z_1+2z_2}{3}}\mathfrak{F}(u)du,$$

and

$$\int_{\frac{2}{3}}^{1}(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)d\mathsf{s} \tag{2.6}$$
$$=\frac{1}{72(z_2-z_1)}\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)+\frac{1}{8(z_2-z_1)^2}\mathfrak{F}(z_2)+\frac{5}{24(z_2-z_1)^2}\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)$$
$$-\frac{1}{(z_2-z_1)^3}\int_{\frac{z_1+2z_2}{3}}^{z_2}\mathfrak{F}(u)du.$$

Combining (2.4)-(2.6), and multiplying the resultant by $(z_2-z_1)^2$, we obtain (2.1). This completes the proof. ∎

2.2. Simpson's Like Integral Inequalities

We now derive results associated with Lemma 2.1.

Theorem 2.2. *Let $\mathfrak{F}:I\subset(0,\infty)\to(0,\infty)$ be a twice differentiable mapping on I°, and $z_1,z_2\in I^\circ$ with $z_1<z_2$ such that $\mathfrak{F}''\in L[z_1,z_2]$ and $|\mathfrak{F}''|$ is φ-convex function on the*

interval $[z_1, z_2]$, *then*

$$\left|\frac{1}{8}\left[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)-\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right.\right.$$
$$\left.\left.+\mathfrak{F}(z_1)+3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)+\mathfrak{F}(z_2)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)\mathrm{d}u\right|$$
$$\leq (z_2-z_1)^2\left(\frac{227}{5184}\right)\left[|\mathfrak{F}''(z_1)|+\frac{1}{2}\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\right]. \tag{2.7}$$

Proof. By making the use of Lemma 2.1 and the φ-convexity of $|\mathfrak{F}|''$, we find

$$\left|\frac{1}{8}\left[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)-\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right.\right.$$
$$\left.\left.+\mathfrak{F}(z_1)+3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)+\mathfrak{F}(z_2)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)\mathrm{d}u\right|$$
$$=\left|(z_2-z_1)^2\int_0^1 \omega(\mathsf{s})\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)\mathrm{d}\mathsf{s}\right|$$
$$\leq (z_2-z_1)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|(|\mathfrak{F}''(z_1)|+\mathsf{s}\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|))\,\mathrm{d}\mathsf{s}\right.$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|(|\mathfrak{F}''(z_1)|+\mathsf{s}\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|))\,\mathrm{d}\mathsf{s}$$
$$\left.+\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right|(|\mathfrak{F}''(z_1)|+\mathsf{s}\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|))\,\mathrm{d}\mathsf{s}\right]$$
$$=(z_2-z_1)^2(M_1+M_2+M_3).$$

where

$$M_1=\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|(|\mathfrak{F}''(z_1)|+\mathsf{s}\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|))\,\mathrm{d}\mathsf{s}$$
$$=\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right||\mathfrak{F}''(z_1)|\mathrm{d}\mathsf{s}+\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|\mathsf{s}\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\mathrm{d}\mathsf{s}$$
$$=|\mathfrak{F}''(z_1)|\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|\mathrm{d}\mathsf{s}+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|\mathsf{s}\mathrm{d}\mathsf{s}$$
$$=\left(\frac{19}{10368}\right)|\mathfrak{F}''(z_1)|+\left(\frac{1}{3072}\right)\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|).$$

$$M_2=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|(|\mathfrak{F}''(z_1)|+\mathsf{s}\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|))\,\mathrm{d}\mathsf{s}$$
$$=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right||\mathfrak{F}''(z_1)|\mathrm{d}\mathsf{s}+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|\mathsf{s}\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\mathrm{d}\mathsf{s}$$

$$= |\mathfrak{F}''(z_1)| \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \frac{\mathsf{s}}{2}(1-\mathsf{s}) \right| \mathrm{ds} + \varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|) \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \frac{\mathsf{s}}{2}(1-\mathsf{s}) \right| \mathsf{s}\mathrm{ds}$$
$$= \left(\frac{13}{324}\right)|\mathfrak{F}''(z_1)| + \left(\frac{13}{648}\right)\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|),$$

and

$$M_3 = \int_{\frac{2}{3}}^{1} \left| (1-\mathsf{s})\left(\frac{\mathsf{s}}{2} - \frac{3}{8}\right) \right| (|\mathfrak{F}''(z_1)| + \mathsf{s}\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|))\, \mathrm{ds}$$
$$= \int_{\frac{2}{3}}^{1} \left| (1-\mathsf{s})\left(\frac{\mathsf{s}}{2} - \frac{3}{8}\right) \right| |\mathfrak{F}''(z_1)|\mathrm{ds} + \int_{\frac{2}{3}}^{1} \left| (1-\mathsf{s})\left(\frac{\mathsf{s}}{2} - \frac{3}{8}\right) \right| \mathsf{s}\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|)\mathrm{ds}$$
$$= |\mathfrak{F}''(z_1)| \int_{\frac{2}{3}}^{1} \left| (1-\mathsf{s})\left(\frac{\mathsf{s}}{2} - \frac{3}{8}\right) \right| \mathrm{ds} + \varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|) \int_{\frac{2}{3}}^{1} \left| (1-\mathsf{s})\left(\frac{\mathsf{s}}{2} - \frac{3}{8}\right) \right| \mathsf{s}\mathrm{ds}$$
$$= \left(\frac{19}{10368}\right)|\mathfrak{F}''(z_1)| + \left(\frac{125}{82944}\right)\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|).$$

Finally, if we have calculated the required integrals and hence obtained the required result.

$$\left| \frac{1}{8}\left[(z_2 - z_1)\left(\mathfrak{F}'\left(\frac{z_1 + 2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1 + z_2}{3}\right)\right) \right.\right.$$
$$\left.\left. + \mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1 + z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1 + 2z_2}{3}\right) + \mathfrak{F}(z_2)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} \mathfrak{F}(u)\mathrm{d}u \right|$$
$$\leq (z_2 - z_1)^2 \left(\frac{125|\mathfrak{F}''(z_1)| + 27\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|)}{82944} \right)$$
$$+ (z_2 - z_1)^2 \left(\frac{26|\mathfrak{F}''(z_1)| + 13|\mathfrak{F}''(z_2)|}{648} \right)$$
$$+ (z_2 - z_1)^2 \left(\frac{152|\mathfrak{F}''(z_1)| + 125\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|)}{82944} \right).$$

This completes the proof. ■

Corollary 2.3. *Inequality* (2.7) *with* $\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|) = |\mathfrak{F}''(z_2)| - |\mathfrak{F}''(z_1)|$ *becomes*

$$\left| \frac{1}{8}\left[(z_2 - z_1)\left(\mathfrak{F}'\left(\frac{z_1 + 2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1 + z_2}{3}\right)\right) \right.\right.$$
$$\left.\left. + \mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1 + z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1 + 2z_2}{3}\right) + \mathfrak{F}(z_2)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} \mathfrak{F}(u)\mathrm{d}u \right|$$
$$\leq (z_2 - z_1)^2 \left(\frac{227}{10368}\right)[|\mathfrak{F}''(z_1)| + |\mathfrak{F}''(z_2)|].$$

Before moving further, let us recall the concept of hypergeometric functions.

$${}_2F_1(x, y; c; z) = \frac{1}{\mathrm{B}(y, c-y)} \int_0^1 \mathsf{s}^{y-1}(1-\mathsf{s})^{c-y-1}(1 - z\mathsf{s})^{-x}\mathrm{ds},$$

for $|z| < 1, c > y > 0$.

Theorem 2.4. *Let $\mathfrak{F} : I \subset (0,\infty) \to (0,\infty)$ be a twice differentiable mapping on I°, and $z_1, z_2 \in I^\circ$ with $z_1 < z_2$ such that $\mathfrak{F}'' \in L[z_1, z_2]$ and $|\mathfrak{F}''|^q$ is φ-convex function on the interval $[z_1, z_2]$ for some fixed $q > 1$ and $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\Bigg|\frac{1}{8}\Bigg[(z_2 - z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right) \\
&\quad +\mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right) + \mathfrak{F}(z_2)\Bigg] - \frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)du\Bigg| \\
&\le (z_2-z_1)^2\Bigg[(\mathfrak{N}_1)^{\frac{1}{r}}\left(\frac{6|\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)}{18}\right)^{\frac{1}{q}} \\
&\quad + \mathfrak{N}_2^{\frac{1}{r}}\left(\frac{2|\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)}{6}\right)^{\frac{1}{q}} \\
&\quad + \mathfrak{N}_3^{\frac{1}{r}}\left(\frac{6|\mathfrak{F}''(z_1)|^q + 5\varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)}{18}\right)^{\frac{1}{q}}\Bigg].
\end{aligned} \tag{2.8}$$

Proof. Using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\Bigg|\frac{1}{8}\Bigg[(z_2 - z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right) \\
&\quad +\mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right) + \mathfrak{F}(z_2)\Bigg] - \frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)du\Bigg| \\
&\le (z_2-z_1)^2\Bigg[\left(\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|^r d\mathsf{s}\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}|\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q d\mathsf{s}\right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|^r d\mathsf{s}\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q d\mathsf{s}\right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right|^r d\mathsf{s}\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}|\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q d\mathsf{s}\right)^{\frac{1}{q}}\Bigg].
\end{aligned} \tag{2.9}$$

Using the φ-convexity of $|\mathfrak{F}''|^q$, we find that

$$\int_0^{\frac{1}{3}}|\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q d\mathsf{s} \le \frac{6|\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)}{18}.$$

Also, we have

$$\int_{\frac{1}{3}}^{\frac{2}{3}}|\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q d\mathsf{s} \le \frac{2|\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)}{6},$$

and

$$\int_{\frac{2}{3}}^{1}|\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q d\mathsf{s} \le \frac{6|\mathfrak{F}''(z_1)|^q + 5\varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)}{18}.$$

One can also obtain

$$\begin{aligned}
\mathfrak{N}_1 &= \int_0^{\frac{1}{3}} \left| \frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|^r d\mathsf{s} \\
&= \int_0^{\frac{1}{4}} \left[\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right]^r d\mathsf{s} + \int_{\frac{1}{4}}^{\frac{1}{3}} \left[\frac{\mathsf{s}}{2}\left(\mathsf{s}-\frac{1}{4}\right)\right]^r d\mathsf{s} \\
&= \frac{1}{8^r}\left[\int_0^{\frac{1}{4}} \left[\mathsf{s}\left(1-4\mathsf{s}\right)\right]^r d\mathsf{s} + \int_{\frac{1}{4}}^{\frac{1}{3}} \left[\mathsf{s}\left(4\mathsf{s}-1\right)\right]^r d\mathsf{s}\right] \\
&= \frac{1}{8^r}\left[\int_0^{\frac{1}{4}} \left[\mathsf{s}\left(1-4\mathsf{s}\right)\right]^r d\mathsf{s} + \int_0^{\frac{1}{3}} \left[\mathsf{s}\left(4\mathsf{s}-1\right)\right]^r d\mathsf{s} - \int_0^{\frac{1}{4}} \left[\mathsf{s}\left(4\mathsf{s}-1\right)\right]^r d\mathsf{s}\right] \\
&= \frac{1}{8^r}\left[\frac{\mathrm{B}(r+1,r+1)}{4^{r+1}} + \frac{(-1)^r}{3^{r+1}(r+1)} \cdot {}_2F_1\left(-r,r+1;r+2;\frac{4}{3}\right)\right. \\
&\quad \left. -\frac{(-1)^r B(r+1,r+1)}{4^{r+1}} \cdot {}_2F_1(-r,r+1;r+2;1)\right],
\end{aligned}$$

$$\begin{aligned}
\mathfrak{N}_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left|\frac{\mathsf{s}}{2}\left(1-\mathsf{s}\right)\right|^r d\mathsf{s} \\
&= \frac{1}{2^r}\left[\int_0^{\frac{2}{3}} \mathsf{s}^r\left(1-\mathsf{s}\right)^r d\mathsf{s} - \int_0^{\frac{1}{3}} \mathsf{s}^r\left(1-\mathsf{s}\right)^r d\mathsf{s}\right] \\
&= \frac{1}{3^{r+1}(r+1)}\left[2 \cdot {}_2F_1\left(-r,r+1;r+2;\frac{2}{3}\right) - \frac{1}{2^r} \cdot {}_2F_1\left(-r,r+1;r+2;\frac{1}{3}\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{N}_3 &= \int_{\frac{2}{3}}^{1} \left|\left(1-\mathsf{s}\right)\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right|^r d\mathsf{s} \\
&= \int_{\frac{2}{3}}^{\frac{3}{4}} \left[\left(1-\mathsf{s}\right)\left(\frac{3}{8}-\frac{\mathsf{s}}{2}\right)\right]^r d\mathsf{s} + \int_{\frac{3}{4}}^{1} \left[\left(1-\mathsf{s}\right)\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right]^r d\mathsf{s} \\
&= \left(\frac{3}{8}\right)^r\left[\int_{\frac{2}{3}}^{\frac{3}{4}} \left(1-\mathsf{s}\right)^r\left(1-\frac{4}{3}\mathsf{s}\right)^r d\mathsf{s} + (-1)^r\int_{\frac{3}{4}}^{1} \left(1-\mathsf{s}\right)^r\left(1-\frac{4}{3}\mathsf{s}\right)^r d\mathsf{s}\right] \\
&= \left(\frac{3}{8}\right)^r\left[\frac{1}{36^r \cdot 12}\mathrm{B}(r+1,1) \cdot {}_2F_1(-r,r+1;r+2;-\frac{1}{3}) + \frac{1}{4^{r+1}3^r}B(r+1,r+1)\right].
\end{aligned}$$

This completes the proof. ■

Corollary 2.5. *Inequality (2.8) with* $\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|) = |\mathfrak{F}''(z_2)| - |\mathfrak{F}''(z_1)|$ *becomes*

$$\left|\frac{1}{8}\left[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right.\right.$$
$$\left.\left. +\mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right) + \mathfrak{F}(z_2)\right] - \frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)du\right|$$

$$\leq (z_2-z_1)^2\Bigg[(\mathfrak{N}_1)^{\frac{1}{r}}\left(\frac{5|\mathfrak{F}''(z_1)|^q+|\mathfrak{F}''(z_2)|^q}{18}\right)^{\frac{1}{q}}+\mathfrak{N}_2^{\frac{1}{r}}\left(\frac{|\mathfrak{F}''(z_1)|^q+|\mathfrak{F}''(z_2)|^q}{6}\right)^{\frac{1}{q}}$$
$$+\mathfrak{N}_3^{\frac{1}{r}}\left(\frac{|\mathfrak{F}''(z_1)|^q+5\mathfrak{F}''(z_2)|^q}{18}\right)^{\frac{1}{q}}\Bigg].$$

Theorem 2.6. *Let $\mathfrak{F}: I\subset(0,\infty)\to(0,\infty)$ be a twice differentiable mapping on I°, and $z_1,z_2\in I^\circ$ with $z_1<z_2$ such that $\mathfrak{F}''\in L[z_1,z_2]$ and $|\mathfrak{F}''|^q$ is φ-convex function on the interval $[z_1,z_2]$ for some fixed $q\geq 1$, then the following inequalities hold:*

$$\Bigg|\frac{1}{8}\Bigg[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)-\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right) \tag{2.10}$$
$$+\mathfrak{F}(z_1)+3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)+\mathfrak{F}(z_2)\Bigg]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)\mathrm{d}u\Bigg|$$
$$\leq (z_2-z_1)^2\Bigg[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152|\mathfrak{F}''(z_1)|^q+27\varphi(|\mathfrak{F}''(z_1)|^q,|\mathfrak{F}''(z_2)|^q)}{82944}\right)^{\frac{1}{q}}$$
$$+\left(\frac{13}{324}\right)^{1-\frac{1}{q}}\left(\frac{26|\mathfrak{F}''(z_1)|^q+13\varphi(|\mathfrak{F}''(z_1)|^q,|\mathfrak{F}''(z_2)|^q)}{648}\right)^{\frac{1}{q}}$$
$$+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152|\mathfrak{F}''(z_1)|^q+125\varphi(|\mathfrak{F}''(z_1)|^q,|\mathfrak{F}''(z_2)|^q)}{82944}\right)^{\frac{1}{q}}\Bigg].$$

Proof. Using Lemma 2.1, power mean inequality, we have

$$\Bigg|\frac{1}{8}\Bigg[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)-\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)$$
$$+\mathfrak{F}(z_1)+3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)+\mathfrak{F}(z_2)\Bigg]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)\mathrm{d}u\Bigg|$$
$$\leq (z_2-z_1)^2\Bigg[\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right||\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|\mathrm{d}\mathsf{s}$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right||\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|\mathrm{d}\mathsf{s}$$
$$+\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right||\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|\mathrm{d}\mathsf{s}\Bigg]$$
$$\leq (z_2-z_1)^2\Bigg[\left(\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|\mathrm{d}\mathsf{s}\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right||\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q\mathrm{d}\mathsf{s}\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|\mathrm{d}\mathsf{s}\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right||\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q\mathrm{d}\mathsf{s}\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right|\mathrm{d}\mathsf{s}\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right||\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|^q\mathrm{d}\mathsf{s}\right)^{\frac{1}{q}}\Bigg]$$

$$\leq (z_2-z_1)^2\left(\frac{19}{10368}\right)^{1-\frac{1}{q}} \times \left(|\mathfrak{F}''(z_1)|^q \int_0^{\frac{1}{3}} \left|\frac{s}{2}\left(\frac{1}{4}-s\right)\right| ds\right.$$
$$\left.+\varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\int_0^{\frac{1}{3}} s\left|\frac{s}{2}\left(\frac{1}{4}-s\right)\right| ds\right)^{\frac{1}{q}}$$
$$+(z_2-z_1)^2\left(\frac{13}{324}\right)^{1-\frac{1}{q}}\left(|\mathfrak{F}''(z_1)|^q\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{s}{2}(1-s)\right| ds\right.$$
$$\left.+\varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\int_{\frac{1}{3}}^{\frac{2}{3}} s\left|\frac{s}{2}(1-s)\right| ds\right)^{\frac{1}{q}}$$
$$+(z_2-z_1)^2\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(|\mathfrak{F}''(z_1)|^q\int_{\frac{2}{3}}^{1}\left|(1-s)\left(\frac{s}{2}-\frac{3}{8}\right)\right| ds\right.$$
$$\left.+\varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\int_{\frac{2}{3}}^{1} s\left|(1-s)\left(\frac{s}{2}-\frac{3}{8}\right)\right| ds\right)^{\frac{1}{q}}.$$

One can easily find the above integrals. This completes the proof. ■

Corollary 2.7. *Inequality* (2.10) *with* $\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|) = |\mathfrak{F}''(z_2)| - |\mathfrak{F}''(z_1)|$ *becomes*

$$\left|\frac{1}{8}\left[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)-\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right.\right. \tag{2.11}$$
$$\left.\left.+\mathfrak{F}(z_1)+3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)+\mathfrak{F}(z_2)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)du\right|$$
$$\leq (z_2-z_1)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{125|\mathfrak{F}''(z_1)|^q+27|\mathfrak{F}''(z_2)|^q}{82944}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{13}{324}\right)^{1-\frac{1}{q}}\left(\frac{13|\mathfrak{F}''(z_1)|^q+13|\mathfrak{F}''(z_2)|^q}{648}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{27|\mathfrak{F}''(z_1)|^q+125|\mathfrak{F}''(z_2)|^q}{82944}\right)^{\frac{1}{q}}\right].$$

Theorem 2.8. *Let* $\mathfrak{F} : I \subset (0,\infty) \to (0,\infty)$ *be a twice differentiable mapping on* I°, *and* $z_1, z_2 \in I^\circ$ *with* $z_1 < z_2$ *such that* $\mathfrak{F}^{''} \in L[z_1, z_2]$ *and* $|\mathfrak{F}^{''}|$ *is* φ*-quasiconvex function on the interval* $[z_1, z_2]$, *then*

$$\left|\frac{1}{8}\left[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)-\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right.\right. \tag{2.12}$$
$$\left.\left.+\mathfrak{F}(z_1)+3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)+\mathfrak{F}(z_2)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)du\right|$$
$$\leq (z_2-z_1)^2\left(\frac{227}{5184}\right)\max\left\{|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|, |\mathfrak{F}''(z_2)|)\right\}.$$

Proof. By making the use of φ-quasiconvexity of $|\mathfrak{F}|''$ and Lemma 2.1, we have

$$\begin{aligned}
&\Bigg|\frac{1}{8}\bigg[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right)-\mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\\
&\quad+\mathfrak{F}(z_1)+3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right)+3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right)+\mathfrak{F}(z_2)\bigg]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)\mathrm{d}u\Bigg|\\
&\le (z_2-z_1)^2\int_0^1|\omega(\mathsf{s})||\mathfrak{F}''((1-\mathsf{s})z_1+\mathsf{s}z_2)|\mathrm{ds}\\
&\le (z_2-z_1)^2\Bigg[\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\mathrm{ds}\\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\mathrm{ds}\\
&\quad+\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right|\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\mathrm{ds}\Bigg]\\
&= (z_2-z_1)^2(\bar{M_1}+\bar{M_2}+\bar{M_3}).
\end{aligned}$$

where

$$\begin{aligned}
\bar{M_1}&=\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\mathrm{ds}\\
&=\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4}-\mathsf{s}\right)\right|\mathrm{ds}\\
&=\left(\frac{19}{10368}\right)\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}.
\end{aligned}$$

$$\begin{aligned}
\bar{M_2}&=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\mathrm{ds}\\
&=\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|\mathrm{ds}\\
&=\left(\frac{13}{324}\right)\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\},
\end{aligned}$$

and

$$\begin{aligned}
\bar{M_3}&=\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right|\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\mathrm{ds}\\
&=\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2}-\frac{3}{8}\right)\right|\mathrm{ds}\\
&=\left(\frac{19}{10368}\right)\max\{|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_1)|+\varphi(|\mathfrak{F}''(z_1)|,|\mathfrak{F}''(z_2)|)\}.
\end{aligned}$$

A simple rearrangement completes the proof. ■

Theorem 2.9. *Let $\mathfrak{F} : I \subset (0,\infty) \to (0,\infty)$ be a twice differentiable mapping on I°, and $z_1, z_2 \in I^\circ$ with $z_1 < z_2$ such that $\mathfrak{F}^{''} \in L[z_1, z_2]$ and $|\mathfrak{F}^{''}|^q$ is φ-quasiconvex function on the interval $[z_1, z_2]$ for some fixed $q \geq 1$, then the following inequalities hold:*

$$\begin{aligned}&\Bigg|\frac{1}{8}\left[(z_2 - z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right. \\ &\quad \left. +\mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right) + \mathfrak{F}(z_2)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)\mathrm{d}u\Bigg| \\ &\leq (z_2 - z_1)^2\left(\frac{227}{5184}\right)\left(\max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\}\right)^{\frac{1}{q}}.\end{aligned} \tag{2.13}$$

Proof. Let $q \geq 1$, then by using Lemma 2.1, we have

$$\begin{aligned}&\Bigg|\frac{1}{8}\left[(z_2 - z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right. \\ &\quad \left. +\mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right) + \mathfrak{F}(z_2)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)\mathrm{d}u\Bigg| \\ &\leq (z_2 - z_1)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4} - \mathsf{s}\right)\right||\mathfrak{F}''((1-\mathsf{s})z_1 + \mathsf{s}z_2)|\mathrm{d}\mathsf{s}\right. \\ &\quad + \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right||\mathfrak{F}''((1-\mathsf{s})z_1 + \mathsf{s}z_2)|\mathrm{d}\mathsf{s} \\ &\quad \left. + \int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2} - \frac{3}{8}\right)\right||\mathfrak{F}''((1-\mathsf{s})z_1 + \mathsf{s}z_2)|\mathrm{d}\mathsf{s}\right].\end{aligned}$$

By making the use of the power mean inequality for the above integrals, we have

$$\begin{aligned}&\Bigg|\frac{1}{8}\left[(z_2 - z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right. \\ &\quad \left. +\mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right) + \mathfrak{F}(z_2)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2}\mathfrak{F}(u)\mathrm{d}u\Bigg| \\ &\leq (z_2 - z_1)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4} - \mathsf{s}\right)\right|\mathrm{d}\mathsf{s}\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\frac{\mathsf{s}}{2}\left(\frac{1}{4} - \mathsf{s}\right)\right||\mathfrak{F}''((1-\mathsf{s})z_1 + \mathsf{s}z_2)|^q\mathrm{d}\mathsf{s}\right)^{\frac{1}{q}}\right. \\ &\quad + \left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right|\mathrm{d}\mathsf{s}\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\mathsf{s}}{2}(1-\mathsf{s})\right||\mathfrak{F}''((1-\mathsf{s})z_1 + \mathsf{s}z_2)|^q\mathrm{d}\mathsf{s}\right)^{\frac{1}{q}} \\ &\quad \left. + \left(\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2} - \frac{3}{8}\right)\right|\mathrm{d}\mathsf{s}\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|(1-\mathsf{s})\left(\frac{\mathsf{s}}{2} - \frac{3}{8}\right)\right||\mathfrak{F}''((1-\mathsf{s})z_1 + \mathsf{s}z_2)|^q\mathrm{d}\mathsf{s}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{2.14}$$

By φ-quasiconvexity of $|\mathfrak{F}''|^q$ for the last three integrals, we get

1. $$\int_0^{\frac{1}{3}} \left| \frac{s}{2}\left(\frac{1}{4}-s\right) \right| \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\} ds$$
$$= \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\} \int_0^{\frac{1}{3}} \left| \frac{s}{2}\left(\frac{1}{4}-s\right) \right| ds$$
$$= \left(\frac{19}{10368}\right) \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\}. \tag{2.15}$$

2. $$\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \frac{s}{2}(1-s) \right| \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\} ds$$
$$= \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\} \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \frac{s}{2}(1-s) \right| ds$$
$$= \left(\frac{13}{324}\right) \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\}. \tag{2.16}$$

3. $$\int_{\frac{2}{3}}^{1} \left| (1-s)\left(\frac{s}{2}-\frac{3}{8}\right) \right| \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\} ds$$
$$= \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\} \int_{\frac{2}{3}}^{1} \left| (1-s)\left(\frac{s}{2}-\frac{3}{8}\right) \right| ds$$
$$= \left(\frac{19}{10368}\right) \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\}. \tag{2.17}$$

By substituting (2.15), (2.16) and (2.17) into (2.14), we have

$$\begin{aligned}
&\left| \frac{1}{8}\left[(z_2-z_1)\left(\mathfrak{F}'\left(\frac{z_1+2z_2}{3}\right) - \mathfrak{F}'\left(\frac{2z_1+z_2}{3}\right)\right)\right.\right. \\
&\quad \left.\left. + \mathfrak{F}(z_1) + 3\mathfrak{F}\left(\frac{2z_1+z_2}{3}\right) + 3\mathfrak{F}\left(\frac{z_1+2z_2}{3}\right) + \mathfrak{F}(z_2)\right] - \frac{1}{z_2-z_1}\int_{z_1}^{z_2} \mathfrak{F}(u)du \right| \\
&\leq (z_2-z_1)^2 \left(\frac{19}{10368}\right)^{1-\frac{1}{q}} \times \left(\left(\frac{19}{10368}\right) \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\}\right)^{\frac{1}{q}} \\
&\quad + (z_2-z_1)^2 \left(\frac{13}{324}\right)^{1-\frac{1}{q}} \left(\left(\frac{13}{324}\right) \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\}\right)^{\frac{1}{q}} \\
&\quad + (z_2-z_1)^2 \left(\frac{19}{10368}\right)^{1-\frac{1}{q}} \left(\left(\frac{19}{10368}\right) \max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\}\right)^{\frac{1}{q}} \\
&= (z_2-z_1)^2 \left(\frac{227}{5184}\right) \left(\max\left\{|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_1)|^q + \varphi(|\mathfrak{F}''(z_1)|^q, |\mathfrak{F}''(z_2)|^q)\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. ∎

3. Applications

In this section, we will discuss applications of our results.

3.1. Applications to Special Means

We recall the following definitions of special means, which are used in our coming results. For arbitrary $z_1, z_2 (z_1 \neq z_2) \in \mathbb{R}_+$, we have

(1) The arithmetic mean:

$$A(z_1, z_2) = \frac{z_1 + z_2}{2}$$

(2) The geometric mean:

$$G(z_1, z_2) = \sqrt{z_1 z_2}$$

(3) The harmonic mean:

$$H(z_1, z_2) = \frac{2z_1 z_2}{z_1 + z_2}$$

(4) The generalized logarithmic mean:

$$L_\rho(z_1, z_2) = \begin{cases} \left[\frac{{z_2}^{\rho+1} - {z_1}^{\rho+1}}{(\rho+1)(z_2 - z_1)}\right]^{\frac{1}{\rho}}, & \rho \neq -1, 0 \\ \frac{z_1 - z_2}{\log z_1 - \log z_2}, & z_1 \neq z_2,\ \rho = -1 \\ \frac{1}{e}\left(\frac{{z_2}^{z_2}}{{z_1}^{z_1}}\right)^{\frac{1}{z_2 - z_1}}, & \rho = 0. \end{cases}$$

Now using our main results, we conclude the following new inequalities.

Proposition 3.1. *Let $z_1, z_2 \in \mathbb{R}$, $0 < z_1 < z_2$, then*

$$\begin{aligned} &\Bigg| \frac{4(z_2 - z_1)}{27}\left[A^3(z_1, 2z_2) - A^3(2z_1, z_2)\right] + \frac{1}{4}A(z_1^4, z_2^4) \\ &\quad + \frac{2}{27}\left[A^4(2z_1, z_2) + A^4(z_1, 2z_2)\right] - L_5^5(z_1, z_2)\Bigg| \\ &\leq (z_2 - z_1)^2\left(\frac{227}{864}\right)[4{z_1}^2 + {z_2}^2]. \end{aligned} \tag{3.1}$$

Proof. The assertion follows from Theorem 2.2 with $\mathfrak{F}(x) = \frac{x^4}{12}$ and a simple computation, where $|\mathfrak{F}''| = x^2$ is φ-convex function with $\varphi(z_1, z_2) = 2z_1 + z_2$. ■

Proposition 3.2. *Let $z_1, z_2 \in \mathbb{R}$, $0 < z_1 < z_2$, then*

$$\begin{aligned} &\Bigg| \frac{10(z_2 - z_1)}{81}\left[A^4(z_1, 2z_2) - A^4(2z_1, z_2)\right] + \frac{1}{4}A(z_1^5, z_2^5) \\ &\quad + \frac{4}{81}\left[A^5(2z_1, z_2) + A^5(z_1, 2z_2)\right] - L_6^6(z_1, z_2)\Bigg| \\ &\leq (z_2 - z_1)^2\left(\frac{1135}{2592}\right)[2{z_1}^3 + {z_1}^9 - z_2^9]. \end{aligned} \tag{3.2}$$

Proof. The assertion follows from Theorem 2.2 with $\mathfrak{F}(x) = \frac{x^5}{20}$ and a simple computation, where $|\mathfrak{F}''| = x^3$ is φ-convex function with $\varphi(z_1, z_2) = 3z_2^2(z_1 - z_2) + 3z_2(z_1 - z_2)^2 + (z_1 - z_2)^3$. ■

Proposition 3.3. *Let $z_1, z_2 \in \mathbb{R}$, $0 < z_1 < z_2$, then we have*

$$\left| \frac{(z_2 - z_1)^2}{12} + \frac{1}{4} A(z_1^2, z_2^2) + \frac{1}{6}\left[A^2(2z_1, z_2) + A^2(z_1, 2z_2)\right] - L_3^3(z_1, z_2) \right| \leq \frac{(z_2 - z_1)^2}{2592}. \tag{3.3}$$

Proof. The assertion follows from Theorem 2.2 with $\mathfrak{F}(x) = x^2$ and a simple computation, where $|\mathfrak{F}''| = 2$ is φ-convex function with $\varphi(z_1, z_2) = z_1 - z_2$. ∎

Proposition 3.4. *Let $z_1, z_2 \in \mathbb{R}$, $0 < z_1 < z_2$, then*

$$\left| \frac{9(z_2 - z_1)}{32}\left[A^{-2}(2z_1, z_2) - A^{-2}(z_1, 2z_2)\right] + \frac{1}{4} H^{-1}(z_1, z_2) + \frac{9}{16}\left[A^{-1}(2z_1, z_2) + A^{-1}(z_1, 2z_2)\right] - L^{-1}(z_1, z_2) \right| \leq (z_2 - z_1)^2 \left(\frac{227}{5184}\right) \max\left\{\frac{2}{z_1^3}, \frac{2}{z_2^3}\right\}. \tag{3.4}$$

Proof. The assertion follows from Theorem 2.8 with $\mathfrak{F}(x) = \frac{1}{x}$ and a simple computation, where $|\mathfrak{F}''| = |\frac{2}{x^3}|$ is φ-convex function with $\varphi(z_1, z_2) = z_1 - z_2$. ∎

3.2. Illustrative Plots

Lastly, for the cases of φ-convex and φ-quasiconvex functions, four three-dimensional plots are given to illustrate the validity of the inequalities.

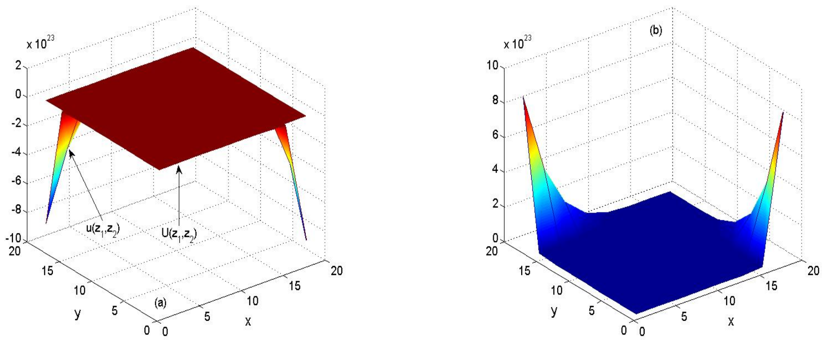

Figure 1. Plot illustrartion for inequality (3.1). (a) For $u(z_1, z_2)$ and $U(z_1, z_2)$. (b) For $U(z_1, z_2) - u(z_1, z_2)$.

From inequality (3.1), we can define

$$u(z_1, z_2) = \left| \frac{4(z_2 - z_1)}{27}\left[A^3(z_1, 2z_2) - A^3(2z_1, z_2)\right] + \frac{1}{4} A(z_1^4, z_2^4) + \frac{2}{27}\left[A^4(2z_1, z_2) + A^4(z_1, 2z_2)\right] - L_5^5(z_1, z_2) \right|$$

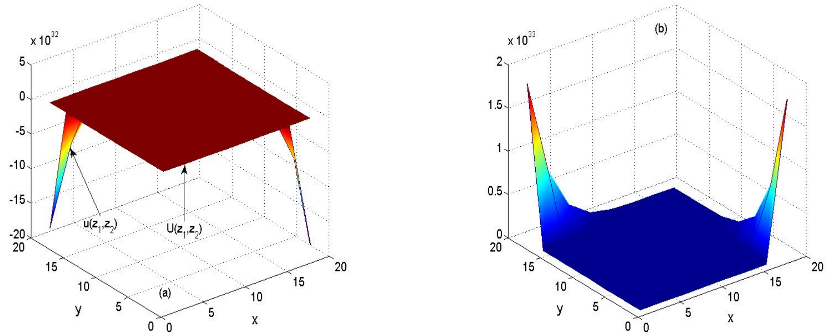

FIGURE 2. Plot illustrartion for inequality (3.2). (a) For $u(z_1, z_2)$ and $U(z_1, z_2)$. (b) For $U(z_1, z_2) - u(z_1, z_2)$.

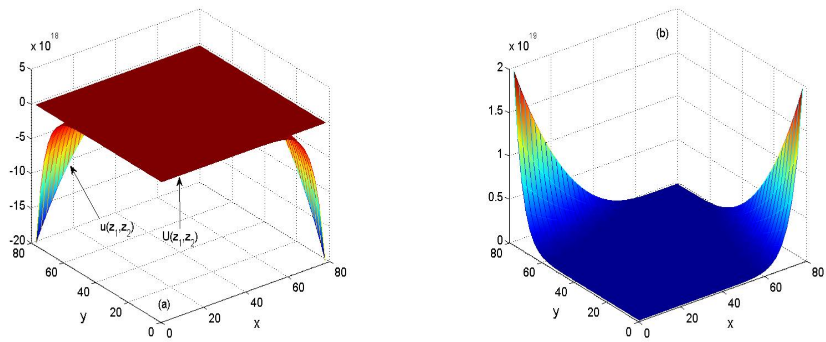

FIGURE 3. Plot illustrartion for inequality (3.3). (a) For $u(z_1, z_2)$ and $U(z_1, z_2)$. (b) For $U(z_1, z_2) - u(z_1, z_2)$.

$$U(z_1, z_2) = (z_2 - z_1)^2 \left(\frac{227}{864}\right)[4{z_1}^2 + {z_2}^2].$$

Hence in Figure 1, the plots of (3.1) and $U(z_1, z_2) - u(z_1, z_2)$ are presented. From inequality (3.2), we can define

$$u(z_1, z_2) = \bigg| \frac{10(z_2 - z_1)}{81} \left[A^4(z_1, 2z_2) - A^4(2z_1, z_2)\right] + \frac{1}{4} A(z_1^5, z_2^5) + \frac{4}{81} \left[A^5(2z_1, z_2) + A^5(z_1, 2z_2)\right] - L_6^6(z_1, z_2) \bigg|$$

$$U(z_1, z_2) = (z_2 - z_1)^2 \left(\frac{1135}{2592}\right)[2{z_1}^3 + {z_1}^9 - z_2^9].$$

Hence in Figure 2, the plots of (3.2) and $U(z_1, z_2) - u(z_1, z_2)$ are presented.

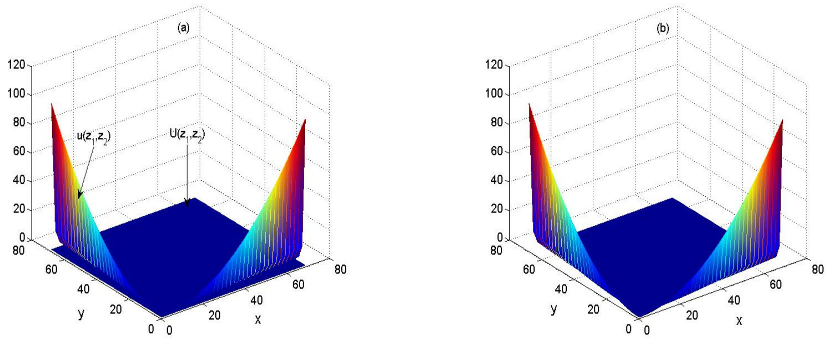

Figure 4. Plot illustrartion for inequality (3.4). (a) For $u(z_1, z_2)$ and $U(z_1, z_2)$. (b) For $U(z_1, z_2) - u(z_1, z_2)$.

From inequality (3.3), we can define

$$u(z_1, z_2) = \left| \frac{(z_2 - z_1)^2}{12} + \frac{1}{4} A(z_1^2, z_2^2) + \frac{1}{6} \left[A^2(2z_1, z_2) + A^2(z_1, 2z_2) \right] - L_3^3(z_1, z_2) \right|$$

$$U(z_1, z_2) = \frac{(z_2 - z_1)^2}{2592}.$$

Hence in Figure 3, the plots of (3.3) and $U(z_1, z_2) - u(z_1, z_2)$ are presented.

From inequality (3.4), we can define

$$u(z_1, z_2) = \left| \frac{9(z_2 - z_1)}{32} \left[A^{-2}(2z_1, z_2) - A^{-2}(z_1, 2z_2) \right] + \frac{1}{4} H^{-1}(z_1, z_2) + \frac{9}{16} \left[A^{-1}(2z_1, z_2) + A^{-1}(z_1, 2z_2) \right] - L^{-1}(z_1, z_2) \right|$$

$$U(z_1, z_2) = (z_2 - z_1)^2 \left(\frac{227}{5184} \right) \max \left\{ \frac{2}{z_1^3}, \frac{2}{z_2^3} \right\}.$$

Hence in Figure 4, the plots of (3.4) and $U(z_1, z_2) - u(z_1, z_2)$ are presented.

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References

[1] M. Alomari, M. Darus, S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, RGMIA Research Report Collection 2 (3) (2009) 1–14.

[2] Y.M. Chu, M.U. Awan, M.Z. Javed, A.G. Khan, Bounds for the remainder in Simpson's inequality via n-polynomial convex functions of higher order using Katugampola fractional integrals, Journal of Mathematics (2020) Article ID 4189036, 10 pages.

[3] M.V. Cortez, T. Abdeljawad, P.O. Mohammed, Y.R. Oliveros, Simpson's integral inequalities for twice differentiable convex functions, Mathematical Problems in Engineering (2020) Article ID 1936461, 15 pages.

[4] Y.P. Deng, M.U. Awan, S.H. Wu, Quantum integral inequalities of Simpson-type for strongly preinvex functions, Mathematics 7 (8) (2019) 751.

[5] S.S. Dragomir, R.P. Agarwal, P. Cerone, On Simpson's inequality and applications, Journal of Inequalities and Applications (2000) 5533-579.

[6] M.E. Gordji, M.R. Delavar, M.D.L. Sen, On ϕ-convex functions, Journal of Mathematical Inequalities 10 (1) (2016) 173–183.

[7] C.Y. Luo, T.S Du, M. Kunt & Y. Zhang, Certain new bounds considering the weighted Simpson–like type inequality and applications, Journal of Inequalities and Applications 2018 (2018) 332.

[8] M. Matloka, Some inequalities of Simpson type for h–convex functions via fractional integrals, Abstract and Applied Analysis (2015) Article ID 956850, 5 pages.

[9] P.O. Mohammed and M.Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, Journal of Computational and Applied Mathematics 372 (2020) 112740.

[10] M.Z. Sarikaya, E. Set, M.Z. Ožde mir, On new inequalities of Simpson's type for s-convex functions, Computers and Mathematics with Applications 60 (2010) 2191–2199.

[11] M.Z. Sarikaya, E. Set, M.E. Ozdemir, On new inequalities of simpsons type for functions whose second derivatives absolute values are convex, Journal of Applied Mathematics, Statistics and Informatics 9 (1) (2013) 37–45.

[12] S. Simic, B.B. Mohsin, Simpson's Rule and Hermite-Hadamard inequality for nonconvex functions, Mathematics 8 (2020) 1248.